SPRINGER'S THEOREM FOR MODULAR COINVARIANTS OF $GL_n(\mathbb{F}_q)$

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ABSTRACT. Two related results are proven in the modular invariant theory of $GL_n(\mathbb{F}_q)$. The first is a finite field analogue of a result of Springer on coinvariants of the symmetric group S_n acting on $\mathbb{C}[x_1, \ldots, x_n]$. It asserts that the following two $\mathbb{F}_{q^n}[GL_n(\mathbb{F}_q) \times \mathbb{F}_{q^n}^\times]$-modules have the same composition factors:

- the coinvariant algebra for $GL_n(\mathbb{F}_q)$ acting on $\mathbb{F}_{q^n}[x_1, \ldots, x_n]$, in which $GL_n(\mathbb{F}_q)$ acts as a subgroup of $GL_n(\mathbb{F}_{q^n})$ by linear substitutions of variables, and $\mathbb{F}_{q^n}^\times$ acts by scalar substitutions of variables,
- the action on the group algebra $\mathbb{F}_{q^n}[GL_n(\mathbb{F}_q)]$ by left and right multiplication.

The second result is a related statement about parabolic invariants and coinvariants.

1. Introduction

This paper concerns two related results in the modular invariant theory of $GL_n(\mathbb{F}_q)$.

The first compares two representations of $G \times C$ over $\mathbb{F}_{q^n}$, where

$$G := GL_n(\mathbb{F}_q)$$
$$C := \mathbb{F}_{q^n}^\times.$$

On one hand, viewing $\mathbb{F}_{q^n}$ as an n-dimensional $\mathbb{F}_q$-vector space, scalar multiplications by elements of $\mathbb{F}_{q^n}^\times$ are invertible $\mathbb{F}_q$-linear maps. Thus C may be viewed as a (cyclic) subgroup[1] of $GL_n(\mathbb{F}_q)$, and $G \times C$ therefore acts on the group algebra $\mathbb{F}_{q^n}[G]$, with G acting by left-multiplication and C by right-multiplication. On the other hand, G acts on polynomials $\mathbb{F}_q[x_1, \ldots, x_n]$ by linear substitutions of variables,

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[1]A generator for C, regarded as an element of $GL_n(\mathbb{F}_q)$, is sometimes called a *Singer cycle*.

and by extending scalars, on $S := \mathbb{F}_{q^n}[x_1, \ldots, x_n]$. Let A denote the *coinvariant algebra* for this G-action, that is, the quotient $S/(S_+^G)$ where (S_G^+) is the ideal generated by G-invariants of positive degree. The quotient algebra A still affords a G-action induced by linear substitutions of variables, and C acts on A by scalar substitutions: $x_i \mapsto cx_i$ for $c \in C = \mathbb{F}_{q^n}^\times$. As these two actions commute, A becomes a representation of $G \times C$.

Theorem 1. *As $\mathbb{F}_{q^n}[G \times C]$-modules, the coinvariant algebra A and the group algebra $\mathbb{F}_{q^n}[G]$ have the same composition factors.*

The proof of Theorem 1 is given in Section 2, using the theory of Brauer characters.

Section 3 proves the second main result, via a similar character computation, concerning the invariants of a parabolic subgroup $P \subset G$. Since the G-invariants S^G form a subalgebra of the P-invariants S^P, one can form a quotient $A^{\langle P \rangle} := S^P/(S_+^G)$. This quotient no longer affords an action of G, however it does afford an action of $C = \mathbb{F}_{q^n}^\times$ as before.

Theorem 2. *As $\mathbb{F}_{q^n}[C]$-modules, the algebra $A^{\langle P \rangle}$ and the P-invariants $\mathbb{F}_{q^n}[G]^P$ in the group algebra are isomorphic.*

The remainder of this introduction gives some context for these results. Theorem 1 is a q-analogue of a well-known result from the theory of reflection groups, where $G = \mathfrak{S}_n$ and C is a cyclic subgroup of $\mathfrak{S}_n$ generated by an n-cycle, as we now explain.

