ORTHOGONALITY OF VERY WELL-POISED SERIES

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Abstract. Rodrigues formulas for very well-poised basic hypergeometric series of any order are given. Orthogonality relations are found for rational functions with an arbitrary number of parameters which generalize the Askey–Wilson polynomials and Rahman's ${}_{10}\phi_9$ biorthogonal rational functions. A pair of orthogonal rational functions of type R_{II} is identified. Elliptic analogues of some of these results are also included.

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1. Introduction

The Askey–Wilson polynomials are classical orthogonal polynomials which depend on five parameters. These polynomials lie at the top of the Askey scheme of classical orthogonal polynomials, and have an expression as a balanced basic hypergeometric series. An alternative representation is given by a very well-poised basic hypergeometric series. Although they have been generalized to polynomials in several variables, in one variable, the only known basic hypergeometric generalization is to a set of biorthogonal rational functions given by Rahman [12], which has six parameters. An elliptic version due to Spiridonov [20] has seven parameters

This work started as an attempt to understand the Rahman biorthogonal rational functions, where they live, and what is the correct level of generality. Our efforts led to a biorthogonal system of very-well poised series with an arbitrary number of parameters. The orthogonality relation, Theorem 4.2, contains the Askey–Wilson and Rahman results as special cases. We also give an elliptic version of Theorem 4.2 in Theorem 8.9.

The key idea is the realization that a general set of very well-poised basic hypergeometric series always have Rodrigues formulas. The Rodrigues formula plays an important

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role in the theory of classical orthogonal polynomials [16], [22]. By a Rodrigues-type representation of a sequence of functions $f_n(x)$, we mean representing f_n as

$$f_n(x) = \frac{c_n}{g_0(x)} T^n g_n(x), \quad n = 0, 1, \cdots, \tag{1.1}$$

where the c_n's are constants and T is a linear operator which does not depend on n.

In [10] it is shown that Watson's transformation of a balanced terminating ${}_4\phi_3$ to a very well-poised ${}_8\phi_7$ is the Rodrigues formula for the Askey–Wilson polynomials. This transformation gives two possible expressions of the Askey-Wilson polynomials. This motivated us to explore Rodrigues type formulas for the ${}_{10}\phi_9$ biorthogonal rational functions R_n and S_n of [12], and consider orthogonality relations for higher order very well-poised series.

In Theorem 3.4 we give a Rodrigues formula of the type (1.1) for a ${}_{2m+8}W_{2m+7}$ function (see Definition 3.3) which generalizes Rahman's rational functions R_n and S_n. We then provide a general orthogonality relation for a ${}_{2m+8}W_{2m+7}$, Theorem 4.2, which generalizes Rahman's biorthogonality relation. A polynomial orthogonality for a ${}_{10}W_9$ is given in Theorem 6.5. Our analysis is completely analogous to polynomials orthogonal with respect to varying weights. There is extensive literature in this area, a sample of which is in [18].

Rahman and Suslov [14] have a Rodrigues type formula for a ${}_{10}\phi_9$ function, but their formulas do not resemble the classical Rodrigues formula. Indeed instead of T^n the Rahman-Suslov formulas involve $T_n T_{n-1} \cdots T_1$, where T_j is linear but depends on j. In [9] Ismail and Rahman gave a three term recurrence relation of type R_{II} for the Rahman functions.

The paper is organized as follows. After preliminary material is introduced in §2, in §3 we define the rational functions and give the Rodrigues formula. The orthogonality relation is established in §4. The polynomial behavior of our rational functions is determined in §5. The special case of ${}_{10}\phi_9$'s is considered in §6, where Rahman's biorthogonality results are reproven. Asymptotics are given in §7. The elliptic generalizations of our main results are given in §8. Section 9 establishes a three term recurrence relation for a system of polynomials $\{U_n(x; \mathbf{t})\}$ we introduce in §6.

The recurrence relations for the biorthogonal rational functions in this work are of the type R_{II} and are associated with R_{II} continued fractions introduced by Ismail and Masson in [8]. Zhedanov [23] pointed out that they arise in a generalized eigenvalue problem and the biorthogonality is between solutions to adjoint generalized eigenvalue problems.

2. Preliminaries

We shall use the notation and terminology in [1], [6], and [7] for basic hypergeometric series. In this section we collect the results to be used in the rest of the paper.

We shall use the inner product associated with the Chebyshev weight $(1-x^2)^{-1/2}$ on $(-1, 1)$, namely

$$< f, g > := \int_{-1}^{1} f(x)\, \overline{g(x)}\, \frac{dx}{\sqrt{1-x^2}}.$$

The operator we iterate for the Rodrigues formulas is the Askey–Wilson operator $\mathcal{D}_q$, (see [7])

$$(\mathcal{D}_q f)(x) = 2\frac{\breve{f}(q^{1/2}z) - \breve{f}(q^{-1/2}z)}{(q^{1/2}-q^{-1/2})(z-1/z)}$$

where $x = (z+1/z)/2 = \cos\theta$, $f(x) = \breve{f}(z)$, $z = e^{i\theta}$. It must be noted that $x = (z+1/z)/2$ makes z and $1/z$ interchangeable. However to specify which branch of the Riemann surface we assume that $|z| \geq |1/z|$. Indeed $|z| = |1/z|$ if and only if $x \in [-1,1]$, in which case we put $x = \cos\theta$ for a unique $\theta \in [0,\pi]$ and $z = e^{i\theta}$. The operator $\mathcal{D}_q$ was first introduced in [2].

Observe that the definition of $\mathcal{D}_q$ requires $\breve{f}(z)$ to be defined for $\left|q^{\pm 1/2}z\right| = 1$ as well as for $|z| = 1$. In particular $\mathcal{D}_q$ is well-defined on $H_{1/2}$, where

$$H_\nu := \left\{ f : f((z+1/z)/2) \text{ is analytic for } q^\nu \leq |z| \leq q^{-\nu} \right\}.$$

The key fact of Cooper [4, Prop. 1q] which we shall use is that the n^{th} iterate of the Askey–Wilson operator may be expanded via very well-poised series.

Proposition 2.1. *The n^{th} iterate of the Askey–Wilson operator $\mathcal{D}_q$ satisfies*

$$\mathcal{D}_q^n f(x) = \frac{(-2/z)^n q^{\frac{1}{2}\binom{n}{2}}}{(q^{1/2}-q^{-1/2})^n (1/z^2;q)_n} \sum_{k=0}^{n} \frac{(q^{-n}, z^2q^{-n};q)_k}{(qz^2,q;q)_k} \frac{1-z^2q^{-n+2k}}{1-z^2q^{-n}} q^{nk} \breve{f}(q^{(2k-n)/2}z).$$

The right side of Proposition 2.1 is invariant under $z \to 1/z$, this is reversing the finite series.

3. Rodrigues formulas and very well-poised series

In this section we give in Theorem 3.4 a Rodrigues formula for the general very well-poised basic hypergeometric series

$${}_{2m+8}W_{2m+7}(q^{-n}z^2; q^{-n}, a_1z, \cdots, a_{m+4}z, q^{1-n}z/b_1, \cdots, q^{1-n}z/b_m; q, Z_1),$$

where

$$Z_1 = q^{2-n} b_1 \cdots b_m / a_1 \cdots a_{m+4}.$$

The first application of Proposition 2.1 uses

$$\breve{f}(z;\mathbf{a},\mathbf{b}) = \prod_{i=1}^{m} \frac{(b_iz, b_i/z;q)_\infty}{(a_iz, a_i/z;q)_\infty},$$

where $\mathbf{a} = (a_1,\ldots,a_m)$ and $\mathbf{b} = (b_1,\ldots,b_m)$. Note that

$$\breve{f}(q^{k-n/2}z; q^{n/2}\mathbf{a}, q^{n/2}\mathbf{b}) = \breve{f}(z;\mathbf{a},\mathbf{b}) \prod_{i=1}^{m} \frac{(a_iz;q)_k (a_i/z;q)_{n-k}}{(b_iz;q)_k (b_i/z;q)_{n-k}}.$$

Proposition 3.1. *The functions*

$$r_n(x;\mathbf{a},\mathbf{b}) = \frac{(-2/z)^n q^{\frac{1}{2}\binom{n}{2}}}{(q^{1/2}-q^{-1/2})^n(1/z^2;q)_n}\prod_{i=1}^m \frac{(a_i/z;q)_n}{(b_i/z;q)_n} \times {}_{2m+4}W_{2m+3}(q^{-n}z^2;q^{-n},a_1z,\cdots,a_mz,q^{1-n}z/b_1,\cdots,q^{1-n}z/b_m;q,Z_2),$$

where

$$Z_2 = \frac{b_1\dots b_m}{a_1\dots a_m}q^n$$

satisfy the Rodrigues formula

$$r_n(x;\mathbf{a},\mathbf{b}) = \frac{1}{\breve{f}(z;\mathbf{a},\mathbf{b})}\mathcal{D}_q^n(\breve{f}(z;q^{n/2}\mathbf{a},q^{n/2}\mathbf{b})).$$

The next application incorporates an Askey–Wilson weight into $\breve{g}$.

