Expansions in the Askey–Wilson Polynomials☆

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Abstract

We give a general expansion formula of functions in the Askey–Wilson polynomials and using Askey–Wilson orthogonality we evaluate several integrals. Moreover we give a general expansion formula of functions in polynomials of Askey–Wilson type, which are not necessarily orthogonal. Limiting cases give expansions in little and big q-Jacobi type polynomials. We also give a new generating function for Askey–Wilson polynomials and a new evaluation for specialized Askey–Wilson polynomials.

Keywords: Andrews formula, Askey–Wilson polynomials, basic hypergeometric series, generating functions, expansion formulas.

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☆Dedicated to George Andrews on his 75th birthday.

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1. Introduction

Andrews [1] proved the terminating basic hypergeometric identity

$$
\begin{aligned}
{}_5\phi_4\left(\begin{matrix} q^{-N},\ \rho_1, \rho_2, b, c \\ \rho_1\rho_2 q^{-N}/a, e, f, g \end{matrix}\,\middle|\, q, q\right) &= \frac{(aq/\rho_1, aq/\rho_2; q)_N}{(aq, aq/\rho_1\rho_2; q)_N} \\
\times \sum_{n=0}^{N} \frac{(q^{-N}, \rho_1, \rho_2, a; q)_n (1-aq^{2n})}{(q, aq/\rho_1, aq/\rho_2, aq^{N+1}; q)_n (1-a)} &\left(\frac{aq^{N+1}}{\rho_1\rho_2}\right)^n u_n,
\end{aligned} \tag{1.1}
$$

where N is a non-negative integer, $qabc = efg$, and

$$
u_n = {}_4\phi_3\left(\begin{matrix} q^{-n}, aq^n, b, c \\ e, \quad f, \quad g \end{matrix}\,\middle|\, q, q\right). \tag{1.2}
$$

(We use the usual basic hypergeometric notation, as in [2], [7] and [10].)

In this paper we show that Andrews' identity (1.1) is one of many similar expansion formulas which follow from expanding an Askey–Wilson basis $(be^{i\theta}, be^{-i\theta}; q)_n$ in the Askey–Wilson polynomials. The expansions established here, Theorem 2.2 and Corollary 2.5, are reminiscent of the Fields and Wimp expansions of hypergeometric functions in hypergeometric polynomials, [6], which are stated and proved in the monographs [10] and [18]. These expansions were extended in [5]. Gessel and Stanton, [8], developed q-Lagrange expansions and applied their results to give q-analogues of some of these results.

Our main tool is an expansion formula due to Ismail and Rahman [11], Proposition 2.1. Section 2 contains expansions which generalize Andrews' result. Section 3 has new generating functions for the Askey–Wilson polynomials. In Section 4 we give the integral evaluations which follow from our expansion formulas. Section 5 is devoted to an expansion of general functions in polynomials of Askey–Wilson type. The main result of §5 is the expansion (5.2). We also show that the expansion (5.2) implies a generalization of earlier results of q-analogues of plane wave expansions, see (5.5). Section 6 contains expansions of little and big Jacobi type polynomials and derived as limiting cases of the expansion of Section 5.

Recall that with $x = \cos\theta$, the Askey–Wilson polynomials are defined by [10]

$$\begin{aligned} p_n(x;\mathbf{t}\,|\,q) &= t_1^{-n}(t_1t_2, t_1t_3, t_1t_4; q)_n \\ &\times {}_4\phi_3\left(\left.\begin{matrix} q^{-n}, t_1t_2t_3t_4q^{n-1}, t_1e^{i\theta}, t_1e^{-i\theta} \\ t_1t_2,\ t_1t_3,\ t_1t_4 \end{matrix}\,\right|\, q, q\right). \end{aligned} \tag{1.3}$$

Throughout this work we will set $z = e^{i\theta}$.

2. Askey-Wilson expansions

In this section we establish a general expansion in Askey-Wilson polynomials, Theorem 2.2, which generalizes Andrews' result (1.1).

We shall use the Ismail-Rahman [11] result alluded to in the introduction, which expands an Askey-Wilson basis in terms of Askey-Wilson polynomials.

Proposition 2.1. *For any non-negative n,*

$$(be^{i\theta}, be^{-i\theta}; q)_n = \sum_{k=0}^{n} f_{n,k}(b,\mathbf{t})p_k(x;\mathbf{t}|q),$$

where

$$\begin{aligned} f_{n,k}(b,\mathbf{t}) &= \frac{(-b)^k q^{\binom{k}{2}}(q;q)_n(b/t_4, bt_4q^k; q)_{n-k}}{(q, t_1t_2t_3t_4q^{k-1}; q)_k(q;q)_{n-k}} \\ &\times {}_4\phi_3\left(\left.\begin{matrix} q^{k-n},\ t_1t_4q^k,\ t_2t_4q^k,\ t_3t_4q^k \\ bt_4q^k, t_1t_2t_3t_4q^{2k}, q^{1+k-n}t_4/b \end{matrix}\,\right|\, q, q\right), \end{aligned} \tag{2.1}$$

When $b = t_4$, the explicit formula for $f_{n,k}$ simplifies considerably. Indeed all the terms in the ${}_4\phi_3$ which appear in $f_{n,k}$ are zero except the last one. In this case we find

$$f_{n,k}(t_4,\mathbf{t}) = \frac{(-t_4)^k(q;q)_n(t_1t_4q^k,\ t_2t_4q^k,\ t_3t_4q^k)_{n-k}}{(q, t_1t_2t_3t_4q^{k-1}; q)_k(q, t_1t_2t_3t_4q^{2k}; q)_{n-k}}\ q^{\binom{k}{2}}. \tag{2.2}$$