For G a finite subgroup of $GL_n(\mathbb{C})$, an element $g \in G$ is called a *reflection* if its fixed subspace is a hyperplane in $\mathbb{C}^n$ (called its *reflecting hyperplane*). One calls G a *reflection group* if it is generated by reflections. Such groups were classified by Shephard and Todd [8], who showed that their polynomial invariants form a polynomial subalgebra:

$$\mathbb{C}[x_1, \ldots, x_n]^G = \mathbb{C}[f_1, \ldots, f_n]$$

where $f_1, \ldots, f_n$ are homogeneous invariants. Chevalley [1] gave a uniform proof of this fact, and showed that, as representations of G, the *coinvariant algebra*

$$A := \mathbb{C}[x_1, \ldots, x_n]/(f_1, \ldots, f_n)$$

and the group algebra $\mathbb{C}[G]$ are isomorphic.

Springer proved a beautiful refinement of this isomorphism in his theory of regular elements. Say that $c \in G$ is *regular* if it has an eigenvector in $\mathbb{C}^n$ which lies on none of the reflecting hyperplanes for reflections in G. Let $C = \langle c \rangle$ be the subgroup of G generated by a regular element c. Springer showed [10, Proposition 4.5] that the following two $G \times C$-representations are isomorphic[2]:

- the coinvariant algebra A, with G acting by linear substitutions, and with C acting by $c(x_i) = \gamma x_i$ for all i, where $\gamma \in \mathbb{C}^\times$ is a root of unity of the same multiplicative order as c.
- the group algebra $\mathbb{C}[G]$, with $G \times C$ acting by left and right multiplication.

In the special case where $G = \mathfrak{S}_n$, the fundamental theorem of symmetric functions asserts $\mathbb{C}[x_1, \ldots, x_n]^G = \mathbb{C}[e_1, \ldots, e_n]$, where e_i is the i^{th} *elementary symmetric function*, having degree i. A result of Dickson [3] gives a q-analogue for $G = GL_n(\mathbb{F}_q)$:

$$\mathbb{F}_q[x_1, \ldots, x_n]^G = \mathbb{F}_q[\mathbf{d}_{n,0}, \ldots, \mathbf{d}_{n,n-1}]$$

where each *Dickson invariant* $\mathbf{d}_{n,i}$ is a homogeneous polynomial of degree $q^n - q^i$; see [9, §8.1]. Chevalley's result was generalized by Mitchell [7, Theorem 1.4], who showed that, as $\mathbb{F}_q[G]$-modules, the coinvariant algebra

$$A := \mathbb{F}_q[x_1, \ldots, x_n]/(\mathbf{d}_{n,0}, \ldots, \mathbf{d}_{n,n-1})$$

and the group algebra $\mathbb{F}_q[G]$ have the same composition factors [3]. Theorem 1 refines this last assertion in the same way that Springer's result refines that of Chevalley.

Theorem 2 may be viewed as the q-analogue of a consequence of Springer's result. When P is a parabolic subgroup of a reflection group G, acting on $S := \mathbb{C}[x_1, \ldots, x_n]$, semisimplicity of the actions implies

$$A^{\langle P \rangle} := S^P/(S^G_+) \cong (S/(S^G_+))^P = A^P$$

where here A^P denotes the P-invariants of the coinvariant algebra A. Then Springer's result that A and $\mathbb{C}[G]$ are isomorphic as $\mathbb{C}[G \times C]$-modules immediately implies that $A^P (\cong A^{\langle P \rangle})$ and $\mathbb{C}[G]^P$ are isomorphic as $\mathbb{C}[C]$-modules. Unfortunately, in characteristic $p > 0$, the situation is not as straightforward, and we see no way of using Theorem 1 to prove Theorem 2.

[2] This phrasing in terms of actions of $G \times C$ is actually borrowed from Kraśkiewicz and Weyman [6]. They proved (independently) a result equivalent to Springer's in the special case where G is a Coxeter group of type $A, B(= C), D$ and c is a Coxeter element.

[3] The authors thank N. Kuhn and L. Smith for pointing this out. The same also holds whenever G is a subgroup of $GL_n(\mathbb{F})$ for which $\mathbb{F}[x_1, \ldots, x_n]^G$ is a polynomial subalgebra.