Definition 3.2. *Let*

$$\breve{g}(z;\mathbf{a},\mathbf{b}) = \frac{2i(z^2,q/z^2;q)_\infty \prod_{i=1}^m(b_iz,b_i/z;q)_\infty}{z\prod_{i=1}^{m+4}(a_iz,a_i/z;q)_\infty},$$

where $\mathbf{a}=(a_1,\dots,a_{m+4})$ *and* $\mathbf{b}=(b_1,\dots,b_m)$.

Note that $\breve{g}(z;\mathbf{a},\mathbf{b})$ involves theta products, $\theta(a):=(a,q/a;q)_\infty$, which satisfy $\theta(aq^p) = (-1)^p q^{-\binom{p}{2}}\theta(a)$, for all $p\in\mathbb{Z}$, see Section 8.

Observe that

$$\breve{g}(q^{k-n/2}z;q^{n/2}\mathbf{a},q^{n/2}\mathbf{b}) = \breve{g}(z;\mathbf{a},\mathbf{b})\frac{\prod_{i=1}^{m+4}(a_iz;q)_k(a_i/z;q)_{n-k}}{\prod_{i=1}^m(b_iz;q)_k(b_i/z;q)_{n-k}}z^{2n-4k}(-1)^n q^{-\binom{2k-n}{2}}q^{n/2-k}.$$

Definition 3.3. *For a non-negative integer n define*

$$p_n(x;\mathbf{a},\mathbf{b}) = \frac{(2z)^n q^{-\frac{1}{2}\binom{n+1}{2}}}{(q^{1/2}-q^{-1/2})^n(1/z^2;q)_n}\frac{\prod_{i=1}^{m+4}(a_i/z;q)_n}{\prod_{i=1}^m(b_i/z;q)_n} \times {}_{2m+8}W_{2m+7}(q^{-n}z^2;q^{-n},a_1z,\cdots,a_{m+4}z,q^{1-n}z/b_1,\cdots,q^{1-n}z/b_m;q,Z_1),$$

where

$$Z_1 = \frac{b_1\dots b_m}{a_1\dots a_{m+4}}q^{2-n}.$$

Theorem 3.4. *The functions $p_n(x;\mathbf{a},\mathbf{b})$ satisfy the Rodrigues formula*

$$p_n(x;\mathbf{a},\mathbf{b}) = \frac{1}{\breve{g}(z;\mathbf{a},\mathbf{b})}\mathcal{D}_q^n(\breve{g}(z;q^{n/2}\mathbf{a},q^{n/2}\mathbf{b})).$$

It must be noted that $\breve{g}(z;\mathbf{a},\mathbf{b})$ is not symmetric in $z\to 1/z$ but is antisymmetric. Moreover if $h(z)$ satisfies $h(1/z)=-h(z)$ the so does the quotient

$$\frac{h(q^{1/2}z)-h(q^{-1/2}z)}{z-1/z}.$$

and its iterates. This implies that the functions $p_n(x;\mathbf{a},\mathbf{b})$ are symmetric in z and $1/z$, hence they are functions of x.

We shall refer to these polynomials as the IRS polynomials.

4. Orthogonality of very-well poised series

In this section we use Theorem 3.4 and q-integration by parts to give an orthogonality relation for $p_n(x;\mathbf{a},\mathbf{b})$ in Theorem 4.2. When $m=0$ Theorem 4.2 is the orthogonality relation for Askey–Wilson polynomials.

To derive orthogonality from a Rodrigues formula we need an appropriate integration by parts formula. Brown, Evans and Ismail proved the following analogue of q-integration by parts in [3, (1.12)].

Theorem 4.1. *The Askey–Wilson operator $\mathcal{D}_q$ satisfies*

$$< \mathcal{D}_q f, g >= \frac{\pi\sqrt{q}}{1-q}\left[f\left(\frac{1}{2}\left(q^{1/2}+q^{-1/2}\right)\right)\overline{g(1)} - f\left(-\frac{1}{2}\left(q^{1/2}+q^{-1/2}\right)\right)\overline{g(-1)}\right] - < f, \sqrt{1-x^2}\,\mathcal{D}_q\left(g(x)\left(1-x^2\right)^{-1/2}\right) >,$$

for $f,\ g\in H_{1/2}$.

Let

$$h(x,a)=\prod_{k=0}^{\infty}(1-2axq^k+a^2q^{2k})=(ae^{i\theta},ae^{-i\theta};q)_\infty,\quad x=\cos\theta,$$

and

$$w(x;\mathbf{a},\mathbf{b})=\frac{h(2x^2-1,1)}{\sqrt{1-x^2}}\frac{\prod_{i=1}^{m}h(x,b_i)}{\prod_{i=1}^{m+4}h(x,a_i)}.$$

If $m=0$, $w(x;\mathbf{a},\varnothing)$ is the Askey–Wilson weight function.

Theorem 4.2. *Assume that $|a_j|<1, 1\le j\le m+4$, the a_j's are real or appear in conjugate pairs. Then for any polynomial $\pi(x)$ of degree at most $n-1$,*

$$\int_{-1}^{1}p_n(x,\mathbf{a},\mathbf{b})\pi(x)w(x;\mathbf{a},\mathbf{b})dx=0.$$

Proof. Note that $w(x;\mathbf{a},\mathbf{b})=\breve{g}(z;\mathbf{a},\mathbf{b})$. We first assume that $|a_jq^{-n/2}|<1$ for $1\le j\le m+4$. Use Theorem 3.4 and Theorem 4.1 n times. Each boundary term in the formula for q-integration by parts is 0 because of the presence of the factor $\sqrt{1-x^2}$. The action of $\mathcal{D}_q$ on products of factors of the type $1/h(x,a)$ produces products of factors of the type $1/h(x,q^{-1/2}a)$. The analyticity assumptions in Theorem 4.1 are satisfied since $\breve{g}(z;\mathbf{a},\mathbf{b})=0$ if $z=q^{j/2}$ for any integer j. The restriction $|a_jq^{-n/2}|<1$ can be removed by analytic continuation since $1/h(x,a)$ is analytic in a in the open unit disc and for all $x\in[-1,1]$. □

5. Polynomial nature of $p_n(x;\mathbf{a},\mathbf{b})$

From Definition 3.3 it would appear that

$$q_n(x;\mathbf{a},\mathbf{b}):=\prod_{j=1}^{m}(b_jz,b_j/z;q)_np_n(x;\mathbf{a},\mathbf{b})$$

has poles at the zeros of $(1/z^2;q)_n$. However these singularities are removable. The main result of this section is Corollary 5.7 which establishes the polynomial character of $q_n(x;\mathbf{a},\mathbf{b})$.

We shall use the Rodrigues formula to give an inductive proof of this fact. First we reformulate the Rodrigues formula as a recursive procedure.

Proposition 5.1. *For any positive integer n we have*

$$q_n(x;\mathbf{a},\mathbf{b}) = \frac{1}{\breve{g}(z;\mathbf{a},q^n\mathbf{b})}\mathcal{D}_q(\breve{g}(z;q^{1/2}\mathbf{a},q^{n-1/2}\mathbf{b})q_{n-1}(x;q^{1/2}\mathbf{a},q^{1/2}\mathbf{b})).$$

Proof. The case $n=1$ is the case $n=1$ of Theorem 3.4. The inductive step follows from

$$\begin{aligned} p_{n+1}(x;\mathbf{a},\mathbf{b}) &= \frac{1}{\breve{g}(z;\mathbf{a},\mathbf{b})}\mathcal{D}_q^{n+1}\left[\breve{g}(z;q^{(n+1)/2}\mathbf{a},q^{(n+1)/2}\mathbf{b})\right] \\ &= \frac{1}{\breve{g}(z;\mathbf{a},\mathbf{b})}\mathcal{D}_q\left[\breve{g}(z;q^{1/2}\mathbf{a},q^{1/2}\mathbf{b})p_n(x;q^{1/2}\mathbf{a},q^{1/2}\mathbf{b})\right] \\ &= \frac{1}{\breve{g}(z;\mathbf{a},\mathbf{b})}\mathcal{D}_q\left[\breve{g}(z;q^{1/2}\mathbf{a},q^{n+1/2}\mathbf{b})q_n(x;q^{1/2}\mathbf{a},q^{1/2}\mathbf{b})\right]. \end{aligned}$$

Multiplying both sides by $\prod_{j=1}^m(b_jz,b_j/z;q)_{n+1}$ gives the desired result for $n+1$. □

In Theorems 5.3 and 5.6 we find the leading term of $q_n(x;\mathbf{a},\mathbf{b})$ and the next lemma enables us to do this.