We first explore applications of (2.2). It is clear from (2.2) that

$$\begin{aligned} &\sum_{n=0}^{\infty} \frac{(t_4z, t_4/z; q)_n}{(q;q)_n}\Lambda_n\zeta^n \\ &= \sum_{k=0}^{\infty} p_k(x;\mathbf{t}|q)\frac{(-t_4\zeta)^k q^{\binom{k}{2}}}{(q, t_1t_2t_3t_4q^{k-1}; q)_k}\sum_{n=0}^{\infty}\Lambda_{n+k}\frac{(t_1t_4q^k,\ t_2t_4q^k,\ t_3t_4q^k)_n}{(q, t_1t_2t_3t_4q^{2k})_n}\zeta^n. \end{aligned} \tag{2.3}$$

An interesting special case is when

$$\Lambda_n = \frac{(a_1, \cdots, a_{p-1}; q)_n}{(t_1t_4, t_2t_4, t_3t_4, b_1, \cdots, b_{p-3}; q)_n}.$$

We state the result as our main theorem.

Theorem 2.2. *We have the following expansion*

$$\begin{aligned}
&{}_{p+1}\phi_p\left(\left.\begin{matrix} a_1, \cdots, a_{p-1}, t_4z, t_4/z \\ t_1t_4, t_2t_4, t_3t_4, b_1, \cdots, b_{p-3} \end{matrix}\right| q, \zeta\right) \\
&= \sum_{k=0}^{\infty} p_k(x; \mathbf{t}|q) \frac{(a_1, \cdots, a_{p-1}; q)_k}{(t_1t_4, t_2t_4, t_3t_4, b_1, \cdots, b_{p-3}; q)_k} \\
&\times \frac{(-t_4\zeta)^k q^{\binom{k}{2}}}{(q, t_1t_2t_3t_4q^{k-1}; q)_k} \, {}_{p-1}\phi_{p-2}\left(\left.\begin{matrix} q^k a_1, \cdots, q^k a_{p-1} \\ q^k b_1, \cdots, q^k b_{p-3}, t_1t_2t_3t_4q^{2k} \end{matrix}\right| q, \zeta\right).
\end{aligned}$$

Remark 2.3. *The Andrews formula* (1.1) *is the case* $p = 4$ *in Theorem 2.2 with the parameter identification*

$$a_1 = q^{-N}, \quad a_2 = \rho_1, \quad a_3 = \rho_2, \quad b_1 = \rho_1\rho_2q^{-N}/a, \quad \zeta = q.$$

In this case the ${}_3\phi_2$ *can be summed by the q-Pfaff–Saalschütz theorem, [7, (II.12)].*

Remark 2.4. *Another application of Theorem 2.2 is to set*

$$a_1 = q^{-N}, \quad a_2 = c_1c_2c_3t_4q^{N-1}, \quad a_j = t_{j-2}t_4 \quad \text{for } 3 \le j \le 5,$$
$$b_k = t_4c_k \quad \text{for } 1 \le j \le 3.$$

Theorem 2.2 solves the connection relation between $p_N(x; t_4, c_1, c_2, c_3|q)$ *and* $p_k(x; \mathbf{t}|q)$. *The connection coefficient is a multiple of a* ${}_5\phi_4$ *and was first found in the Askey–Wilson memoir [3].*

Upon setting $z = t_1$ in Theorem 2.2, we have the next corollary.

Corollary 2.5. *We have the following identity*

$$
\begin{aligned}
&{}_p\phi_{p-1}\left(\left.\begin{matrix} a_1,\cdots,a_{p-1},t_4/t_1 \\ t_2t_4,t_3t_4,b_1,\cdots,b_{p-3} \end{matrix}\right| q,\zeta\right) \\
&= \sum_{k=0}^{\infty} \frac{(a_1,\cdots,a_{p-1},t_1t_2,t_1t_3;q)_k}{(t_2t_4,t_3t_4,b_1,\cdots,b_{p-3};q)_k} \\
&\times \frac{(-t_4\zeta/t_1)^k q^{\binom{k}{2}}}{(q,t_1t_2t_3t_4q^{k-1};q)_k}\, {}_{p-1}\phi_{p-2}\left(\left.\begin{matrix} q^k a_1,\cdots,q^k a_{p-1} \\ q^k b_1,\cdots,q^k b_{p-3},t_1t_2t_3t_4q^{2k} \end{matrix}\right| q,\zeta\right).
\end{aligned}
$$

We note that by equating coefficients of ζ^n on both sides of the equation in Corollary 2.5 is equivalent to the sum of a terminating very well poised ${}_6\phi_5$, [7, (II.21)]

Remark 2.6. *One may take* Λ_n *to be* 0 *unless* $n \equiv a \pmod{b}$ *for fixed integers* $a, b,\ a > 0, b \geq 0$. *This leads to hypergeometric expansions where the differences of consecutive parameters in a certain group is* $1/b$.

We now return to the general expansion formula Proposition 2.1. Since the coefficient of $p_k(x;\mathbf{t}|q)$ is a single sum, if we multiply by a function of n, and then sum over n, the coefficient of $p_k(x;\mathbf{t}|q)$ is a double sum. We state this result.