2. Proof of Theorem 1

The proof of Theorem 1 relies on the theory of Brauer characters; see [2, §82]. For a finite group H, an $\mathbb{F}_{q^n}[H]$-module W, and $h \in H$ a p-regular element (where p is the characteristic of $\mathbb{F}_{q^n}$), let $\phi^H_W(h) \in \mathbb{C}$ denote the Brauer character value of h on W. If $W = \bigoplus_k W_k$ is a graded $\mathbb{F}_{q^n}[H]$-module, define its *graded Brauer character* by

$$\phi^H_W(h;t) := \sum_k \phi^H_{W_k}(h)\ t^k.$$

When $R = \bigoplus_k R_k$ is a graded k-algebra, define its *Hilbert series*

$$\mathrm{Hilb}(R,t) := \sum_k \dim_k R_k\ t^k.$$

To prove the theorem, we must show that for every p-regular element $(g,c) \in G \times C$ there is an equality of the Brauer character values

$$\phi^{G\times C}_{\mathbb{F}_{q^n}[G]}(g,c) = \phi^{G\times C}_A(g,c). \tag{2.1}$$

We begin by computing the left side of (2.1).

Proposition 3. *For $c \in \mathbb{F}_{q^n}^\times$ and $g \in G = GL_n(\mathbb{F}_q)$, the Brauer character $\phi^{G\times C}_{\mathbb{F}_{q^n}[G]}(g,c)$ vanishes unless g^{-1} is G-conjugate to c.*

When g^{-1} is G-conjugate to c, one has

$$\phi^{G\times C}_{\mathbb{F}_{q^n}[G]}(g,c) = |GL_{\frac{n}{r}}(\mathbb{F}_{q^r})| = \prod_{\substack{0\leq i\leq n-1\\ i\equiv 0 \mod r}} (q^n - q^i).$$

where the integer r is defined by $\mathbb{F}_q(c) = \mathbb{F}_{q^r} \subset \mathbb{F}_{q^n}$.

Proof. Note that $\mathbb{F}_{q^n}[G]$ is a permutation representation of $G \times C$, and therefore lifts to a representation defined over $\mathbb{Z}$. Hence its Brauer character is its usual character, namely $\phi^{G\times C}_{\mathbb{F}_{q^n}[G]}(g,c)$ is the number of points fixed as (g,c) permutes G. Therefore

$$\begin{aligned}\phi^{G\times C}_{\mathbb{F}_{q^n}[G]}(g,c) &= |\{h \in G : ghc = h\}| \\ &= |\{h \in G : c = h^{-1}g^{-1}h\}| \\ &= \begin{cases} |C_G(c)| & \text{if } g^{-1} \text{ is } G\text{-conjugate to } c, \\ 0 & \text{else,}\end{cases}\end{aligned}$$

where $C_G(c)$ is the centralizer of c in G. Note that an invertible $\mathbb{F}_q$-linear transformation of $\mathbb{F}_{q^n}$ centralizes c if and only if it is an $\mathbb{F}_q(c)$-linear transformation. Hence if $\mathbb{F}_q(c) = \mathbb{F}_{q^r}$, then $C_G(c) \cong GL_{\frac{n}{r}}(\mathbb{F}_{q^r})$. $\square$

We next turn to computing the right side of (2.1). For this, we need some notation about the Brauer lifting process. Assume that $g \in G = GL_n(\mathbb{F}_q)$ is p-regular, and let V be the span of linear forms in $\mathbb{F}_{q^n}[x_1, \ldots, x_n]$ (or in A). Since g is p-regular, it acts semisimply on V with eigenvalues $\overline{\lambda}_1, \ldots, \overline{\lambda}_n$ lying in an extension of $\mathbb{F}_{q^n}$ by appropriate roots of unity. Under the Brauer lifting process, these eigenvalues lift to roots of unity $\lambda_1, \ldots, \lambda_n \in \mathbb{C}^\times$. Given $c \in C = \mathbb{F}_{q^n}^\times$, which acts on V as the scalar c, let γ be the root of unity in $\mathbb{C}^\times$ which lifts it.