We denote the coefficient of z^m in a Laurent polynomial $c(z)$ by $[z^m]c(z)$.

Lemma 5.2. *Let $c(z)$ be a Laurent polynomial with degrees bounded between $-m-2$ and $m+2$, and let $\breve{f}(z)$ be a symmetric Laurent polynomial of degree k. Then*

$$\frac{c(z)\breve{f}(q^{1/2}z)-c(1/z)\breve{f}(q^{-1/2}z)}{z-1/z}$$

is a symmetric Laurent polynomial of degree $m+1+k$, with leading coefficient

$$(q^{k/2}[z^{m+2}]c(z)-q^{-k/2}[z^{-m-2}]c(z))[z^k]\breve{f}(z).$$

Proof. The poles at $z=1$ and $z=-1$ are cancelled by zeros of the numerator (since $\breve{f}(\pm q^{1/2}) = \breve{f}(\pm q^{-1/2})$ by the symmetry of $\breve{f}$), and thus the result is a Laurent polynomial, the symmetry of which follows from the symmetry of $\breve{f}$. The claim about the leading coefficient follows by dividing by z^{m+1+k} and taking the limit $z\to\infty$. □

Theorem 5.3. *For any $\mathbf{a}$, $\mathbf{b}$, $q_n(x;\mathbf{a},\mathbf{b})$ is a polynomial in x of degree at most $(m+1)n$. The inequality on the degree is strict if and only*

$$b_1\cdots b_m = q^{n-1+s}a_1\cdots a_{m+4},$$

for some $0\le s\le n-1$.

Proof. First it is useful to note that

$$c(z) = \frac{\breve{g}(zq^{1/2};\mathbf{a}q^{1/2},\mathbf{b}q^{n-1/2})}{\breve{g}(z;\mathbf{a},\mathbf{b}q^n)},$$

implies that

$$c(1/z) = \frac{\breve{g}(zq^{-1/2}; \mathbf{a}q^{1/2}, \mathbf{b}q^{n-1/2})}{\breve{g}(z; \mathbf{a}, \mathbf{b}q^n)}.$$

A straightforward induction using Proposition 5.1 and Lemma 5.2 with

$$c(z) = -\frac{q^{-1/2}}{z^2} \prod_{i=1}^{m} (1 - b_i q^{n-1}/z) \prod_{j=1}^{m+4} (1 - a_j z),$$

$$\breve{f}(z) = q_{n-1}(x; q^{1/2}\mathbf{a}, q^{1/2}\mathbf{b}), \qquad k = (n-1)(m+1)$$

shows that $q_n(x; \mathbf{a}, \mathbf{b})$ has degree $\leq (m+1)n$, with leading coefficient

$$2^{(m+1)n} q^{m\binom{n}{2}} \prod_{s=0}^{n-1} (a_1 \cdots a_{m+4} q^{n-1+s} - b_1 \cdots b_m),$$

and the result follows. □

If one experiments with the special cases when the degree bound is not attained, one finds that for otherwise generic parameters such that

$$b_1 \cdots b_m = q^{n-1+s} a_1 \cdots a_{m+4}$$

for some $0 \leq s \leq n-1$, $q_n(x; \mathbf{a}, \mathbf{b})$ has degree $nm+s$. Unfortunately, the above inductive argument does not suffice to give this stronger bound, though it does allow one to reduce to the case $s = 0$. To resolve this case, we need a stronger version of Proposition 5.1. This requires an operator identity satisfied by the Askey–Wilson operator, which we now state.

Lemma 5.4. *Let*

$$\breve{\phi}_n(z; a) = (az, a/z; q)_n.$$

The Askey–Wilson operator satisfies the operator identity

$$\mathcal{D}_q^{l+m} = \frac{1}{\breve{\phi}_m(z; q^{-m/2}v)} \mathcal{D}_q^l \breve{\phi}_{l+m}(z; q^{-(l+m)/2}v) \mathcal{D}_q^m \frac{1}{\breve{\phi}_l(z; q^{-l/2}v)}.$$

Proof. We verify that both sides give the same result when applied to $\breve{\phi}_s(z; q^{-l/2}v)$. We use

$$\mathcal{D}_q \breve{\phi}_n(z; a) = \frac{2a(1-q^n)}{q-1} \breve{\phi}_{n-1}(z; aq^{1/2}), \qquad \mathcal{D}_q \frac{1}{\breve{\phi}_n(z; a)} = \frac{2a(1-q^n)}{1-q} \frac{1}{\breve{\phi}_{n+1}(z; aq^{-1/2})},$$

and

$$\breve{\phi}_l(z; q^{-l/2}v) \breve{\phi}_s(z; q^{l/2}v) = \breve{\phi}_{s+l}(z; q^{-l/2}v),$$

$$\frac{\breve{\phi}_{l+m}(z; q^{-(l+m)/2}v)}{\breve{\phi}_{l+m+s}(z; q^{-(l+m)/2}v)} = \frac{1}{\breve{\phi}_s(z; q^{(l+m)/2}v)}$$

$$\breve{\phi}_{s+l}(z; q^{m/2}v) \breve{\phi}_m(z; q^{-m/2}v) = \breve{\phi}_{s+l+m}(z; q^{-m/2}v).$$

After treating the cases $s \geq l$ and $s < l$ separately one completes the proof. □

Remark 5.5. *Apply Lemma 5.4 to a function f. Comparing coefficients of $f(q^{j-(l+m)/2})$ on both sides of Proposition 2.1 gives a special case of Jackson's ${}_8\phi_7$ summation,* [6, (II.22)].

This allows us to prove the following result, from which the claim about $q_n(x;\mathbf{a},\mathbf{b})$ follows immediately.

Theorem 5.6. *Let $f(z)$ be a symmetric Laurent polynomial of degree k, and define*

$$h_n(z;\mathbf{a},\mathbf{b}) = \frac{1}{\breve{g}(z;\mathbf{a},q^n\mathbf{b})}\mathcal{D}_q^n(\breve{g}(z;q^{n/2}\mathbf{a},q^{n/2}\mathbf{b})f(z)).$$

Then $h_n(z;\mathbf{a},\mathbf{b})$ is a symmetric Laurent polynomial of degree at most $(m+1)n+k$. Moreover, if for some s with $0\le s\le n-1$,

$$b_1b_2\cdots b_m = q^{n-1+s+k}a_1a_2\cdots a_{m+4}.$$

then $h_n(z;\mathbf{a},\mathbf{b})$ has degree at most $mn+s+k$.

Proof. If $s\neq 0$, the claim follows by the argument of the previous induction. Proposition 5.1 holds with $q_n(x;\mathbf{a},\mathbf{b})$ replaced by $h_n(z;\mathbf{a},\mathbf{b})$. Thus the leading term of $h_n(z;\mathbf{a},\mathbf{b})$ is a nonzero multiple of $a_1\cdots a_{m+4}q^{n+1}-b_1\cdots b_m$ times the leading term of $h_{n-1}(z;q^{1/2}\mathbf{a},q^{1/2}\mathbf{b})$. If $\mathbf{a}$ and $\mathbf{b}$ satisfy the hypothesized restriction with parameters n and s, then $q^{1/2}\mathbf{a}$ and $q^{1/2}\mathbf{b}$ satisfy the same relation with parameters $n-1$ and $s-1$. Thus by induction $h_{n-1}(z;q^{1/2}\mathbf{a},q^{1/2}\mathbf{b})$ has degree at most $m(n-1)+s-1+k$. Lemma 5.2 implies $h_n(z;\mathbf{a},\mathbf{b})$ has degree at most

$$m(n-1)+s-1+k+m+1 = mn+s+k.$$

It remains only to establish the $s=0$ case, namely to show that if

$$b_1b_2\cdots b_m = q^{n-1+k}a_1a_2\cdots a_{m+4},$$

then $h_n(z;\mathbf{a},\mathbf{b})$ has degree at most $mn+k$.