Proposition 2.7. *We have the expansion*

$$
\begin{aligned}
\sum_{n=0}^{\infty} \frac{\Lambda_n}{(q,bt_4;q)_n}(bz,b/z;q)_n &= \sum_{k=0}^{\infty} p_k(x;\mathbf{t}|q) \frac{(-b)^k q^{\binom{k}{2}}}{(q,bt_4,t_1t_2t_3t_4q^{k-1};q)_k} \\
\times \sum_{s=0}^{\infty} &\frac{(t_1t_4q^k,\ t_2t_4q^k,t_3t_4q^k;q)_s}{(q,bt_4q^k,t_1t_2t_3t_4q^{2k};q)_s}\left(\frac{b}{t_4}\right)^s \sum_{n=0}^{\infty} \frac{(b/t_4;q)_n}{(q;q)_n}\Lambda_{n+k+s}.
\end{aligned}
$$

We next make two different choices for Λ_n's in Proposition 2.7. First let

$$
\Lambda_n = \frac{(A;q)_n}{(B;q)_n}\left(\frac{Bt_4}{bA}\right)^n.
$$

Then the sum over n on the right side of Proposition 2.7 is evaluable by the q-analogue of Gauss's theorem [7, (II.7)], and the coefficient of $p_k(x;\mathbf{t}|q)$ is a single sum. Thus we have established Theorem 2.8 which we will now state.

Theorem 2.8. *We have the expansion formula*

$$
\begin{aligned}
&{}_3\phi_2\left(\left.\begin{matrix} A, bz, b/z \\ bt_4, B \end{matrix}\,\right|\, q, \frac{Bt_4}{bA}\right) \\
&= \frac{(B/A, Bt_4/b; q)_\infty}{(B, Bt_4/bA; q)_\infty} \sum_{k=0}^{\infty} \frac{(A;q)_k}{(bt_4, Bt_4/b;q)_k} \left(\frac{Bt_4}{bA}\right)^k \frac{(-b)^k q^{\binom{k}{2}}}{(q, t_1t_2t_3t_4q^{k-1};q)_k} \\
&\quad \times {}_4\phi_3\left(\left.\begin{matrix} Aq^k, t_1t_4q^k, t_2t_4q^k, t_3t_4q^k \\ bt_4q^k, t_1t_2t_3t_4q^{2k}, q^kBt_4/b \end{matrix}\,\right|\, q, \frac{B}{A}\right) p_k(x;\mathbf{t}|q).
\end{aligned}
$$

The second choice for Λ_n in Proposition 2.7 is

$$
\Lambda_n = \frac{(q^{-N}, A;q)_n}{(B, q^{1-N}Ab/Bt_4;q)_n} q^n.
$$

This time the n-sum is evaluable by the q-Pfaff-Saalschütz theorem, [7, (II.12)].

Theorem 2.9. *The expansion of a general terminating ${}_4\phi_3$ in the Askey-Wilson polynomials is given by*

$$
\begin{aligned}
&{}_4\phi_3\left(\left.\begin{matrix} q^{-N}, A, bz, b/z \\ bt_4, B, bAq^{1-N}/Bt_4 \end{matrix}\,\right|\, q, q\right) \\
&= \frac{(B/A, Bt_4/b;q)_N}{(B, Bt_4/Ab;q)_N} \sum_{k=0}^{N} \frac{(-t_4)^k q^{\binom{k+1}{2}} (q^{-N}, A;q)_k}{(q, bt_4, t_1t_2t_3t_4q^{k-1}, Bt_4/b, q^{1-N}A/B;q)_k} p_k(x;\mathbf{t}) \\
&\quad \times {}_5\phi_4\left(\left.\begin{matrix} q^{-N+k}, Aq^k, t_1t_4q^k, t_2t_4q^k, t_3t_4q^k \\ bt_4q^k, Bt_4q^k/b, Aq^{k+1-N}/B, t_1t_2t_3t_4q^{2k}, \end{matrix}\,\right|\, q, q\right).
\end{aligned}
$$

In Theorem 2.9 if we replace A by Aq^{N-1}, we can then identify parameters a_2, a_3 such that the ${}_4\phi_3$ in Theorem 2.9 is a multiple of $p_N(x; b, a_2, a_3, t_4)$. As such Theorem 2.9 is equivalent to a connection coefficient problem solved in [3]. We also note that although Theorem 2.8 is the limiting case $N \to \infty$ of Theorem 2.9, Theorem 2.8 is not available in the literature.

Remark 2.10. *If we specialize Theorem 2.9 to*

$$
b = t_2, \quad B = t_1t_2, \quad z = t_3.
$$

the ${}_5\phi_4$ in Theorem 2.9 reduces to a balanced ${}_3\phi_2$ which is again evaluable by the q-Pfaff-Saalschütz theorem [7, (II.12)]. *The resulting identity is the terminating* *case of the Watson transformation* [7, (III.18)]. *The nonterminating case Watson transformation* [7, (III.18)] *follows by analytic continuation in the variable* $d = q^N$.

We record an inverse to the expansion formula Proposition 2.1. The proof uses the connection relation for the Askey–Wilson basis, which is [9], [10],

$$\frac{(az, a/z; q)_m}{(q, ab; q)_m} = \sum_{k=0}^{m} \frac{(bz, b/z; q)_k}{(q, ab; q)_k} \frac{(a/b; q)_{m-k}}{(q; q)_{m-k}} \left(\frac{a}{b}\right)^k. \tag{2.4}$$

Theorem 2.11. *The inverse relation to Proposition 2.1 is*

$$p_n(x; \mathbf{t}) = t_1^{-n} \prod_{j=2}^{4} (t_1 t_j; q)_n \sum_{k=0}^{n} \frac{(q^{-n}, t_1t_2t_3t_4q^{n-1}, bz, b/z; q)_k}{(q, t_1t_2, t_1t_3, t_1t_4; q)_k} \left(\frac{qt_1}{b}\right)^k$$
$$\times {}_4\phi_3 \left(\left. \begin{matrix} q^{k-n}, bt_1q^k, t_1t_2t_3t_4q^{n+k-1},\ t_1/b \\ t_1t_2q^k, t_1t_3q^k, t_1t_4q^k, \end{matrix} \right| q, q \right).$$

Proof. We take $a = t_1$ in (2.4) and use (1.3). □

3. Askey-Wilson generating functions

In this section we give two generating functions for Askey-Wilson polynomials: Theorem 3.1, which follows from Proposition 2.1, and Theorem 3.2, for which we provide an independent proof.