Lemma 4. *With notation as above,*

$$\phi_A^{G\times C}(g,c) = \left[\frac{\prod_{i=0}^{n-1}(1-t^{q^n-q^i})}{\prod_{i=1}^{n}(1-\lambda_i t)}\right]_{t=\gamma}.$$

Proof. This is essentially a calculation along the lines of Molien's Theorem [9, Proposition 4.3.1]. We start by computing the graded Brauer character for g on $S = \mathbb{F}_{q^n}[x_1, \ldots, x_n]$, which we identify with the symmetric algebra $\mathrm{Sym}(V)$. Note that the eigenvalues of g on $\mathrm{Sym}^k(V)$ will be all the monomials $\overline{\lambda}_1^{k_1} \cdots \overline{\lambda}_n^{k_n}$ with $\sum_i k_i = n$. Consequently,

$$\phi^G_{\mathrm{Sym}^k(V)}(g) = \sum_{\sum_i k_i = k} \lambda_1^{k_1} \cdots \lambda_n^{k_n}$$

$$\phi^G_{\mathrm{Sym}(V)}(g;t) = \prod_{i=1}^{n} \frac{1}{(1-\lambda_i t)}.$$

Let $\mathcal{D}^*(n)$ denote the extension of the Dickson algebra $\mathbb{F}_q[x_1, \ldots, x_n]^G$ by the scalars $\mathbb{F}_{q^n}$, that is

$$\begin{aligned}\mathcal{D}^*(n) &= \mathbb{F}_{q^n} \otimes_{\mathbb{F}_q} \mathbb{F}_q[x_1, \ldots, x_n]^G \\ &= \mathbb{F}_{q^n}[x_1, \ldots, x_n]^G = \mathrm{Sym}(V)^G = \mathbb{F}_{q^n}[\mathbf{d}_{n,0}, \ldots, \mathbf{d}_{n,n-1}].\end{aligned}$$

Since $\deg(\mathbf{d}_{n,i}) = q^n - q^i$, one has that

$$\phi^G_{\mathcal{D}^*(n)}(g;t) = \mathrm{Hilb}(\mathcal{D}^*(n), t) = \prod_{i=0}^{n-1} \frac{1}{1-t^{q^n-q^i}}.$$

Let $g \in G$ be a p-regular element. Observing the following three facts

- $\mathrm{Sym}(V) = \mathbb{F}_{q^n}[x_1, \ldots, x_n]$ is a free $\mathcal{D}^*(n)$-module (see [9, Cor. 6.7.13]),
- A is a semisimple $\mathbb{F}_{q^n}[\langle g\rangle]$-module, and
- g acts trivially on $\mathcal{D}^*(n)$,

we see that there is an isomorphism of graded $\mathbb{F}_{q^n}[\langle g\rangle]$-modules

$$\mathrm{Sym}(V) \cong A \otimes_{\mathbb{F}_{q^n}} \mathcal{D}^*(n).$$

This implies

$$\phi^G_{\mathrm{Sym}(V)}(g;t) = \phi^G_A(g;t)\ \phi^G_{\mathcal{D}^*(n)}(g;t).$$

Therefore

$$\phi^G_A(g;t) = \frac{\phi^G_{\mathrm{Sym}(V)}(g;t)}{\phi^G_{\mathcal{D}^*(n)}(g;t)} = \frac{\prod_{i=0}^{n-1}(1-t^{q^n-q^i})}{\prod_{i=1}^{n}(1-\lambda_i t)}. \tag{2.2}$$

To understand $\phi^{G\times C}_A((g,c);t)$ from (2.2), note that c acts on the k^{th}-graded piece A_k by the scalar c^k. Hence $\phi^{G\times C}_{A_k}(g,c)$ is γ^k times the coefficient of t^k in (2.2). Then $\phi^{G\times C}_A(g,c)$ comes from summing this over all k, which is the same as setting $t=\gamma$ in (2.2). □

In analyzing the right side of Lemma 4, one needs to know about the zeros at $t=\gamma$ of factors like $t^{q^j-q^i}-1$. This is equivalent to knowledge of when $c^{q^j-q^i}=1$, which is easily characterized.

Lemma 5. *For $c \in \mathbb{F}^\times_{q^n}$, define r by $\mathbb{F}_q(c)=\mathbb{F}_{q^r}$. Then for any two integers i,j, one has $c^{q^j-q^i}=1$ if and only $j \equiv i \mod r$. Consequently, the same holds replacing c by γ.*

Proof. Without loss of generality, assume $i \le j$.