We shall use Lemma 5.4 to derive another recurrence for $h_n(z;\mathbf{a},\mathbf{b})$

$$\begin{aligned} h_{n+1}(z;\mathbf{a},\mathbf{b}) &= \frac{1}{\breve{g}(z;(q^na_1,\mathbf{a}'),q^{n+1}\mathbf{b})} \\ &\times\mathcal{D}_q(\breve{g}(z;(q^{n+1/2}a_1,q^{1/2}\mathbf{a}'),q^{n+1/2}\mathbf{b})h_n(z;(q^{-1/2}a_1,q^{1/2}\mathbf{a}'),q^{1/2}\mathbf{b})), \end{aligned} \tag{5.1}$$

where $\mathbf{a}=(a_1,\mathbf{a}')=(a_2,a_2,\cdots,a_{m+4})$. It must be noted that (5.1) is a raising operator relation for h_n.

To prove (5.1) apply Lemma 5.4 with $(l,m)=(1,n)$ and $v=q^{n/2}a_1$ to give the operator identity

$$\begin{aligned} &\frac{1}{\breve{g}(z;\mathbf{a},q^{n+1}\mathbf{b})}\mathcal{D}_q^{n+1}\breve{g}(z;q^{n/2+1/2}\mathbf{a},q^{n/2+1/2}\mathbf{b}) \\ &\quad= \left(\frac{1}{\breve{g}(z;(q^na_1,\mathbf{a}),q^{n+1}\mathbf{b})}\mathcal{D}_q\breve{g}(z;(q^{n+1/2}a_1,q^{1/2}\mathbf{a}),q^{n+1/2}\mathbf{b})\right) \\ &\quad\quad \left(\frac{1}{\breve{g}(z;(q^{-1/2}a_1,q^{1/2}\mathbf{a}),q^{n+1/2}\mathbf{b})}\mathcal{D}_q^n\breve{g}(z;(q^{n/2-1/2}a_1,q^{n/2+1/2}\mathbf{a}),q^{n/2+1/2}\mathbf{b})\right). \end{aligned}$$

Suppose by induction that $h_n(z;\mathbf{A},\mathbf{B})$ has degree $mn+k$ whenever

$$B_1B_2\cdots B_m = q^{n-1+k}A_1A_2\cdots A_{m+4}.$$

Suppose that

$$b_1b_2\cdots b_m = q^{n-1+k}a_1a_2\cdots a_{m+4}.$$

holds. We must show that $h_{n+1}(z;\mathbf{a},\mathbf{b})$ has degree at most $m(n+1)+k$.

We now use Lemma 5.2 with

$$c(z) = -\frac{q^{-1/2}}{z^2}(1-a_1q^nz)\prod_{i=2}^{m+4}(1-a_iz)\prod_{j=1}^{m}(1-b_jq^n/z)$$

$$\breve{f}(z) = h_n(z;(q^{-1/2}a_1, q^{1/2}\mathbf{a}'), q^{1/2}\mathbf{b})$$

to conclude that $h_{n+1}(z;\mathbf{a},\mathbf{b})$ has degree at most $m+1$ more than $h_n(z;(q^{-1/2}a_1,q^{1/2}\mathbf{a}'),q^{1/2}\mathbf{b})$. Moreover the leading term of $h_{n+1}(z;\mathbf{a},\mathbf{b})$ is a multiple of $b_1\dots b_m - q^{n+k}a_1\cdots a_{m+4}$, which is zero. So the degree of $h_{n+1}(z;\mathbf{a},\mathbf{b})$ is at most m more than $h_n(z;(q^{-1/2}a_1,q^{1/2}\mathbf{a}'),q^{1/2}\mathbf{b})$. We see that $(\mathbf{A},\mathbf{B}) = ((q^{-1/2}a_1,q^{1/2}\mathbf{a}'),q^{1/2}\mathbf{b})$ satisfies the hypothesized relation for n. So by induction the degree of $h_n(z;(q^{-1/2}a_1,q^{1/2}\mathbf{a}'),q^{1/2}\mathbf{b})$ is at most $mn+k$, and the degree of $h_{n+1}(z;\mathbf{a},\mathbf{b})$ has degree at most $mn+k+m = m(n+1)+k$. □

Corollary 5.7. *If $b_1b_2\cdots b_m = q^{n-1+s}a_1a_2\cdots a_{m+4}$ for some $s, 0\le s\le n-1$, then $q_n(x;\mathbf{a},\mathbf{b})$ is a polynomial in x of degree at most $mn+s$.*

6. Rahman's biorthogonal ${}_{10}\phi_9$'s

In this section we derive Rodrigues formulas and the biorthogonality relation [12, §3] for Rahman's very-well poised ${}_{10}\phi_9$'s from Theorem 3.4 and Theorem 4.2. We also give a polynomial orthogonality in Theorem 6.5.

Rahman's [12] biorthogonal rational functions which depend upon five parameters $\mathbf{t} = (t_1,t_2,t_3,t_4,t_5)$. The functions are denoted by R_n and S_n and given by

$$R_n(x;\mathbf{t}\,|\,q) = {}_{10}W_9(t_1^2t_2t_3t_4t_5/q; t_1t_3t_4t_5, t_1t_2t_4t_5, t_1t_2t_3t_5, t_1z, t_1/z, t_1t_2t_3t_4q^{n-1}, q^{-n}; q; q),$$
$$S_n(x;\mathbf{t}\,|\,q) = {}_{10}W_9(t_1/t_5; q/t_2t_5, q/t_3t_5, q/t_4t_5, t_1z, t_1/z, t_1t_2t_3t_4q^{n-1}, q^{-n}; q; q).$$

First we rewrite Rahman's functions as another ${}_{10}W_9$ function, our $p_n(x;\mathbf{a},\mathbf{b})$. Bailey's ${}_{10}\phi_9$ transformation [6, (III.28)] is

$$\begin{aligned}
&{}_{10}\phi_9\left(\begin{matrix} A,\ qA^{1/2},\ -qA^{1/2},\ B,\ C,\ D,\ E,\ F,\ \lambda Aq^{n+1}/EF,\ q^{-n} \\ A^{1/2}, -A^{1/2}, qA/B, qA/C, qA/D, qA/E, qA/F, EFq^{-n}/\lambda, Aq^{n+1}\end{matrix}\,\middle|\, q,q\right)\\
&= \frac{(qA, qA/EF, q\lambda/E, q\lambda/F;q)_n}{(qA/E, qA/F, q\lambda/EF, q\lambda;q)_n} \qquad (6.1)\\
&\times{}_{10}\phi_9\left(\begin{matrix} \lambda,\ q\lambda^{1/2},\ -q\lambda^{1/2},\ \lambda B/A,\ \lambda C/A,\ \lambda D/A,\ E,\ F,\ \lambda Aq^{n+1}/EF,\ q^{-n} \\ \lambda^{1/2}, -\lambda^{1/2}, qA/B, qA/C, qA/D, q\lambda/E, q\lambda/F, EFq^{-n}/A, \lambda q^{n+1}\end{matrix}\,\middle|\, q,q\right),
\end{aligned}$$

where $\lambda = qA^2/BCD$.

Proposition 6.1. *The Rahman functions are given by*

$$
\begin{aligned}
R_n(x;\mathbf{t}\,|\,q) =& c_n p_n(x;\mathbf{a},b_1), \quad \mathbf{a}=(t_1,t_2,t_3,t_4,t_5q^{1-n}), \quad b_1=t_1t_2t_3t_4t_5=T,\\
c_n(x) =& \left(\frac{q-1}{2}\right)^n q^{\frac{1}{2}\binom{n}{2}} \frac{(t_1T;q)_n}{(t_1t_2,t_1t_3,t_1t_4,q^{1-n}t_1t_5;q)_n} t_1^n\\
S_n(x;\mathbf{t}\,|\,q) =& d_n p_n(x;\mathbf{a},b_1), \quad \mathbf{a}=(t_1,t_2,t_3,t_4,t_5q^{-n}), \quad b_1=t_1t_2t_3t_4t_5/q=T/q,\\
d_n(x) =& \left(\frac{q-1}{2}\right)^n q^{\frac{1}{2}\binom{n}{2}} \frac{(Tz/q,T/qz;q)_n}{(q^{-n}zt_5,q^{-n}t_5/z;q)_n} \frac{(q^{-n}t_5/t_1;q)_n}{(t_1t_2,t_1t_3,t_1t_4,T/qt_1;q)_n} t_1^n.
\end{aligned}
$$

Proof. Both assertions follow from two applications of (6.1).