Theorem 3.1. *The Askey–Wilson polynomials have the generating function*

$$\begin{aligned} \frac{(be^{i\theta}, be^{-i\theta}; q)_\infty}{(bt_4, b/t_4; q)_\infty} &= \sum_{k=0}^{\infty} \frac{(-b)^k q^{\binom{k}{2}}}{(q, bt_4, t_1t_2t_3t_4q^{k-1}; q)_k}\, p_k(x; \mathbf{t}|q) \\ &\quad \times {}_3\phi_2 \left(\left. \begin{matrix} t_1t_4q^k,\ t_2t_4q^k,\ t_3t_4q^k \\ bt_4q^k, t_1t_2t_3t_4q^{2k} \end{matrix} \right| q, \frac{b}{t_4} \right) \end{aligned} \tag{3.1}$$

and satisfy the relationship

$$\begin{aligned} &\frac{(t_1z, t_1/z, t_1t_2t_3t_4; q)_\infty}{(t_1t_2, t_1t_3, t_1t_4; q)_\infty} \\ &= \sum_{k=0}^{\infty} \frac{(-t_1)^k (t_1t_2t_3t_4/q; q)_k}{(q, t_1t_2, t_1t_3, t_1t_4; q)_k}\, q^{\binom{k}{2}}\, \frac{1 - t_1t_2t_3t_4q^{2k-1}}{1 - t_1t_2t_3t_4/q}\, p_k(x; \mathbf{t}|q) \end{aligned} \tag{3.2}$$

Proof. To prove (3.1) we let $n \to \infty$ in Proposition 2.1. Taking the limit inside the sum is justified by Tannery's theorem, [4], the discrete analogue of the Lebesgue dominated convergence theorem. We omit the details. The identity (3.2) is the case $b = t_1$ of (3.1), because the ${}_3\phi_2$ becomes a ${}_2\phi_1$ and is summed by the q-Gauss theorem [7, (II.8)]. □

One may ask for a version of Theorem 3.1 in which the infinite products in z are in the denominator.

Theorem 3.2. *The Askey–Wilson polynomials have the generating function*

$$\frac{1}{(be^{i\theta}, be^{-i\theta}; q)_\infty} = \sum_{n=0}^{\infty} p_n(x; \mathbf{t}|q) c_n(\mathbf{t}, b), \tag{3.3}$$

where

$$\begin{aligned} c_n(\mathbf{t}, b) = {} & \frac{b^n \, (t_2t_3t_4bq^n; q)_\infty}{(q, t_1t_2t_3t_4q^{n-1}; q)_n \prod_{j=2}^4 (t_jb; q)_\infty} \\ & \times {}_3\phi_2\left(\left. \begin{matrix} q^n t_2t_3, q^n t_2t_4, q^n t_3t_4 \\ q^{2n}t_1t_2t_3t_4, \; q^n t_2t_3t_4b \end{matrix} \right| q, t_1 b \right). \end{aligned} \tag{3.4}$$

Proof of Theorem 3.2. We use two facts to prove Theorem 3.2.

The first fact is the orthogonality relation [10] for Askey-Wilson polynomials

$$\int_{-1}^{1} p_m(x; \mathbf{t} \,|\, q) \, p_n(x; \mathbf{t} \,|\, q) \, w(x; \mathbf{t} \,|\, q) \, dx = A(\mathbf{t}) h_n(\mathbf{t}) \, \delta_{m,n}, \tag{3.5}$$

$$h_n(\mathbf{t}) = \frac{(q;q)_n \prod\limits_{1 \le j < k \le 4} (t_jt_k; q)_n (t_1t_2t_3t_4q^{n-1}; q)_n}{(t_1t_2t_3t_4; q)_{2n}}, \tag{3.6}$$

$$w(x; \mathbf{t}) = w(x; t_1, t_2, t_3, t_4 \,|\, q) = \frac{(e^{2i\theta}, e^{-2i\theta}; q)_\infty}{\prod\limits_{j=1}^{4} (t_je^{i\theta}, t_je^{-i\theta}; q)_\infty} \frac{1}{\sqrt{1-x^2}}, \quad -1 < x < 1. \tag{3.7}$$

Here we have assumed that $max\{|t_1|, |t_2|, |t_3|, |t_4|\} < 1$.

The second fact is Theorem 3.5 in [14] (with $x_4 \leftrightarrow x_5$)

$$\begin{aligned}
&\frac{(q;q)_\infty}{2\pi}\int_0^\pi \frac{w(\cos\theta;x_1,x_2,x_3,x_4)}{(x_5e^{i\theta},x_5e^{-i\theta};q)_\infty}\,\sin\theta\,d\theta \\
&= \frac{(x_1x_2x_3x_4,x_2x_3x_4x_5,x_1x_5;q)_\infty}{\prod_{1\le r<s\le 5}(x_rx_s;q)_\infty}\,{}_3\phi_2\left(\left.\begin{matrix} x_2x_3,x_2x_4,x_3x_4 \\ x_1x_2x_3x_4,\ x_2x_3x_4x_5\end{matrix}\right| q, x_1x_5\right).
\end{aligned} \tag{3.8}$$

The integral (3.8) is a special case of the Nassrallah–Rahman integral [7, (6.3.2)] but the form (3.8) is more convenient to use.