Assuming $j \equiv i \mod r$, then $q^j - q^i$ is an integer multiple of q^r-1. Since $c \in \mathbb{F}^\times_{q^r}$, one has $c^{q^r}=c$, and hence $c^{q^r-1}=1$. Thus $c^{q^j-q^i}=1$ also.

Conversely, assume $1 = c^{q^j-q^i}$. Raising both sides of this equation to the $(q^{n-i})^{th}$ power gives

$$\begin{aligned} 1 &= (c^{q^j-q^i})^{q^{n-i}} \\ &= c^{(q^{j-i}-1)q^n} \\ &= (c^{q^{j-i}-1})^{q^n} \\ &= c^{q^{j-i}-1}. \end{aligned}$$

This implies that c lies in the splitting field for $x^{q^{j-i}}-x$ over $\mathbb{F}_q$, that is, the extension $\mathbb{F}_{q^{j-i}}$. But then $\mathbb{F}_{q^r}=\mathbb{F}_q(c)\subset\mathbb{F}_{q^{j-i}}$, forcing r to divide $j-i$. □

Theorem 1 will now follow by comparing Proposition 3 with the following proposition.

Proposition 6. *For $c \in \mathbb{F}^\times_{q^n}$ and $g \in GL_n(\mathbb{F}_q)$, the Brauer character $\phi^{G\times C}_A(g,c)$ vanishes unless g^{-1} is G-conjugate to c. When g^{-1} is G-conjugate to c, one has*

$$\phi^{G\times C}_A(g,c) = \prod_{\substack{0\le i\le n-1 \\ i\equiv 0 \mod r}} (q^n-q^i)$$

where r is defined by $\mathbb{F}_q(c) = \mathbb{F}_{q^r}$.

Proof. By Lemma 5 one has that γ is a (simple) root of each of the $\frac{n}{r}$ factors of the form $1 - t^{q^n - q^i}$ with $i \equiv 0 \mod r$ that appear in the numerator of the rational function in Lemma 4. Suppose $\phi_A^{G\times C}(g, c) \neq 0$. Since the Brauer character value comes from substituting $t = \gamma$ in this rational function, γ must appear as a root of at least $\frac{n}{r}$ of the factors in the denominator. Thus at least $\frac{n}{r}$ of $\lambda_1^{-1}, \ldots, \lambda_n^{-1}$ must equal γ, or equivalently, at least $\frac{n}{r}$ of the eigenvalues $\overline{\lambda}_1^{-1}, \ldots, \overline{\lambda}_n^{-1}$ of g^{-1} must equal c. Denoting by $m_{c,\mathbb{F}_q}$ the minimal polynomial of c, which has c as a simple root by separability, this implies $(m_{c,\mathbb{F}_q})^{\frac{n}{r}}$ must divide the characteristic polynomial of g^{-1}. But this minimal polynomial has degree r, so $(m_{c,\mathbb{F}_q})^{\frac{n}{r}}$ *equals* the characteristic polynomial of g^{-1}. The minimal polynomial of g^{-1} equals $m_{c,\mathbb{F}_q}$ since g^{-1} is diagonalizable, and this now completely determines the rational canonical form of g^{-1}. It follows that g^{-1} is uniquely determined up to conjugacy within $G = GL_n(\mathbb{F}_q)$, and since c is an element with the same rational canonical form, g^{-1} is G-conjugate to c. This proves the first assertion of the proposition.

Now assume g^{-1} is G-conjugate to c. Since the characteristic polynomial of g has c^{-1} as a root, and the Galois group of $\mathbb{F}_{q^n}$ over $\mathbb{F}_q$ is cyclic of order n, generated by the Frobenius automorphism, the eigenvalues of g will be $\{c^{-q^i}\}_{i=0}^{n-1}$. As these eigenvalues lift to $\{\lambda_i\}_{i=1}^{n} = \{\gamma^{-q^i}\}_{i=0}^{n-1}$, Lemma 4 tells us in this case that

