# School Mathematics Education

## A Status Report in 2013

R. A. Askey's 80th Birthday, Madison, WI

December 6, 2013

H. Wu

You come to this conference undoubtedly expecting to hear about Askey-Wilson polynomials or the Rogers-Ramanujan identities, but perhaps not about K–12 math education.

Let me therefore try to make it easier for you to get though this lecture.

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**Nah, just kidding!**

If Dick chose to spend a good deal of his time in the last twenty years on school math education, it could not possibly be this trivial.

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*Why not?*

**Example 1.** *What is a fraction?*

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School mathematics' inability to say what
a fraction is has precipitated a crisis
in education for decades.

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We have the set $\mathbb{Q}$ of fractions, and we are determined to make $\mathbb{Q}$ into a field. The definition

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Would you like to explain this to an eleven-year old?

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For example, why is two-thirds of $2\frac{1}{2}$ cups of punch equal to

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They can't care less what a "field" is and whether its multiplication is "distributive" or not.

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Problems of constant speed are a staple of grades 6–11, and we must find ways to explain to twelve-year olds what it means.

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Can we get thirteen-year olds to accept this when they are still struggling to grasp what a linear equation of two variables is, and what the *graph of an equation* means?

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Fourteen year-olds have to solve optimization problems involving quadratic functions. They know nothing about calculus.

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We cannot improve K–12 education by teaching prospective teachers and educators the same old, same old. The process of *customization* is too arduous to be performed individually by teachers and educators.

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School mathematics education is, in this sense, **mathematical engineering**: It customizes abstract mathematics to meet the needs of K–12 students.

School mathematics is an engineered product, and this is why

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He conveniently forgot Kolmogorov's disastrous math education reform in the former Soviet Union in the sixties. (V. Arnold reportedly contemplated severing relations with his revered teacher over this issue.)

My conjecture is that the success of mathematicians It is not first rate or second rate; it is the engineering that matters.

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What *we have* is abstract mathematics.
What *they need,* desperately, is instruction on well-engineered mathematics.

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Let us call it **TSM** (Textbook School Mathematics).

TSM is what has been taught by teachers and promulgated by educators for more or less the past four decades, because *this is what they have been getting from the math community*.

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In musical terms, we in the math community are the composers, and math teachers and math educators are the performers.

If the performance of a composition falls flat, are the performers entirely to blame?

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(More about the reform later.)

So what is TSM all about? Let us look at the previous five examples of mathematics that needed good engineering to make them usable in K–12.

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**The reform:** Leave all three concepts undefined but improve on the math instruction by giving more examples, analogies, and heuristic arguments.

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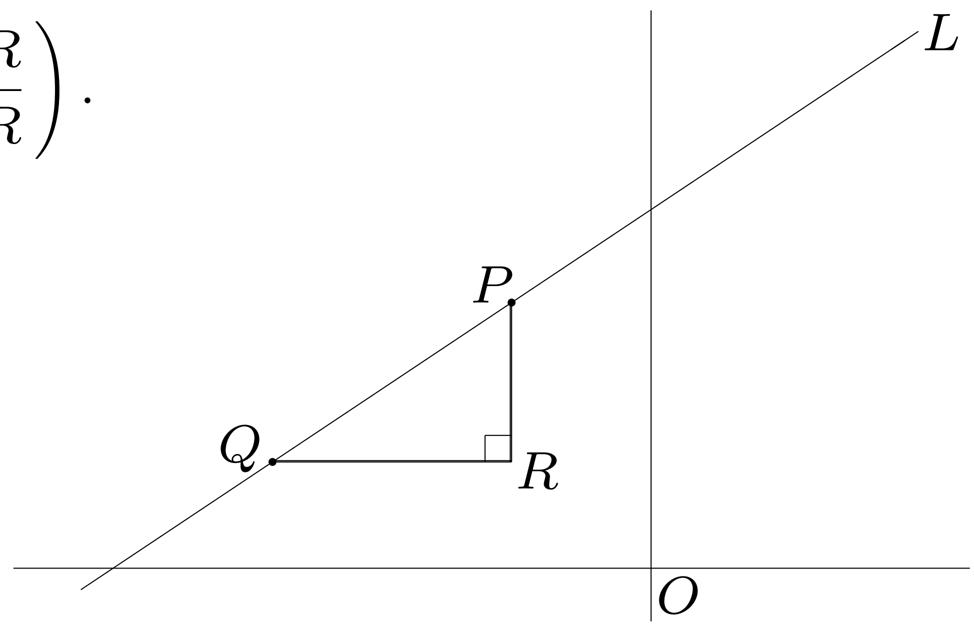
We should explain what "defining slope by rote" means.