For symmetry we replace b by t_5. We shall find the coefficient $c_n(\mathbf{t},t_5)$ of $p_n(x;\mathbf{t}|q)$ using orthogonality, setting

$$\sum_{n=0}^\infty c_n(\mathbf{t},t_5)p_n(x;\mathbf{t}|q) = \frac{1}{(t_5e^{i\theta},t_5e^{-i\theta})_\infty}.$$

Such a formula exists because the right-hand side is $\in L^2[w,[-1,1]]$. Moreover

$$c_n(\mathbf{t},t_5)h_n(\mathbf{t})A(\mathbf{t}) = \int_{-1}^1 \frac{w(x,\mathbf{t})}{(t_5e^{i\theta},t_5e^{-i\theta};q)_\infty}\,p_n(x;\mathbf{t}|q)dx$$

Therefore, using (1.3) we see that

$$\begin{aligned}
c_n(\mathbf{t},t_5)h_n(\mathbf{t})A(\mathbf{t}) &= \frac{(t_1t_2,t_1t_3,t_1t_4;q)_n}{t_1^n}\sum_{k=0}^n \frac{(q^{-n},t_1t_2t_3t_4q^{n-1};q)_k}{(q,t_1t_2,t_1t_3,t_1t_4;q)_k}q^k \\
&\times \int_0^\pi \frac{1}{(t_5e^{i\theta},t_5e^{-i\theta};q)_\infty} w(\cos\theta;t_1q^k,t_2,t_3,t_4)\sin\theta d\theta.
\end{aligned}$$

The integral is now evaluated by (3.8) and we obtain

$$\begin{aligned}
&\frac{(q;q)_\infty}{2\pi}c_n(\mathbf{t},t_5)h_n(\mathbf{t})A(\mathbf{t}) = \frac{(t_1t_2,t_1t_3,t_1t_4;q)_n}{t_1^n}\sum_{k=0}^n \frac{(q^{-n},t_1t_2t_3t_4q^{n-1};q)_k}{(q,t_1t_2,t_1t_3,t_1t_4;q)_k}q^k \\
&\times \frac{(q^kt_1t_2t_3t_4,t_2t_3t_4t_5,q^kt_1t_5;q)_\infty}{\prod_{j=2}^5(q^kt_1t_j;q)_\infty\prod_{2\le r<s\le 5}(t_rt_s;q)_\infty}\,{}_3\phi_2\left(\left.\begin{matrix} t_2t_3,t_2t_4,t_3t_4 \\ q^kt_1t_2t_3t_4,\ t_2t_3t_4t_5\end{matrix}\right| q, q^kt_1t_5\right).
\end{aligned}$$

Write the ${}_3\phi_2$ as a sum over s and interchange the k and s sums to see that

$$\begin{aligned}
&\frac{(q;q)_\infty}{2\pi}c_n(\mathbf{t},t_5)h_n(\mathbf{t})A(\mathbf{t}) = \frac{(t_1t_2,t_1t_3,t_1t_4;q)_n(t_1t_2t_3t_4,t_2t_3t_4t_5;q)_\infty}{t_1^n\prod_{j=2}^4(t_1t_j;q)_\infty\prod_{2\le r<s\le 5}(t_rt_s;q)_\infty} \\
&\times \sum_{s=0}^\infty \frac{(t_2t_3,t_2t_4,t_3t_4;q)_s}{(q,t_1t_2t_3t_4,t_2t_3t_4t_5;q)_s}(t_1t_5)^s\sum_{k=0}^n \frac{(q^{-n},t_1t_2t_3t_4q^{n-1};q)_k}{(q,q^st_1t_2t_3t_4;q)_k}q^{k(s+1)}.
\end{aligned}$$

The k sum is an evaluable terminating ${}_2\phi_1$, [7, (II.7)], and we obtain

$$\frac{(q;q)_\infty}{2\pi}c_n(\mathbf{t},t_5)h_n(\mathbf{t})A(\mathbf{t}) = \frac{(t_1t_2,t_1t_3,t_1t_4;q)_n(t_1t_2t_3t_4,t_2t_3t_4t_5;q)_\infty}{t_1^n\prod_{j=2}^4(t_1t_j;q)_\infty\prod_{2\le r<s\le 5}(t_rt_s;q)_\infty}$$
$$\times\sum_{s=0}^\infty\frac{(t_2t_3,t_2t_4,t_3t_4;q)_s}{(q,t_1t_2t_3t_4,t_2t_3t_4t_5;q)_s}(t_1t_5)^s\frac{(q^{s+1-n};q)_n}{(q^st_1t_2t_3t_4;q)_n}.$$

Thus $s\ge n$, so shift s by n. Therefore the left-hand side in the above equation is the statement of the theorem. □

An attractive special case of Theorem 3.2 is a corollary due to Kim and Stanton [16].