$$
\begin{aligned}
\phi_A^{G\times C}(g, c) &= \left[\prod_{i=0}^{n-1} \frac{(1 - t^{q^n - q^i})}{(1 - \gamma^{-q^i} t)}\right]_{t=\gamma} \\
&= \left[\prod_{\substack{0\le i\le n-1 \\ i\not\equiv 0 \mod r}} \frac{(1 - t^{q^n - q^i})}{(1 - \gamma^{-q^i} t)}\right]_{t=\gamma} \prod_{\substack{0\le i\le n-1 \\ i\equiv 0 \mod r}} \lim_{t\to\gamma} \frac{(1 - t^{q^n - q^i})}{(1 - \gamma^{-q^i} t)} \\
&= \prod_{\substack{0\le i\le n-1 \\ i\not\equiv 0 \mod r}} \frac{(1 - \gamma^{q^n - q^i})}{(1 - \gamma^{-q^i + 1})} \prod_{\substack{0\le i\le n-1 \\ i\equiv 0 \mod r}} \frac{-(q^n - q^i)\gamma^{q^n - q^i - 1}}{-\gamma^{-q^i}} \\
&= \prod_{\substack{0\le i\le n-1 \\ i\equiv 0 \mod r}} (q^n - q^i).
\end{aligned}
$$

where the third equality above uses L'Hôpital's Rule. □

3. Proof of Theorem 2

The proof of Theorem 2 is a character computation, very much analogous to that of the previous section.

Let P be the parabolic subgroup which fixes a particular flag of $\mathbb{F}_q$-subspaces

$$0 \subset V^{m_1} \subset \cdots \subset V^{m_\ell} = \mathbb{F}_{q^n} \tag{3.1}$$

in which $\dim_{\mathbb{F}_q} V^{m_s} = m_s$ for each s. Given the sequence $(m_1, \ldots, m_\ell)$, define the sequence of integers $\mathbf{n} := (n_1, \ldots, n_\ell)$ that sums to $n(= m_\ell)$ by

$$m_s = n_1 + n_2 + \cdots + n_s \text{ for } 1 \leq s \leq \ell,$$

and call any flag as in (3.1) an $\mathbf{n}$-*flag*. Recall that the total number of $\mathbf{n}$-flags is counted by a *q-multinomial coefficient*

$$\begin{bmatrix} n \\ n_1, \ldots, n_\ell \end{bmatrix}_q := \frac{\prod_{i=1}^{n}(q^n - q^{n-i})}{\prod_{s=1}^{\ell}\prod_{i=1}^{n_s}(q^{m_s} - q^{m_s - i})}.$$

Since $C = \mathbb{F}_{q^n}^{\times}$ has order $q^n - 1$, which is coprime to the characteristic of $\mathbb{F}_{q^n}$, it acts semisimply, and a calculation in $\mathbb{C}$ with Brauer characters suffices to prove the isomorphism asserted in Theorem 2. Fix once and for all an isomorphism of C with the $(q^n-1)^{st}$ roots of unity in $\mathbb{C}$. Given $c \in C$, let $\gamma \in \mathbb{C}$ denotes its lift under this isomorphism.

One must show that for every $c \in C$, there is an equality of $\mathbb{F}_{q^n}[C]$ Brauer character values

$$\phi_{\mathbb{F}_{q^n}[G]^P}(c) = \phi_{A\langle P \rangle}(c). \tag{3.2}$$

We begin by computing the left side of (3.2).

Proposition 7. *For $c \in \mathbb{F}_{q^n}^{\times}$ the Brauer character $\phi_{\mathbb{F}_{q^n}[G]^P}(c)$ vanishes unless the integer r defined by $\mathbb{F}_q(c) = \mathbb{F}_{q^r}$ divides every n_i (or equivalently, every m_i). When this divisibility occurs, one has*

$$\phi_{\mathbb{F}_{q^n}[G]^P}(c) = \begin{bmatrix} \frac{n}{r} \\ \frac{n_1}{r}, \ldots, \frac{n_\ell}{r} \end{bmatrix}_{q^r}.$$