It must be noted that one of the parameters in $\mathbf{a}$ in Proposition 6.1 depends on n.

For R_n use $A=t_1^2t_2t_3t_4t_5/q$, $B=t_1/z$, $C=t_1t_3t_4t_5$, $D=t_1t_2t_3t_5$ followed by $A=t_1t_2t_4z/q$, $B=q^{n-1}t_1t_2t_3t_4$, $C=1/t_3t_5$, $D=t_1t_2t_4t_5$.

For S_n use $A=t_1/t_5$, $B=t_1/z$, $C=q/t_2t_5$, $D=q/t_3t_5$ followed by $A=t_1t_2t_3z/q$, $B=q/t_4t_5$, $C=q^{n-1}/t_1t_2t_3t_4$, $D=t_1t_2t_3t_5/q$. □

Thus we have Rodrigues formulas for the Rahman functions.

Theorem 6.2. *The Rahman functions have the Rodrigues formulas*

$$
\begin{aligned}
c_n^{-1}R_n(x;\mathbf{t}\,|\,q) =& \frac{1}{\breve{g}(z;\mathbf{a},b_1)}\mathcal{D}_q^n(\breve{g}(z;q^{n/2}\mathbf{a},q^{n/2}b_1)),\\
\mathbf{a} =& (t_1,t_2,t_3,t_4,t_5q^{1-n}), \quad b_1=t_1t_2t_3t_4t_5\\
d_n^{-1}S_n(x;\mathbf{t}\,|\,q) =& \frac{1}{\breve{g}(z;\mathbf{a},b_1)}\mathcal{D}_q^n(\breve{g}(z;q^{n/2}\mathbf{a},q^{n/2}b_1)),\\
\mathbf{a} =& (t_1,t_2,t_3,t_4,t_5q^{-n}), \quad b_1=t_1t_2t_3t_4t_5/q
\end{aligned}
$$

We next show that Rahman's biorthogonality follows from Theorem 4.2.

Theorem 6.3. *If $n\neq m$, $T=t_1t_2t_3t_4t_5$, then*

$$\int_{-1}^{1} w(x;(t_1,t_2,t_3,t_4,t_5),T)R_n(x;\mathbf{t}\,|\,q)S_m(x;\mathbf{t}\,|\,q)dx=0,$$

holds for $|t_j|<1, 1\le j<5$, and max$\{|t_5q^{-m}|,|t_5q^{-n}|\}<1$.

Proof. First assume that $n>m$. Then from Theorem 4.2 and Proposition 6.1 we have

$$
\begin{aligned}
0 &= \int_{-1}^{1} w(x;(t_1,t_2,t_3,t_4,t_5q^{1-n}),T)R_n(x;\mathbf{t}\,|\,q)\pi(x)dx\\
&= \int_{-1}^{1} w(x;(t_1,t_2,t_3,t_4,t_5),T)R_n(x;\mathbf{t}\,|\,q)\frac{\pi(x)}{(q^{1-n}t_5z,q^{1-n}t_5/z;q)_{n-1}}dx
\end{aligned}
$$

for any polynomial $\pi(x)$ of degree at most $n-1$. By Corollary 5.7 with $m=1$ and $s=0$, we can choose $\pi(x)$ to be a multiple of

$$p_m(x;(t_1,t_2,t_3,t_4,q^{-m}t_5),T/q)(Tz/q,T/qz;q)_m(q^{1-n}t_5z,q^{1-n}t_5/z;q)_{n-1-m}$$

which by Proposition 6.1 is a multiple of

$$S_m(x;\mathbf{t}\,|\,q)(q^{1-n}t_5z, q^{1-n}t_5/z;q)_{n-1}.$$

Next suppose that $n<m$. From Theorem 4.2 and Proposition 6.1 we have

$$\begin{aligned} 0 &= \int_{-1}^{1} w(x;(t_1,t_2,t_3,t_4,t_5q^{-m}),T/q)S_m(x;\mathbf{t}\,|\,q)\frac{(q^{-m}zt_5, q^{-m}t_5/z;q)_m}{(Tz/q,T/qz;q)_m}\pi(x)dx \\ &= \int_{-1}^{1} w(x;(t_1,t_2,t_3,t_4,t_5),T)S_m(x;\mathbf{t}\,|\,q)\frac{\pi(x)}{(Tz,T/z;q)_{m-1}}dx \end{aligned}$$

for any polynomial $\pi(x)$ of degree at most $m-1$. This time use Corollary 5.7 with $m=1$ and $s=0$, and choose $\pi(x)$ to be a multiple of

$$p_n(x;(t_1,t_2,t_3,t_4,q^{1-n}t_5),T)(Tz,T/z;q)_n(Tzq^n,Tq^n/z;q)_{m-1-n}$$

which by Proposition 6.1 is a multiple of

$$R_n(x;\mathbf{t}\,|\,q)(Tz,T/z;q)_{m-1},$$

and the proof is complete. □

Rahman [13] gave the orthogonality relation when the parameters are not necessarily small, and in general the orthogonality relation has a discrete part. One can derive such a relation using contour integration instead of integration on $[-1,1]$.

One may ask for a polynomial orthogonality relation for a ${}_{10}W_9$. We provide one for polynomials in x of degree n (see Theorem 6.5).

Definition 6.4. *Let*

$$\begin{aligned} U_n(x;\mathbf{t}\,|\,q) =& z^n\frac{(Tzq^{n-1};q)_n}{(1/z^2;q)_n}\prod_{i=1}^{5}(t_i/z;q)_n \\ &\times {}_{10}W_9(q^{-n}z^2;t_1z,t_2z,t_3z,t_4z,t_5z,zq^{2-2n}/T,q^{-n};q;q) \end{aligned}$$

Note that Definition 6.4 is equivalent to

$$U_n(x;\mathbf{t}\,|\,q) = \alpha_n p_n(x;\mathbf{t},q^{n-1}T)(Tzq^{n-1},Tq^{n-1}/z;q)_n, \quad T=t_1t_2t_3t_4t_5,$$

where α_n is a non-zero constant. Thus Corollary 5.7 with $m=1$ and $s=0$ shows that $U_n(x;\mathbf{t}\,|\,q)$ is a polynomial in x of degree n.

Theorem 6.5. *Assume that $|t_j|<1, 1\le j\le 5$, the t_j's are real or appear in conjugate pairs. Then the polynomials $U_n(x;\mathbf{t}\,|\,q)$ of degree n satisfy the orthogonality relation*

$$\int_{-1}^{1} U_n(x;\mathbf{t}\,|\,q)U_m(x;\mathbf{t}\,|\,q)\frac{w(x;\mathbf{t},T)}{\prod_{k=0}^{2n-2}(1-2Txq^k+T^2q^{2k})}dx = 0, \quad n>m.$$

Proof. This follows directly from Theorem 4.2 using

$$w(x;\mathbf{t},Tq^{n-1}) = \frac{w(x;\mathbf{t},T)}{(Tz,T/z;q)_{n-1}}, \quad \alpha_n p_n(x;\mathbf{t},q^{n-1}T) = \frac{U_n(x;\mathbf{t}\,|\,q)}{(Tq^{n-1}z,Tq^{n-1}/z;q)_n}.$$

This completes the proof. □

The polynomials U_n are symmetric in the parameters t_i. If $t_5 = 0$ (and thus $T = 0$) Theorem 6.5 is the orthogonality relation for Askey–Wilson polynomials. We note that $t_5 = 0$ does give the Askey–Wilson polynomials with the symmetric normalization

$$U_n(x; (t_1, t_2, t_3, t_4, 0), 0) = p_n(x; \mathbf{t}|q).$$

A three-term relation for U_n, generalizing the Askey–Wilson three-term recurrence, is given in §9.

7. Asymptotics

In this section we give in Theorem 7.2 the large n asymptotics of the polynomials $U_n(x; \mathbf{t}, T)$. Although these polynomials are not, strictly speaking, orthogonal polynomials, we show that orthogonal polynomial techniques can lead one to guess an orthogonality relation such as Theorem 6.5.

The following theorem relates the asymptotics of orthonormal polynomials to the weight function, see [22, Chapter 12].