Corollary 3.3. *The continuous dual q-Hahn polynomials $p_n(x;t_1,t_2,t_3|q)$ have the generating function*

$$\sum_{k=0}^\infty\frac{p_k(x;t_1,t_2,t_3|q)}{(q,bt_1t_2t_3;q)_k}b^k = \frac{(bt_1,bt_2,bt_3;q)_\infty}{(be^{i\theta},be^{-i\theta},bt_1t_2t_3;q)_\infty}$$

Proof. Take $t_2=0$ in Theorem 3.2 and relabel the t_j's. The ${}_3\phi_2$ becomes a ${}_1\phi_0$ which we sum by the q-binomial theorem. □

Corollary 3.4. *For any positive integer n, $p_n(z,\mathbf{t}|q)=0$ if*

$$z=-\gamma,\quad t_1=\gamma,\quad t_2=\gamma^3,\quad t_3=\gamma^5,\quad t_4=0,\quad \gamma=e^{2\pi i/6}.$$

Corollary 3.5. *Let ω be a primitive cubic root of unity. Then*

$$\sum_{k=0}^n\begin{bmatrix}n\\k\end{bmatrix}_q\frac{1}{(c^3;q)_k}p_k(-1/2;c,\omega c,\omega^2c|q) = \begin{cases}0 & \text{if } 3\nmid n,\\ \frac{(q,q^2;q^3)_{n/3}}{(c^3q,c^3q^2;q^3)_{n/3}} & \text{if } 3|n.\end{cases}\tag{3.9}$$

Proof. Multiply the equation in Corollary 3.3 by $(bt_1t_2t_3;q)_\infty/(b;q)_\infty$ then expand $(q^kbt_1t_2t_3;q)_\infty/(b;q)_\infty$ by the q-binomial theorem. Set

$$t_1=c,\quad t_2=c\omega,\quad t_3=c\omega^2,\quad x=\cos(2\pi/3)$$

and equate the coefficients of like powers of b. □

4. Integrals

The expansions in §2 can be changed into integral evaluations using the orthogonality relation (3.5), and equations (3.6)-(3.7).

Proposition 2.1 becomes

$$\int_0^\pi \frac{(be^{i\theta}, be^{-i\theta}; q)_n (e^{2i\theta}, e^{-2i\theta}; q)_\infty}{\prod_{j=1}^4 (t_j e^{i\theta}, t_j e^{-i\theta}; q)_\infty} p_k(\cos\theta; \mathbf{t}|q)\, d\theta$$

$$= \frac{(-b)^k q^{\binom{k}{2}} (q;q)_n (b/t_4, bt_4 q^k; q)_{n-k}}{(q;q)_{n-k}} \frac{2\pi (t_1t_2t_3t_4 q^{2k}; q)_\infty}{(q;q)_\infty \prod_{1\le r<s\le 4} (t_r t_s q^k; q)_\infty} \tag{4.1}$$

$$\times {}_4\phi_3 \left(\begin{matrix} q^{k-n},\ t_1t_4q^k,\ t_2t_4q^k,\ t_3t_4q^k \\ bt_4q^k, t_1t_2t_3t_4q^{2k}, q^{1+k-n}t_4/b \end{matrix} \middle| q, q \right).$$

In view of (3.1), the limiting case $n \to \infty$ of (4.1) is

$$\int_0^\pi \frac{(be^{i\theta}, be^{-i\theta}, e^{2i\theta}, e^{-2i\theta}; q)_\infty}{\prod_{j=1}^4 (t_j e^{i\theta}, t_j e^{-i\theta}; q)_\infty} p_k(\cos\theta; \mathbf{t}|q)\, d\theta$$

$$= (-b)^k q^{\binom{k}{2}} (b/t_4, bt_4q^k; q)_\infty \frac{2\pi (t_1t_2t_3t_4 q^{2k}; q)_\infty}{(q;q)_\infty \prod_{1\le r<s\le 4} (t_r t_s q^k; q)_\infty} \tag{4.2}$$

$$\times {}_3\phi_2 \left(\begin{matrix} t_1t_4q^k,\ t_2t_4q^k,\ t_3t_4q^k \\ bt_4q^k, t_1t_2t_3t_4q^{2k} \end{matrix} \middle| q, \frac{b}{t_4} \right).$$

When $b = t_1$, for example, the ${}_3\phi_2$ in (4.2) sums. The result is known because it is the constant term in the expansion of $p_k(x; t_1, t_2, t_3, t_4)$ in $p_k(x; 0, t_2, t_3, t_4)$, see [3], or [10].

We record the analogous results for Proposition 2.7 and Theorem 2.2.

Theorem 4.1. *We have the integral evaluation*

$$\int_0^\pi \left[\sum_{n=0}^\infty \frac{\Lambda_n(bz, b/z; q)_n}{(q, bt_4; q)_n} \right] p_k(\cos\theta; \mathbf{t}|q) \frac{(e^{2i\theta}, e^{-2i\theta}; q)_\infty}{\prod_{j=1}^4 (t_j e^{i\theta}, t_j e^{-i\theta}; q)_\infty} d\theta$$

$$= \frac{2\pi (-b)^k q^{\binom{k}{2}} (t_1t_2t_3t_4q^{2k}; q)_\infty}{(bt_4; q)_k (q;q)_\infty \prod_{1\le r<s\le 4} (q^k t_r t_s; q)_\infty}$$

$$\times \sum_{s=0}^\infty \frac{(t_1t_4q^k,\ t_2t_4q^k, t_3t_4q^k; q)_s}{(q, bt_4q^k, t_1t_2t_3t_4q^{2k}; q)_s} \left(\frac{b}{t_4}\right)^s \sum_{n=0}^\infty \frac{(b/t_4; q)_n}{(q;q)_n} \Lambda_{k+n+s}.$$