Proof. Note that G acts transitively on $\mathbf{n}$-flags and P is the stabilizer of a particular $\mathbf{n}$-flag. Hence the G-action by right-multiplication on $\mathbb{F}_{q^n}[G]^P \cong \mathbb{F}_{q^n}[P \backslash G]$ is isomorphic to the permutation representation in which G permutes $\mathbf{n}$-flags. Restricting this to the subgroup C, one concludes that the Brauer character value $\phi_{\mathbb{F}_{q^n}[G]^P}(c)$ is the number of $\mathbf{n}$-flags stabilized by c. Since c generates the subfield $\mathbb{F}_{q^r}$ over $\mathbb{F}_q$, an $\mathbf{n}$-flag of $\mathbb{F}_q$-subspaces is stabilized by c if and only if it consists of $\mathbb{F}_{q^r}$-subspaces, and the result follows. $\square$

We next turn to computing the right side of (3.2). For this, we employ a result of Kuhn and Mitchell [5] (rediscovered by Hewett [4]) generalizing Dickson's Theorem, that describes the parabolic invariants S^P.

Theorem 8. *(*[5, Theorem 2.2], [4, Theorem 1.4]*) Let $P \subset G$ be the parabolic subgroup stabilizing a chosen $\mathbf{n}$-flag. Then the P-invariants $\mathbb{F}_q[x_1, \ldots, x_n]^P$ form a polynomial subalgebra, with homogeneous generators of degree*

$$q^{m_s} - q^{m_s - i} \text{ for } 1 \leq s \leq \ell \text{ and } 1 \leq i \leq n_s.$$

This allows one to calculate the Hilbert series for $A^{\langle P \rangle}$, and hence also its $\mathbb{F}_{q^n}[C]$ Brauer character.

Corollary 9. *Let $S = \mathbb{F}_{q^n}[x_1, \ldots, x_n]$ as before. Then the quotient ring $A^{\langle P \rangle} := S^P/(S^G_+)$ has Hilbert series*

$$\begin{aligned} \mathrm{Hilb}(A^{\langle P \rangle}, t) &= \frac{\prod_{i=1}^{n}(1 - t^{q^n - q^{n-i}})}{\prod_{s=1}^{\ell}\prod_{i=1}^{n_s}(1 - t^{q^{m_s} - q^{m_s - i}})} \\ &=: \begin{bmatrix} n \\ n_1, \ldots, n_\ell \end{bmatrix}_{q,t} \end{aligned}$$

which we will call a "(q,t)-multinomial coefficient".

Consequently, the $\mathbb{F}_{q^n}[C]$ Brauer character value $\phi_{A^{\langle P \rangle}}(c)$ is obtained from $\begin{bmatrix} n \\ n_1, \ldots, n_\ell \end{bmatrix}_{q,t}$ by setting $t = \gamma$, where $\gamma \in \mathbb{C}$ is the lift of c.

Proof. By Theorem 8, we know that

$$S^P = \mathbb{F}_{q^n}[x_1, \ldots, x_n]^P \cong \mathbb{F}_{q^n} \otimes_{\mathbb{F}_q} \mathbb{F}_q[x_1, \ldots, x_n]^P$$

is a polynomial algebra, and hence is a Cohen-Macaulay ring. Therefore since S^P is an integral extension of the polynomial subalgebra $S^G = \mathcal{D}^*(n)$ [9, §2.3], it is free as a $\mathcal{D}^*(n)$-module. Thus

$$\mathrm{Hilb}(S^P, t) = \mathrm{Hilb}(\mathcal{D}^*(n), t)\ \mathrm{Hilb}(S^P/(S^G_+), t), \text{ so that}$$

$$\mathrm{Hilb}(A^{\langle P \rangle}, t) = \frac{\mathrm{Hilb}(S^P, t)}{\mathrm{Hilb}(\mathcal{D}^*(n), t)}.$$

Using the degrees of the generators of S^P given in Theorem 8, one then has

$$\mathrm{Hilb}(S^P, t) = \frac{1}{\prod_{s=1}^{\ell}\prod_{i=1}^{n_s}(1 - t^{q^{m_s} - q^{m_s - i}})}$$

and the result follows. □

Theorem 2 will now follow by comparing Proposition 7 with the following proposition.