Theorem 7.1. *Assume that $\{p_n(x)\}$ are orthonormal with respect to a weight function w on $[-1, 1]$ and that $\int_{-1}^{1} |\ln f(\cos\theta)| d\theta < \infty, f(x) = w(x)\sqrt{1-x^2}$. Let*

$$D(z) = \exp\left[\frac{1}{4}\int_{-\pi}^{\pi} \ln f(\cos\theta) \frac{1 + ze^{-i\theta}}{1 - ze^{i\theta}}\, d\theta\right], |z| < 1.$$

Then

$$\lim_{n\to\infty} z^{-n} p_n(x) = \frac{1}{\sqrt{2\pi}\, D(1/z)},$$

where $x \in \mathbb{C} \setminus [-1, 1]$ and $x = (z + 1/z)/2, |z| > 1$. Moreover the radial limit exists, $\lim_{r\to 1^-} D(re^{i\theta}) = D(e^{i\theta})$ and $w(\cos\theta) = |D(e^{i\theta})|^2/\sin\theta$.

It must be noted that Theorem 7.1 contains information only on the absolutely continuous component of the orthogonality measure and gives no information on the discrete part.

When we do not know the weight function w but do know asymptotics of $p_n(x)$, Theorem 7.1 will provide a good candidate for the weight function.

First we transform $U_n(x; \mathbf{t}, T)$ using (6.1) with $A = q^{-n}z^2$, $B = t_1 z$, $C = t_2 z$, and $D = t_3 z$. The resulting expression may be written as

$$U_n(x; \mathbf{t}, T) = \frac{z^n (q, t_1t_2, t_1t_3, t_2t_3, t_4t_5, q^n t_1t_2t_3/z, Tq^{n-1}/z; q)_n}{(t_1t_1t_3/qz; q)_{2n}}$$
$$\times \sum_{k=0}^{n} \frac{(t_1/z, t_2/z, t_3/z, t_1t_2t_3/qz, q^{n-1}T/t_4, q^{n-1}T/t_5; q)_{n-k}}{(q, t_1t_2, t_1t_3, t_2t_3, q^n t_1t_2t_3/z, Tq^{n-1}/z; q)_{n-k}} \frac{t_1t_2t_3 - zq^{1-2n+2k}}{t_1t_2t_3 - zq^{1-2n}}$$
$$\times \frac{(t_4z, t_5z; q)_k}{(q, t_4t_5; q)_k} (qz)^{-2k}.$$

Therefore for $|z| > 1$ we have the limiting relation

$$\lim_{n\to\infty} z^{-n}\, U_n(x;\mathbf{t}) = (t_4t_5;q)_\infty \prod_{j=1}^{3}(t_j/z;q)_\infty\ {}_2\phi_1(t_4z,t_5z;t_4t_5;q,1/z^2)$$

Applying the q-analogue of Gauss's theorem [6] we establish the following theorem.

Theorem 7.2. *The large n asymptotics of U_n for $|z| > 1$ is given by*

$$\lim_{n\to\infty} z^{-n}\, U_n(x;\mathbf{t}) = \frac{\prod_{j=1}^{5}(t_j/z;q)_\infty}{(1/z^2;q)_\infty}.$$

If the U_n's were orthogonal polynomials Theorem 7.1 would give

$$D(z) = \frac{(1/z^2;q)_\infty}{\prod_{j=1}^{5}(t_j/z;q)_\infty}$$

one would expect the weight function to be

$$\frac{(e^{2i\theta},e^{-2i\theta};q)_\infty}{\sin\theta\prod_{j=1}^{5}(t_je^{i\theta},t_je^{-i\theta};q)_\infty}, \quad \theta\in[0,\pi]. \tag{7.1}$$

This agrees with Theorem 6.5 except for the part of the weight function which depends on n. In fact the weight function in Theorem 6.5 is exactly analogous to the the problem of varying weights in orthogonal polynomials, see for example [18]. If we let $n \to \infty$ in the weight function in Theorem 6.5 we indeed get the quantity in (7.1). So the asymptotics seems to give the n independent part of the weight function.

8. Elliptic analogues

In this section we give elliptic analogues of the main results of the previous sections. An elliptic version of the iterated Askey–Wilson operator is given in Proposition 8.1. An elliptic analogue of the polynomials $q_n(x;\mathbf{a},\mathbf{b})$ is given in Definition 8.4, and their orthogonality relation is Theorem 8.9.

The operator identity, Lemma 5.4 above, satisfied by the Askey–Wilson operator, is a limiting case of an identity satisfied by a family of elliptic difference operators. We recall the elliptic analogue of the infinite q-shifted factorials, the elliptic Gamma function of [17]

$$\Gamma_{p,q}(z) = \prod_{j,k\geq 0}\frac{1-p^{j+1}q^{k+1}/z}{1-p^jq^kz},$$

which satisfies the recurrence

$$\Gamma_{p,q}(qz) = \theta_p(z)\Gamma_{p,q}(z),$$

where

$$\theta_p(z) = \prod_{j\geq 0}(1-p^{j+1}/z)(1-p^jz).$$

This function, in turn, satisfies the following quasiperiodicity property:

$$\theta_p(pz) = -z^{-1}\theta_p(z),$$

making $\theta_p(\exp(2\pi\sqrt{-1}z))$ a theta function for the lattice $\langle 1, \log(p)/2\pi\sqrt{-1}\rangle$. We also define finite elliptic shifted factorial by

$$\theta_p(z;q)_m = \frac{\Gamma_{p,q}(q^m z)}{\Gamma_{p,q}(z)} = \prod_{0\le j<m} \theta_p(q^j z).$$

The analogue of a polynomial of degree n in the elliptic context is a "symmetric theta function" of degree n, a holomorphic function f such that $f(1/z) = f(z)$ and $f(pz) = (pz^2)^{-n} f(z)$. As with polynomials, these form a vector space of dimension $n+1$.

The most natural analogue of the Askey–Wilson operator is the operator $\mathcal{D}_{q;p}$ which acts by

$$(\mathcal{D}_{q;p}f)(z) = \frac{f(q^{1/2}z)}{\theta_p(z^2)} + \frac{f(q^{-1/2}z)}{\theta_p(1/z^2)},$$

which manifestly preserves the space of functions invariant under $z \mapsto 1/z$. This is not quite a direct analogue, in so far as the limit as $p \to 0$ is not quite the same operator (the coefficients differ by powers of z), but is more convenient for dealing with questions of orthogonality. The key point is that this operator is formally self-adjoint with respect to the density

$$\Delta(z;p,q) = \frac{(p;p)_\infty (q;q)_\infty}{2} \frac{1}{\Gamma_{p,q}(z^2)\Gamma_{p,q}(1/z^2)} \frac{dz}{2\pi\sqrt{-1}z},$$

the fixed part of the density of the elliptic beta integral [19]. To be precise, if f and g are invariant under $z \mapsto 1/z$, then the integral

$$\int_{S^1} (\mathcal{D}_{q;p}f)(z) g(z) \Delta(z)$$

is, by symmetry, equal to

$$2\int_{S^1} f(q^{1/2}z) g(z) \theta_p(z^2)^{-1} \Delta(z).$$

The change of variables $z \mapsto q^{-1/2}/z$ makes this equal to

$$2\int_{|z|=q^{-1/2}} f(z) g(q^{1/2}z) \theta_p(z^2)^{-1} \Delta(z).$$

Moving the contour back to the unit circle and symmetrizing gives

$$\int_{S^1} f(z) (\mathcal{D}_{q;p}g)(z) \Delta(z),$$

assuming that there are no poles for $1 \le |z| \le q^{-1/2}$. (The possibility of poles is why we refer to this as "formal" self-adjointness above.)

Unlike the Askey–Wilson operator, however, the powers of this operator are not well-behaved; if we try to square the operator, we find that the two contributions to the constant term are quasiperiodic, but with different multipliers, and thus the sum is not even a (meromorphic) theta functions. However, in [15, §9], the second author introduced a family of multivariate operators satisfying an analogue of Lemma 5.4 above; in the univariate case, this is the operator identity

$$\mathcal{D}_{l+m}(q;p) = \frac{\Gamma_{p,q}(q^{-m/2}vz)\Gamma_{p,q}(q^{-m/2}v/z)}{\Gamma_{p,q}(q^{m/2}vz)\Gamma_{p,q}(q^{m/2}v/z)}\,\mathcal{D}_l(q;p)\,\frac{\Gamma_{p,q}(q^{(l+m)/2}vz)\Gamma_{p,q}(q^{(l+m)/2}v/z)}{\Gamma_{p,q}(q^{-(l+m)/2}vz)\Gamma_{p,q}(q^{-(l+m)/2}v/z)}$$
$$\times\mathcal{D}_m(q;p)\,\frac{\Gamma_{p,q}(q^{-l/2}vz)\Gamma_{p,q}(q^{-l/2}v/z)}{\Gamma_{p,q}(q^{l/2}vz)\Gamma_{p,q}(q^{l/2}v/z)},$$

where $\mathcal{D}_1(q;p) = \mathcal{D}_{q;p}$. The above identity is not explicitly stated in [15].