In particular we get the following corollary

Corollary 4.2. *The following evaluation holds*

$$\int_0^\pi {}_{p+1}\phi_p\left(\begin{matrix} a_1,\cdots,a_{p-1},t_4e^{i\theta},t_4e^{-i\theta} \\ t_1t_4,t_2t_4,t_3t_4,b_1,\cdots,b_{p-3}\end{matrix}\middle| q,\zeta\right) p_k(\cos\theta;\mathbf{t}|q)\ \frac{(e^{2i\theta},e^{-2i\theta};q)_\infty}{\prod\limits_{j=1}^4(t_je^{i\theta},t_je^{-i\theta};q)_\infty}d\theta$$

$$= \frac{2\pi(a_1,\cdots,a_{p-1};q)_k}{(b_1,\cdots,b_{p-3};q)_k}\ \frac{(-t_4\zeta)^k q^{\binom{k}{2}}(t_1t_2,t_1t_3,t_2t_3;q)_k}{\prod_{1\le r<s\le 4}(t_rt_s;q)_\infty}$$

$$\times(t_1t_2t_3t_4q^{2k};q)_\infty\ {}_{p-1}\phi_{p-2}\left(\begin{matrix} q^ka_1,\ \cdots,\ q^ka_{p-1} \\ q^kb_1,\cdots,q^kb_{p-3},t_1t_2t_3t_4q^{2k}\end{matrix}\middle| q,\zeta\right).$$

5. An Expansion with Arbitrary Coefficients

We consider Proposition 2.1 when $b = t_4$ so $f_{n,k}$ is given by (2.2). In this section we give another proof of this result and generalize it to sums involving arbitrary sequences. This extends the following formula of Verma [20]

$$\sum_{m=0}^\infty a_mb_m\frac{(zw)^m}{m!}$$
$$= \sum_{n=0}^\infty \frac{(-z)^n}{n!\,(\gamma+n)_n}\left(\sum_{r=0}^\infty\frac{b_{n+r}\,z^r}{r!\,(\gamma+2n+1)_r}\right)\left[\sum_{s=0}^n\frac{(-n)_s(n+\gamma)_s}{s!}\,a_sw^s\right], \tag{5.1}$$

from Jacobi type polynomials to Askey–Wilson type polynomials. Verma also noted a Laguerre type expansion where w is replaced by w/γ, b_n is replaced by γb_n and $\gamma\to\infty$.

We now go back to (2.3) and observe that $\{(t_4z,t_4/z;q)_n\}$ is a basis for the space of polynomials, hence we can replace $(t_4z,t_4/z;q)_n$ by $A_n(t_4z,t_4/z;q)_n$ and (2.3) will remain valid as long as the series on both sides converge. This establishes the following expansion theorem.

Proposition 5.1. *We have the general expansion*

$$\begin{aligned}
&\sum_{n=0}^{\infty} \frac{(az, a/z;q)_n}{(q;q)_n} A_n B_n \zeta^n \\
&= \sum_{k=0}^{\infty} \frac{(-\zeta)^k q^{\binom{k}{2}}}{(q, Cq^{k-1};q)_k} \left[\sum_{j=0}^{k} \frac{(q^{-k}, Cq^{k-1};q)_j}{(q;q)_j} A_j (az, a/z;q)_j q^j \right] \\
&\quad \times \left[\sum_{n=0}^{\infty} \frac{B_{n+k}\zeta^n}{(q, Cq^{2k};q)_n} \right].
\end{aligned} \tag{5.2}$$

Proposition 5.1 writes a triple sum as a single sum. Another way to prove Proposition 5.1 is to use matrix inversion. In [8, Theorem 3.2] the explicit matrix A has an explicit inverse, namely

$$A_{k,j} = \frac{(Cq^{2j-1};q)_{k-j}}{(q;q)_{k-j}} q^{-kj}, \qquad (A^{-1})_{s,k} = \frac{(C;q)_{2s-1}(1-Cq^{2k-1})}{(q;q)_{s-k}(C;q)_{s+k}} q^{\binom{s-k+1}{2}}(-1)^{s-k}.$$

Indeed the product $A^{-1}A$ is the identity matrix. Using this result, the right side of Proposition 5.1 reduces to a single sum, which is the left side.

Ismail and Zhang [15] introduced the q-exponential function

$$\mathcal{E}_q(\cos\theta;\alpha) := \frac{(\alpha^2;q^2)_\infty}{(q\alpha^2;q^2)_\infty} \sum_{n=0}^{\infty} (-ie^{i\theta}q^{(1-n)/2}, -ie^{-i\theta}q^{(1-n)/2};q)_n \frac{(-i\alpha)^n}{(q;q)_n} q^{n^2/4}. \tag{5.3}$$

In [13, (6.7)] the following expansion for $\mathcal{E}_q$ was established,

$$\mathcal{E}_q(\cos\theta;\alpha) = \frac{\left(-\alpha;q^{1/2}\right)_\infty}{\left(q\alpha^2;q^2\right)_\infty} {}_2\phi_1\left(\left. \begin{matrix} q^{1/4}e^{i\theta}, q^{1/4}e^{-i\theta} \\ -q^{1/2} \end{matrix} \right| q^{1/2}, -\alpha \right). \tag{5.4}$$

For proofs and details see Chapter 14 of [10].