Let $L$ be a nonvertical line in the coordinate plane and let $P = (p_1, p_2)$ and $Q = (q_1, q_2)$ be distinct points on $L$. According to TSM, the definition of the **slope of $L$** is:

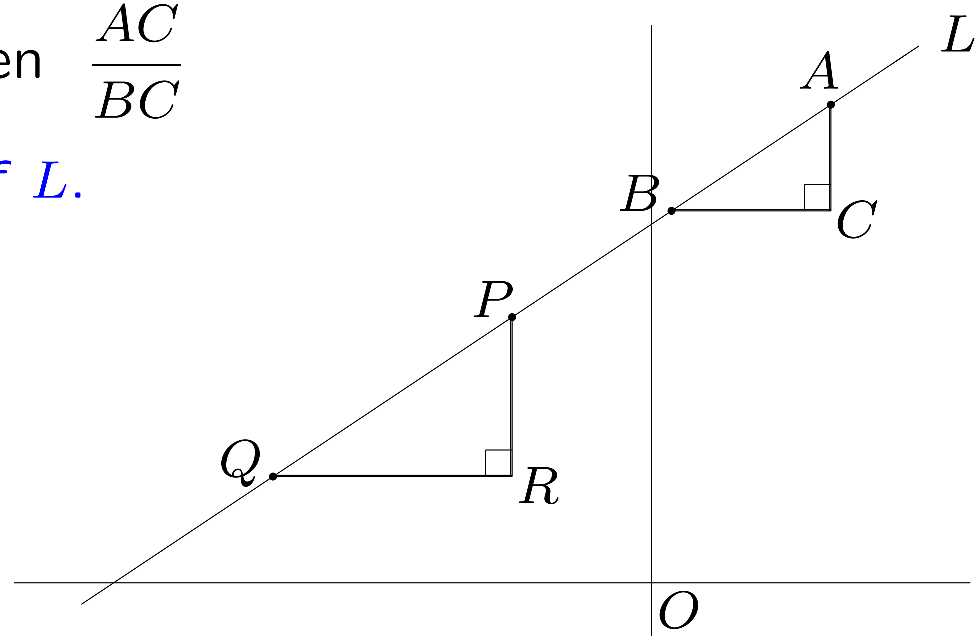
$$\frac{p_2 - q_2}{p_1 - q_1} \left(= \frac{PR}{QR}\right).$$



Lines have lots of points. What if two different points $A$ and $B$ on $L$ are chosen instead?

With $A$ and $B$ chosen, then $\dfrac{AC}{BC}$

would also be the slope of $L$.