From this, it is straightforward to deduce the analogue of Proposition 2.1. Indeed, when $m = 1$, the simple fact that the right-hand side is independent of v means that the residue at $v = q^{1/2}z$ must be 0, but this gives a first-order recurrence for the coefficients of $\mathcal{D}_l(q;p)$. The leading coefficient is also straightforward to compute.

Proposition 8.1. *The operator $\mathcal{D}_n(q;p)$ has the expansion*

$$(\mathcal{D}_n(q;p)f)(z) = \sum_{0\le j\le n}(-1)^j q^{-j(j-1)/2}z^{2j}\frac{\theta_p(q^{n-2j}z^2)}{\theta_p(q^{-j}z^2;q)_{n+1}}\frac{\theta_p(q;q)_n}{\theta_p(q;q)_j\theta_p(q;q)_{n-j}}f(q^{n/2-j}z)$$

As remarked after Lemma 5.4, this expression turns the operator identity into a (Zariski dense) special case of the Frenkel-Turaev summation [5], the elliptic analogue of Jackson summation.

Note that just as for $n = 1$, $\mathcal{D}_n(q;p)$ is formally self-adjoint with respect to the density $\Delta(z;p,q)$.

Theorem 8.2. *Let f be a symmetric theta function of degree k, and let* **a** *be a sequence of length $2m+4$ satisfying*

$$q^{(m+1)(n-1)+k}a_1\cdots a_{2m+4} = p^{m+1}.$$

Then

$$\frac{1}{\prod_{1\le i\le 2m+4}\Gamma_{p,q}(a_iz)\Gamma_{p,q}(a_i/z)}\mathcal{D}_n(q;p)\prod_{1\le i\le 2m+4}\Gamma_{p,q}(q^{n/2}a_iz)\Gamma_{p,q}(q^{n/2}a_i/z)f(z)$$

is a symmetric theta function of degree $mn+k$.

Proof. As before, this reduces easily to the case $n = 1$. In that case, we verify that the resulting function has the correct symmetry and quasi-periodicity, so the only obstruction to being a theta function is the potential poles coming from zeros of $\theta_p(z^2)$. By symmetry, however, the function must have *even* order at such points, and thus the apparent simple poles are in fact removable singularities as required. □

Remark 8.3. *To relate this to Theorem 5.6 above, note that the elliptic Gamma function satisfies the reflection principle $\Gamma_{p,q}(x)\Gamma_{p,q}(pq/x) = 1$, and thus we can write a ratio of elliptic Gamma functions as a product. Moreover, we can shift one of the parameters by a factor of p at the cost of introducing some powers of q and z to the coefficients of the operator (as $\Gamma_{p,q}(px)/\Gamma_{p,q}(x)$ is quasiperiodic under q-shifts!).*

Applying the operator when $k = 0$ in Theorem 8.2 to 1 gives a symmetric theta function of degree mn.

Definition 8.4. *Let*

$$q_n(z;\mathbf{a};p) = \frac{1}{\prod_{1\le i\le 2m+4}\Gamma_{p,q}(a_i z)\Gamma_{p,q}(a_i/z)}\mathcal{D}_n(q;p)\prod_{1\le i\le 2m+4}\Gamma_{p,q}(q^{n/2}a_i z)\Gamma_{p,q}(q^{n/2}a_i/z)(1)$$

Note that Proposition 8.1 implies that $q_n(z;\mathbf{a};p)$ has an expression as a "very-well-poised balanced" elliptic hypergeometric series.

If we take $m=-1$, then we find that the operator decreases the degree. Since theta functions of negative degree do not exist, we obtain the following result.

Corollary 8.5. *If $f(z)$ is a symmetric theta function of degree $n-1$, then for any a,*

$$\mathcal{D}_n(q;p)\left[\frac{f(z)}{\theta_q(az)\theta_q(a/z)}\right]=0.$$

Together with self-adjointness of $\mathcal{D}_n(q;p)$, Corollary 8.5 allows us to prove orthogonality results. The simplest interesting case is a Rodrigues formula for the biorthogonal functions of Spiridonov and Zhedanov, which are now defined.

Definition 8.6. *For parameters satisfying $t_0t_1t_2t_3u_0u_1=pq$, let*

$$\begin{aligned}&f_n(z;t_0,t_1,t_2,t_3;u_0,u_1)\\&=\prod_{j=0}^{3}\frac{1}{\Gamma_{p,q}(t_jz,t_j/z)}\,\frac{1}{\Gamma_{p,q}(u_0z)\Gamma_{p,q}(u_0/z)\Gamma_{p,q}(pu_1q^{1-n}z)\Gamma_{p,q}(pu_1q^{1-n}/z)}\\&\qquad\times\mathcal{D}_n(q;p)H(z),\end{aligned}$$

where

$$\begin{aligned}H(z)&=\Gamma_{p,q}(q^{-n/2}u_0z)\Gamma_{p,q}(q^{-n/2}u_0/z)\Gamma_{p,q}((pu_1q^{1-n/2})z)\Gamma_{p,q}((pu_1q^{1-n/2})/z)\\&\quad\times\prod_{j=0}^{3}\Gamma_{p,q}(q^{n/2}t_jz)\Gamma_{p,q}(q^{n/2}t_j/z).\end{aligned}\tag{8.1}$$

Also, define a family of densities by

$$\Delta(z;\mathbf{a}):=\prod_{1\le i\le 2m+4}\Gamma_{p,q}(a_iz)\Gamma_{p,q}(a_i/z)\Delta(z).$$

Lemma 8.7. *The function*

$$\theta_p((pq/u_0)z;q)_n\theta_p((pq/u_0)/z;q)_nf_n(z;t_0,t_1,t_2,t_3;u_0,u_1)$$

is a symmetric theta function of degree n, so that $f_n(z;t_0,t_1,t_2,t_3;u_0,u_1)$ is a symmetric elliptic function. Next, suppose $|t_0|,|t_1|,|t_2|,|t_3|,|q^{-n}u_0|,|q^{1-n}u_1|<1$. Then for any symmetric elliptic function g such that

$$\theta_p((pq/u_1)z;q)_{n-1}\theta_p((pq/u_1)/z;q)_{n-1}g(z)$$

is holomorphic,

$$\int_{S^1}f_n(z;t_0,t_1,t_2,t_3;u_0,u_1)g(z)\Delta(z;t_0,t_1,t_2,t_3,u_0,u_1)=0.$$

In particular, $f_n(z;t_0,t_1,t_2,t_3;u_0,u_1)$ is proportional to the elliptic biorthogonal function of [21].

Proof. We have

$\theta_p((pq/u_0)z;q)_n\theta_p((pq/u_0)/z;q)_n f_n(z;t_0,t_1,t_2,t_3;u_0,u_1) = q_n(z;t_0,t_1,t_2,t_3,q^{-n}u_0,q^{1-n}pu_1;p)$,

which is indeed a symmetric theta function of degree n. We can also write

$$\int_{S^1} f_n(z;t_0,t_1,t_2,t_3;u_0,u_1)g(z)\Delta(z;t_0,t_1,t_2,t_3,u_0,u_1)$$
$$= \int_{S^1}[\mathcal{D}_n(q;p)H(z)]\frac{\Gamma_{p,q}(u_1z)\Gamma_{p,q}(u_1/z)}{\Gamma_{p,q}((pu_1/q^{n-1})z)\Gamma_{p,q}((pu_1/q^{n-1})/z)}g(z)\Delta(z),$$

where $H(z)$ is given by (8.1).