Proposition 5.2. *The function $\mathcal{E}_q(\cos\theta;\alpha)$ has the expansion*

$$\begin{aligned}
\frac{(q^2t^4;q^4)_\infty}{(-t;q)_\infty} \mathcal{E}_{q^2}(x;t) &= \sum_{k=0}^{\infty} \frac{t^k q^{k^2/2}}{(q, -q, t_2t_3t_4q^{k-1/2};q)_k} p_k(x;q^{1/2}, t_2, t_3, t_4|q) \\
&\quad \times {}_3\phi_2\left(\left. \begin{matrix} q^{k+1/2}t_2, q^{k+1/2}t_3, q^{k+1/2}t_4 \\ -q^{k+1}, t_2t_3t_4q^{2k+1/2} \end{matrix} \right| q, -t \right).
\end{aligned}$$

Proof. In (1.3) and (5.2) we set

$$a = t_1 = q^{1/2}, \quad \zeta = -t, \quad C = q^{1/2} t_2 t_3 t_4,$$
$$A_j = \frac{1}{(q^{1/2}t_2, q^{1/2}t_3, q^{1/2}t_4; q)_j}, \quad B_j = \frac{(q^{1/2}t_2, q^{1/2}t_3, q^{1/2}t_4; q)_j}{(-q; q)_j},$$

and establish the desired expansion ☐

Rahman [19] chose a set of continuous q-Jacobi polynomials with

$$\mathbf{t} = (q^{1/2}, q^{\alpha+1/2}, -q^{\beta+1/2}, -q^{1/2}),$$

defined by

$$P_n^{(\alpha,\beta)}(\cos\theta; q) = \frac{(q^{\alpha+1}, -q^{\beta+1}; q)_n}{(q, -q; q)_n}, {}_4\phi_3\left(\begin{matrix} q^{-n}, q^{n+\alpha+\beta+1}, q^{1/2}e^{i\theta}, q^{1/2}e^{-i\theta}, \\ q^{\alpha+1}, -q^{\beta+1}, -q \end{matrix} \middle| q, q \right).$$

Askey and Wilson [3] defined a set of continuous q-Jacobi polynomials by choosing

$$\mathbf{t} = (q^{1/4+\alpha/2}, q^{3/4+\alpha/2}, -q^{1/4+\beta/2}, q^{3/4+\beta/2})$$

$$P_n^{(\alpha,\beta)}(\cos\theta | q) = \frac{(q^{\alpha+1}; q)_n}{(q; q)_n} {}_4\phi_3\left(\begin{matrix} q^{-n}, q^{n+\alpha+\beta+1}, q^{1/4+\alpha/2}e^{i\theta}, q^{1/4+\alpha/2}e^{-i\theta}, \\ q^{\alpha+1}, -q^{1/2+\alpha/2+\beta/2}, -q^{1+\alpha/2+\beta/2} \end{matrix} \middle| q, q \right).$$

These polynomials are related by [3, (4.20),(4.21)]

$$P_n^{(\alpha,\beta)}(x|q^2) = q^{\alpha n} \frac{(q; q)_n}{(-q^{\alpha+\beta+1}; q)_n} P_n^{(\alpha,\beta)}(x; q)$$

In [12, (6.1.3)], $\mathcal{E}_q(x; t)$ is expanded in continuous q-Jacobi polynomials. It is clear that Proposition 5.2 generalizes such an expansion because it contains one more free parameter.

Another interesting case of Proposition 5.1 is

$$a = t_1, \quad C = t_1 t_2 t_3 t_4, \quad A_j = \frac{1}{\prod_{k=2}^4 (t_1 t_k; q)_j}, \quad B_j = \Lambda_j \prod_{k=2}^4 (t_1 t_k; q)_j.$$

The result is the expansion (2.3).

6. Big and Little q-Jacobi Polynomials

The Askey-Wilson polynomials contain, as special and limiting cases, many other sets of polynomials. This is done in detail in [17]. Here we record two limiting cases of Proposition 5.1.

Recall that the big and little q-Jacobi polynomials are defined by

$$P_n(x;\alpha,\beta,\gamma) = {}_3\phi_2\left(\left.\begin{matrix} q^{-n}, \alpha\beta q^{n-1}, x \\ \alpha q\gamma q \end{matrix}\,\right|\, q, q\right), \tag{6.1}$$

$$p_n(x;\alpha,\beta) = {}_2\phi_1\left(\left.\begin{matrix} q^{-n}, q^{n+1}\alpha\beta \\ q\alpha \end{matrix}\,\right|\, q, qx\right), \tag{6.2}$$

respectively, see [10, (18.4.7)&(18.4.11)]. We now derive expansions in

Theorem 6.1. *We have the following expansions in big q-Jacobi type polynomials*

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{(x;q)_n}{(q;q)_n}A_nB_n\zeta^n = \sum_{k=0}^{\infty}\frac{(-\zeta)^k q^{\binom{k}{2}}}{(q,Cq^{k-1};q)_k}&\left[\sum_{j=0}^{k}\frac{(q^{-k},Cq^{k-1};q)_j}{(q;q)_j}A_j(x;q)_jq^j\right] \\ &\times\left[\sum_{n=0}^{\infty}\frac{B_{n+k}\zeta^n}{(q,Cq^{2k};q)_n}\right],\end{aligned} \tag{6.3}$$

and little q-Jacobi type polynomial expansion

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{x^n}{(q;q)_n}A_nB_n\zeta^n = \sum_{k=0}^{\infty}\frac{(-\zeta)^k q^{\binom{k}{2}}}{(q,Cq^{k-1};q)_k}&\left[\sum_{j=0}^{k}\frac{(q^{-k},Cq^{k-1};q)_j}{(q;q)_j}A_jx^jq^j\right] \\ &\times\left[\sum_{n=0}^{\infty}\frac{B_{n+k}\zeta^n}{(q,Cq^{2k};q)_n}\right],\end{aligned} \tag{6.4}$$

Proof. In (5.2) we set $z = a/x$, then replace A_n by $(-1)^nx^nq^{-\binom{n}{2}}a^{-2n}A_n$, then let $a\to\infty$. The result is (6.3). In (6.3) replace x by λx and A_n by $(-1)^nq^{-\binom{n}{2}}/\lambda^n$ and let $\lambda\to\infty$. This proves (6.4). □

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