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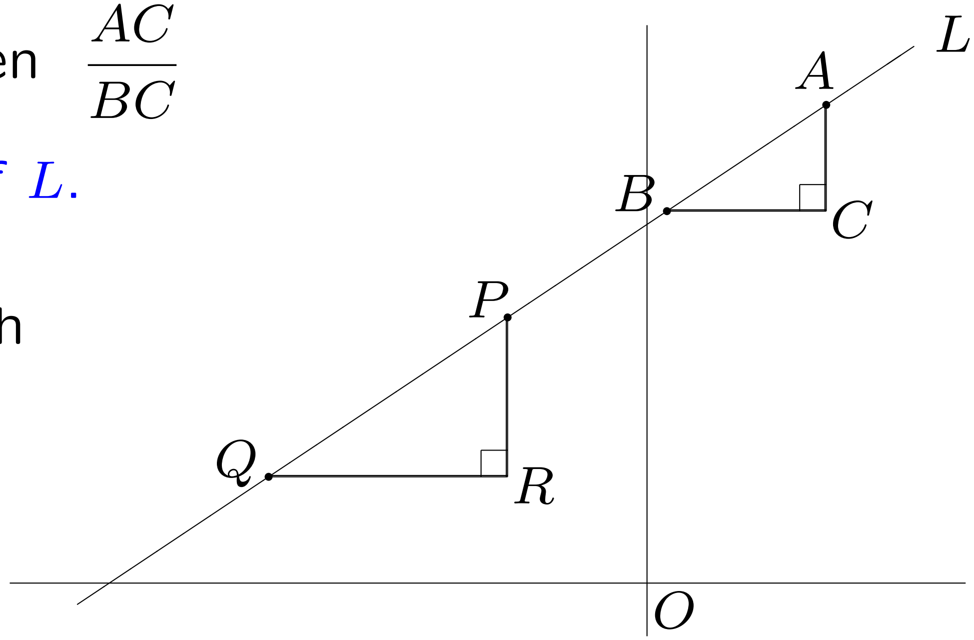


With $A$ and $B$ chosen, then $\dfrac{AC}{BC}$ would also be the slope of $L$.

But if $\dfrac{PR}{QR} \neq \dfrac{AC}{BC}$, which of the two ratios should be *the slope of $L$?*

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Doesn't this confuse students? Absolutely! *Slope* is a major impasse in the learning of algebra.

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**TSM engineering:** Students are taught to use completing the square to express $ax^2 + bx + c$ "in vertex form",

$$a\left(x - \frac{b}{2a}\right) + \frac{4ac - b^2}{4a},$$

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Perhaps a few more examples of TSM would further clarify what it is.

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*Profound flaw:* Students are confused about what the *equal sign* "=" is supposed to mean. There is data to show that this confusion adversely affects student learning in algebra.

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*Profound flaw:* It confuses ten-year olds about what "addition" is. They managed to learn that addition is *combining things* when adding whole numbers, but now addition is something *inscrutable.* This is the beginning of math phobia.

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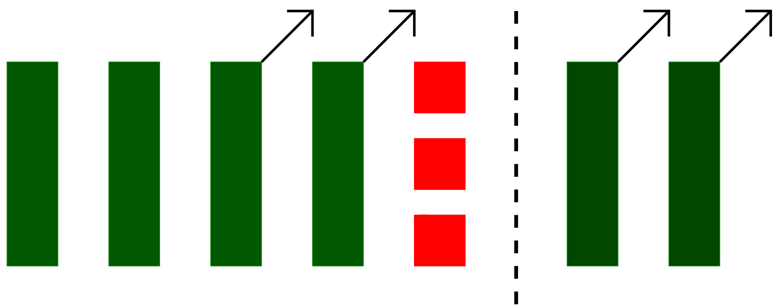
$(i)$ Appeal to Euclid: *equals added to equals remain equal.*

(*"Equal" as what?*)

$(ii)$ Use algebra tiles to "model" this equation $4x - 3 = 2x$. Thus let a **green** rectangle model a variable and a red square model $-1$. Then if we remove two green tiles on the left (i.e., adding $-2x$), we should be able to also remove two green tiles on the right without destroying the equality:



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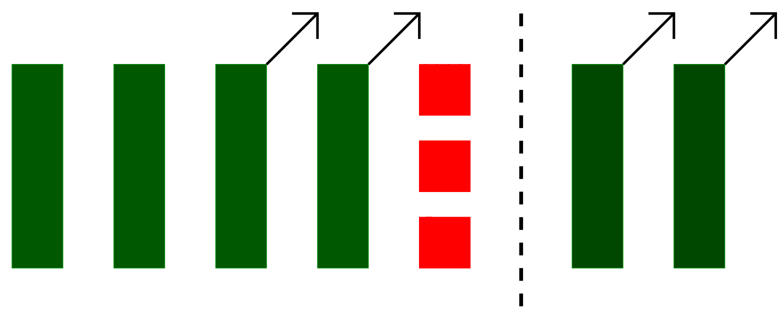
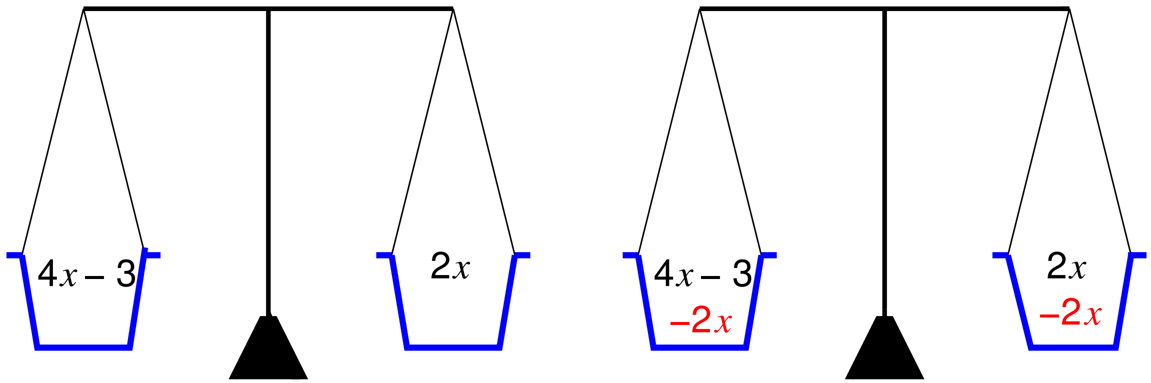


(*Solving-an-equation is supposed to be done by reasoning, not by making an analogy using a manipulative.*)

$(iii)$ Use a balance scale to "model" the equation $4x - 3 = 2x$. If we remove $2x$ (whatever it is) from both weighing pans, the pans will stay in balance.



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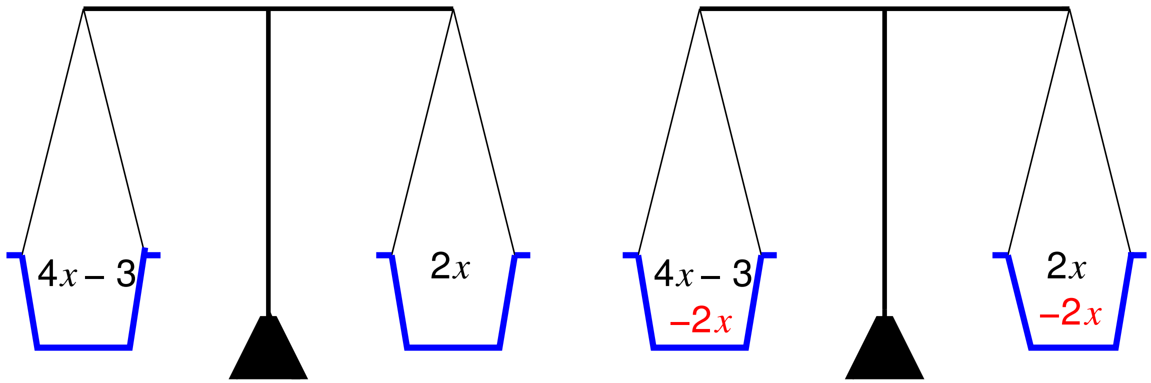


(*Again, solving-an-equation has to be done by reasoning, not by making an analogy.*)

Of course, once one gets past

$$(-2x) + (4x - 3) \;=\; (-2x) + 2x,$$

one can repeat the same procedure to get the usual solution:

$$\begin{aligned} 2x - 3 &= 0 \\ (2x - 3) + 3 &= 0 + 3 \\ 2x &= 3 \\ x &= \frac{3}{2} \end{aligned}$$

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*Rather, if $x$ is a solution (thus a number), then the usual computation shows that,* necessarily, *$x$ is equal to a finite number of possibilities. One then goes back to verify that these possibilities are honest solutions.*

Think about what this kind of teaching does to students' perception of "equality". They no longer know what "$A = B$" means anymore.

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Don't you think that if some competent mathematicians had taken the slightest bit of interest in education way back, this kind of phony reasoning about algebra tiles and balance scale would have been stopped dead on its tracks?