Proposition 10. *For $c \in \mathbb{F}_{q^n}^\times$, the Brauer character $\phi_{A\langle P\rangle}(c)$ vanishes unless the integer r defined by $\mathbb{F}_q(c) = \mathbb{F}_{q^r}$ divides every n_i. When this divisibility occurs,*

$$\phi_{A\langle P\rangle}(c) = \begin{bmatrix} \frac{n}{r} \\ \frac{n_1}{r}, \ldots, \frac{n_\ell}{r} \end{bmatrix}_{q^r}.$$

Proof. We proceed similarly to the proof of Proposition 6: since $\phi_{A\langle P\rangle}(c)$ is obtained from the (q,t)-multinomial coefficient by setting $t = \gamma$, we analyze the zeros at $t = \gamma$ among the factors in numerator and denominator of this rational function.

As before, Lemma 5 implies that γ appears as a root $\frac{n}{r}$ times in the numerator, in the factors $1 - t^{q^n - q^{n-i}}$ for $i \equiv 0 \mod r$.

In the denominator, the factor $1 - t^{q^{m_s} - q^{m_s - i}}$ has γ as a (simple) root if and only $i \equiv 0 \mod r$, again by Lemma 5. Thus each product $\prod_{i=1}^{n_s}(1 - t^{q^{m_s} - q^{m_s - i}})$ in the denominator can contribute at most $\frac{n_s}{r}$ zeros at $t = \gamma$, with equality if and only if $r | n_s$. Since $\sum_{s=1}^{\ell} \frac{n_s}{r} = \frac{n}{r}$, this means that the rational function will vanish at $t = \gamma$ unless $r | n_s$ for each s.

Now assume $r | n_s$ for each s. Then

$$\begin{aligned}
\phi_{A\langle P\rangle}(c) &= \left[\frac{\prod_{i=1}^{n}(1 - t^{q^n - q^{n-i}})}{\prod_{s=1}^{\ell}\prod_{i=1}^{n_s}(1 - t^{q^{m_s} - q^{m_s - i}})} \right]_{t=\gamma} \\
&= \frac{\prod_{\substack{1 \le i \le n \\ i \not\equiv 0 \mod r}} (1 - \gamma^{q^n - q^{n-i}})}{\prod_{s=1}^{\ell} \prod_{\substack{1 \le i \le n_s \\ i \not\equiv 0 \mod r}} (1 - \gamma^{q^{m_s} - q^{m_s - i}})} \\
&\qquad \times \lim_{t \to \gamma} \frac{\prod_{\substack{1 \le i \le n \\ i \equiv 0 \mod r}} (1 - t^{q^n - q^{n-i}})}{\prod_{s=1}^{\ell} \prod_{\substack{1 \le i \le n_s \\ i \equiv 0 \mod r}} (1 - t^{q^{m_s} - q^{m_s - i}})}
\end{aligned} \tag{3.3}$$

Note that since r divides n and each m_s, one has $\gamma = \gamma^{q^n} = \gamma^{q^{m_s}}$, and hence

$$\gamma^{q^n - q^{n-i}} = \gamma^{q^{m_s} - q^{m_s - j}} \Leftrightarrow \gamma^{-q^{n-i}} = \gamma^{-q^{m_s - j}} \Leftrightarrow j \equiv i \mod r$$

by Lemma 5. This implies that the first quotient of products on the right side of the last equation in (3.3) is 1. Using L'Hôpital's Rule on the second quotient of products yields

$$\begin{aligned}
\phi_{A\langle P\rangle}(c) &= \frac{\prod_{\substack{1\leq i\leq n \\ i\equiv 0 \mod r}} (-(q^n - q^{n-i})\gamma^{q^n-q^{n-i}-1})}{\prod_{s=1}^{\ell}\prod_{\substack{1\leq i\leq n_s \\ i\equiv 0 \mod r}} (-(q^{m_s} - q^{m_s-i})\gamma^{q^{m_s}-q^{m_s-i}-1})} \\
&= \frac{\prod_{\substack{1\leq i\leq n \\ i\equiv 0 \mod r}} (q^n - q^{n-i})}{\prod_{s=1}^{\ell}\prod_{\substack{1\leq i\leq n_s \\ i\equiv 0 \mod r}} (q^{m_s} - q^{m_s-i})} \\
&= \begin{bmatrix} \frac{n}{r} \\ \frac{n_1}{r}, \ldots, \frac{n_\ell}{r} \end{bmatrix}_{q^r}.
\end{aligned}$$

□

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