The conditions on the parameters ensure that the residue terms in the formal self-adjointness do not appear (i.e., the integrand has no poles between the shifted contour and the unit circle), so the integrand is

$$\Delta(z)\mathcal{D}_n(q;p)\frac{\Gamma_{p,q}(u_1z)\Gamma_{p,q}(u_1/z)}{\Gamma_{p,q}((pu_1/q^{n-1})z)\Gamma_{p,q}((pu_1/q^{n-1})/z)}g(z).$$

may be rewritten as

$$\Delta(z)\mathcal{D}_n(q;p)\frac{\Gamma_{p,q}(u_1z)\Gamma_{p,q}(u_1/z)}{\Gamma_{p,q}(pu_1z)\Gamma_{p,q}(pu_1/z)}\theta_p((pq/u_1)z;q)_{n-1}\theta_p((pq/u_1)/z;q)_{n-1}g(z).$$

Because

$$\frac{\Gamma_{p,q}(u_1z)\Gamma_{p,q}(u_1/z)}{\Gamma_{p,q}(pu_1z)\Gamma_{p,q}(pu_1/z)} = \frac{1}{\theta_q(u_1z)\theta_q(u_1/z)},$$

the integrand vanishes by Corollary 8.5. □

Remark 8.8. *The constant can be recovered from the fact that when $z = t_0$, only one of the $n+1$ terms in the Rodrigues formula is nonzero, so that*

$$f_n(t_0;t_0,t_1,t_2,t_3;u_0,u_1) = \frac{\theta_p(t_0t_1,t_0t_2,t_0t_3,1/t_0u_1;q)_n}{\theta_p(pqt_0/u_0;q)_n}.$$

Essentially the same argument gives the following more general orthogonality result. When $m=1$, this recovers the above orthogonality (for the numerator of f_n, to be precise).

Theorem 8.9. *Let $q^{(m+1)(n-1)}\prod_{0\le i<2m+4} a_i = p^m$, so that the function*

$$q_n(z;pa_0,a_1,\dots,a_{2m+3};p)$$

is a symmetric theta function of degree mn, and suppose $|a_0|,\dots,|a_{2m+3}|<1$. Then for any symmetric theta function g of degree $n-1$,

$$\int_{S^1} q_n(z;pa_0,a_1,\dots,a_{2m+3};p)g(z)\Delta(z;a_0,\dots,a_{2m+3}) = 0.$$

Note that since the function

$$\frac{\theta_q(bz)\theta_q(b/z)}{\theta_q(az)\theta_q(a/z)}$$

is periodic in q, it follows from Proposition 8.1 that the operator

$$a^{-n}\theta_q(q^{-n/2}az)\theta_q(q^{-n/2}a/z)\mathcal{D}_n(q;p)\left[\frac{1}{\theta_q(az)\theta_q(a/z)}\right]$$

is independent of a.

In fact, the same theta functions satisfy a number of different orthogonality relations, arising from the fact that

$$a_0^n q_n(z;pa_0,a_1,\dots,a_{2m+3};p) = a_1^n q_n(z;a_0,pa_1,\dots,a_{2m+3};p),$$

which in turn follows immediately from the above observation that

$$a^{-n}\theta_q(q^{-n/2}az)\theta_q(q^{-n/2}a/z)\mathcal{D}_n(q;p)\left[\frac{1}{\theta_q(az)\theta_q(a/z)}\right]$$

is independent of a. It seems likely that these orthogonality relations determine q_n (indeed, it should typically be enough to take the first m such relations), though to prove this in general requires the computation of a fairly complicated determinant. The orthogonality relations impose linear conditions on q_n, so for q_n to be uniquely determined requires that the restriction of these equations to some complement of $\langle q_n\rangle$ has a minor of full rank.

We also have the following analogue of Theorem 6.5.

Corollary 8.10. *Let* $U_n(z;t_1,t_2,t_3,t_4,t_5;p) := q_n(z;t_1,t_2,t_3,t_4,t_5,p^2q^{2-2n}/t_1t_2t_3t_4t_5)$. *Then for* $m<n$, *and* $|t_j|<1$ *for all* $j, 1\le j\le 6$, *we have*

$$\int_{S^1} U_n(z)U_m(z)\theta_p(az,a/z;q)_{n-1-m}\Delta(z;t_1,t_2,t_3,t_4,t_5,pq^{2-2n}/t_1t_2t_3t_4t_5)=0.$$

Remark 8.11. *Note here that the density depends on* m *in a crucial way, via the factor* $\theta_p(az,a/z;q)_{n-1-m}$ *as part of the density. Of course, in the limit* $p\to 0$, *one may as well take* $a=0$, *and thus recover Theorem 6.5.*

9. A Recursion Relation

Proposition 9.1. *The polynomial* $U_n(x)=U_n(x_n;\mathbf{t}|q)$ *satisfies the following 3-term recurrence relation*

$$\begin{aligned}U_{n+1} - A_{n+1}(1-q^{2n-2}Tz)(1-q^{2n-2}T/z)(1-q^{2n-3}Tz)(1-q^{2n-3}T/z)U_{n-1}\\ -\left(B_n(1-t_1/z)(1-t_1z)+C_n\right)U_n=0,\end{aligned}\tag{9.1}$$

where

$$A_n=\prod_{i=1}^{4}\prod_{j=i+1}^{5}(1-t_it_jq^{n-2})(1-q^{n-1})(-q^8)/\prod_{i=1}^{5}(1-e_5q^{n-2}/t_i)\frac{N}{D},\ \textit{where}$$

$$\begin{aligned}N=&(q^{10}-e_5^4q^{8n}+e_5^2(q^{3n+5}-q^{5n+5})-e_4q^{2n+8}+e_5^2e_4(q^{6n+3}-q^{5n+3})+e_5e_3q^{3n+6}\\&+e_5^3e_1q^{6n+2}-e_5^2e_2q^{5n+4}+e_5e_1(q^{3n+7}-q^{2n+7}))\\ D=&(q^{18}-e_5^4q^{8n}+e_5^2(q^{3n+10}-q^{5n+8})-e_4q^{2n+14}+e_5^2e_4(q^{6n+5}-q^{5n+6})\\&+e_5e_3q^{3n+11}+e_5^3e_1q^{6n+4}-e_5^2e_2q^{5n+7}+e_5e_1(q^{3n+12}-q^{2n+13}));\end{aligned}$$

$$C_n = \prod_{j=2}^{5}(1-q^n t_1 t_j)\frac{(1-q^{2n-1}t_2t_3t_4t_5)(1-q^{2n}t_2t_3t_4t_5)}{t_1(1-q^{n-1}t_2t_3t_4t_5)} \\ - A_{n+1}\frac{(1-t_1^2t_2t_3t_4t_5q^{2n-2})(1-t_1^2t_2t_3t_4t_5q^{2n-3})(1-t_2t_3t_4t_5q^{n-2})t_1}{\prod_{j=2}^{5}(1-q^{n-1}t_1t_j)}.$$

$$B_n = \Big((1-t_1t_2q^n)\prod_{j=3}^{5}(1-q^n t_2 t_j)\frac{(1-q^{2n-1}t_1t_3t_4t_5)(1-q^{2n}t_1t_3t_4t_5)}{(1-q^{n-1}t_1t_3t_4t_5)t_2} \\ - A_{n+1}\frac{(1-t_1t_2^2t_3t_4t_5q^{2n-2})(1-t_1t_2^2t_3t_4t_5q^{2n-3})(1-t_1t_3t_4t_5q^{n-2})t_2}{\prod_{j=3}^{5}(1-q^{n-1}t_2t_j)(1-t_1t_2q^{n-1})} - C_n\Big) \\ /(1-t_1/t_2)(1-t_1t_2).$$

and e_j *is the elementary symmetric function of* t_1, t_2, t_3, t_4, t_5 *of degree* j*.*

This was verified using computer algebraic techniques by Christoph Koutschan [11].

It must be noted that equation (9.1) when written in $x, x = (z+1/z)/2$ is of the form of a recurrence relation of an R_{II} fraction, [8, (3.1)]. In the notation of §3 of [8], the interpolation points are

$$a_{n+1} = [Tq^{2n-2} + q^{2-2n}/T]/2, \quad b_{n+1} = [Tq^{2n-3} + q^{3-2n}/T]/2.$$

The interpolation points are also manifested in the orthogonality relation in Theorem 6.5. Moreover $U_n(x;\mathbf{t}|q)$ can be evaluated at these special points. To see this use the symmetry of U_n in z and $1/z$ to put $1/z = Tq^{2n-2}$ in (6.2). Indeed we have

$$U_n(x_n;\mathbf{t}|q) = \frac{q^{2n(n-1)}}{T^n}\prod_{j=1}^{5}(Tq^{n-1}/t_j;q)_n, \quad x_n = \frac{1}{2}[Tq^{2n-2} + q^{2-2n}/T].$$

Similarly we may just set $z = Tq^{2n-3}$ in (6.2) and the ${}_8W_7$ is now a sum of two terms, so we can find a closed form expression for $U_n(y_n;\mathbf{t}|q)$, where $y_n = [Tq^{2n-3} + q^{3-2n}/T]/2$.

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