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However, in high school geometry,

*TSM defines* **congruent polygons** *and* **similar polygons** *anew in terms of equal angles and proportional sides.*

*First, students are told that vague statements such as* same size and same shape *and* same shape but not necessarily the same size *are acceptable definitions in mathematics. Then they are victimized a second time by being told,* with no explanations, *to*

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*discard these definitions, and*

*replace them with definitions that work only for polygons.*

*They end up not knowing what congruence is or what similarity means.*

The list goes on and on, but there is no need.

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Any kind of improvement in school math education must begin with the eradication of TSM, period.

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*But*, here is the catch: If teachers only know TSM, they wouldn't reject TSM-based textbooks.

**Obvious question:** To get rid of **Textbook** School Mathematics, isn't it simpler to just get better textbooks written?

In a capitalist society, nobody can tell publishers what to write except the *customers*, i.e., the teachers and students, and ultimately, only the teachers.

*But*, here is the catch: If teachers only know TSM, they wouldn't reject TSM-based textbooks.

They would reject TSM-based textbooks only when we teach them non-TSM school mathematics. *And we haven't started to do that yet.*

This is also the place to clarify the statement that "school math education must begin with the eradication of TSM".

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It is not even a *sufficient* condition for good education: Beyond mathematical content, there is pedagogy and all the political considerations relate to equity.

However, eradicating TSM is a *necessary* condition for good math education.

A lesson on fundamentally flawed mathematics has no redeeming social values.

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**TSM is self-reproducing**, in the following sense.

School students of today will be the *teachers* of tomorrow. They learn TSM in school, but when they go to college, *they do not get any new knowledge to replace TSM.* It comes to pass that when they begin teaching, they teach—yes, TSM to their own students.

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School students today will also be the *educators* of tomorrow. They retrace the exact same intellectual journey as teachers, and when they do their research or make recomendations, they will do it on the basis of TSM.

When teachers and educators are doomed to recycle TSM, TSM will be the de facto school mathematics in perpetuity.

*Is this so bad?*

Here are three trivial examples of why the domination of math education research by TSM is harmful:

(1) Research on "students' misunderstanding of the equal sign".

We have seen (cf. division-with-remainder and solving equations) how TSM corrupts students' conception of equality. To educators brought up in TSM, the damage done to students by a defective curriculum would not be apparent. Consequently, the blame for students' misconception is likely to be placed on the students themselves.

This is how TSM can warp a researcher's perception. The real culprit is the curriculum, not the students.

(2) Research on the teaching and learning of fractions, thus far, accepts the TSM version (a fraction is a piece of pizza) as given and almost all efforts have been spent on making *pizzas* more palatable and more teachable.

There seems to be little awareness that better engineering can produce a *fully developed* version of the *mathematics* of fractions that is usable in K-12. In fact, this version has been incorporated into the recent **Common Core Standards**.

The research on fractions would have been relevant to school math education had it not been distorted by TSM.

**Digression:**

The Common Core Standards are the outcome of an initiative sponsored by the National Governors' Association (NGA) and the Council of chief State School Officers (CCSSO) that tries to introduce uniformity in the nation's education. So far, there are Mathematics Standards and English Language Arts Standards (2010).

Forty-five states have adopted the standards.

The Common Core State Standards in Mathematics **(CCSSM)** are the only set of standards to show any awareness of the deleterious effects of TSM. In many (though not all) places, the CCSSM make an effort to prescribe ways to get around TSM.

There have been many criticisms of the Common Core movement, usually about the assessment or the policies. However, the criticisms of the *mathematics* in the CCSSM are mostly misinformed.

(3) In the 1989 NCTM Standards, it is stated:

> The proficiency in the addition, subtraction, and multiplication of fractions and mixed numbers should be limited to those with simple denominators that can be visualized concretely or pictorially and are apt to occur in real-world settings; such computation promotes conceptual understanding of the operations. This is not to suggest, however, that valuable instructional time should be devoted to exercises like $17/24 + 5/18$ or $5\frac{3}{4} \times 4\frac{1}{4}$, which are much harder to visualize and unlikely to occur in real-life situations. (page 96)

This is a recommendation that already assumes that the *school mathematics* of fractions is what TSM has to offer. It is so intrinsicaly defective that one shouldn't bother to make sense of it.

To the writers of the NCTM Standards, adding and multiplying fractions were rote procedures that had to be memorized without a trace of reasoning. Since they saw that the TSM of fractions was unlearnable, they decided to undercut the whole subject.

But there are more substantive reasons why we should break TSM's stranglehold on school education.

You may have noticed that, up to this point, I have not said anything about why the math community should pay some attention to school math education.

I will try to do so now.

In our immediate past, the event that first galvanized national attention on math education was the 1995 **TIMSS** (Trends in Mathematics and Science Study) result.

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In 1995, the U.S. was above average in grade 4, below average in grade 8, but the result for the 12th grade was devastating: *In a field of 21 countries, the U.S. students' achievement ranked ahead of only 3 nations: Lithuania, Cyprus, and South Africa.*

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(Dick succeeded in convincing many people that things are worse than they'd like to believe. This is typical of Dick's contributions to school mathematics education: he is content to work behind the scenes.)

Which shows a correct method for finding $\frac{1}{3} - \frac{1}{4}$?

**A** $\dfrac{(1-1)}{(4-3)}$

**B** $\dfrac{1}{(4-3)}$

**C** $\dfrac{(3-4)}{3 \times 4}$

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| | Correct | A | B | C | D |
|---|---|---|---|---|---|
| Average | 37.1 | 25.4 | 26.0 | 9.4 | 37.1 |
| Hong Kong | 77.0 | 4.0 | 8.7 | 10.0 | 77.0 |
| Korea | 86.0 | 2.7 | 6.9 | 4.2 | 86.0 |
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*This clearly shows the effect of TSM on student learning!*

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What it pointed to was the dwindling pool of students proficient in mathematics. The dearth of scientists and engineers was the inevitable consequence.

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For FY 2008, the entire annual H-1B quota was *exhausted in the first day*, and for FY 2014, the quota was used up within a few days.

In 2005, Congress duly took note and commissioned the National Academies to look into how we might "enhance the science and technology enterprise so that the United States can successfully compete, prosper, and be secure in the global community of the 21st century".

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The result was the volume **Rising Above the Gathering Storm** of 2007. Remarkably, a second edition followed in 2010:

**Rising Above the Gathering Storm, Revisited.**
*Rapidly Approaching Category 5.*

RAGS is what is called a *high-profile volume.*

Included in the 20 committee members who wrote the 2005 volume were

CEOs of DuPont, Exxon Mobile, Intel, Lockheed Martin,
and Merck,
three Noble Prize winners,
two recipients of the Natonal Medal of Technology, and
five university presidents.

RAGS envisions the end of American leadership in science and technology in the coming decades. It makes four recommendations for change, and the first is to **“Increase America’s talent pool by vastly improving K-12 science and mathematics education.”**

The recommended action of highest priority is to **place knowledgeable math and science teachers in the classrooms.**

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The recommended action of highest priority is to **place knowledgeable math and science teachers in the classrooms.**

*And the reason for a second edition in three years?*

". . . our overall public school system—or more accurately 14,000 systems—has shown little sign of improvement, particularly in mathematics and science."

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Competent mathematicians must begin to teach our teachers and educators non-TSM *school mathematics* in order to break the vicious cycles.

We as a community have never paid much attention to school education. It is time to change.

I am not saying **all** of us have to change, but a few competent ones should. If we mathematicians don't teach the mathematics, who would?

There is another good reason for us to begin teaching prospective math teachers better.

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The CCSSM will be officially implemented in the fall of 2014. As of this moment, it looks to be an eerie reenactment of the New Math scenario: teachers will be mouthing words that they don't understand because many parts of the Common Core Standards ask that *school mathematics*—not TSM—be taught.

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But we in institutions of higher learning have never helped prospective teachers unlearn their TSM.

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We can kill two birds with one stone:

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**Why don't we do the right thing this time around?**