REMINISCENCES ABOUT RICHARD ASKEY AND HIS CONTRIBUTIONS TO COMBINATORICS

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Abstract. We tell some stories about our association with Askey and how he influenced the development of the combinatorial theory of orthogonal polynomials and constant term identities as well as the multivariate special functions.

Richard Askey was an analyst but in this paper we only discuss his contributions to combinatorics. None of his important contributions to the analytic aspects of special functions will be discussed here. Dick was a gifted teacher, mathematician and educator. He had this great insight into people's tastes and personalities. He had great success in attracting mathematicians with different background to his favorite subject, "Special Functions". It takes a very special person to achieve this goal.

The authors were very privileged to have met and worked with Dick Askey.

Ismail spent the academic year 1974–1975 at the University of Wisconsin as an assistant to Dick Askey. When he arrived, Askey told him that his duties are to go over the manuscript of a book on orthogonal polynomials that Dick is writing. Askey felt that the addition theorem of Jacobi polynomials was the last major open problem in the area and Tom Koornwinder's solution essentially closes the subject.

He also said that they can also look at other problems involving positivity questions. Then Dick received a paper [18] by Even and Gillis about representing the number of derangements as an integral involving products of Laguerre polynomials. Askey and Ismail started looking at this problem and extended the Even–Gillis result to Meixner polynomials [5] and it became clear that there is a lot of combinatorics behind the linearization of products of functions. By the end of Ismail's stay they had interesting work but the book manuscript did not materialize because Dick rightly felt that there is more going on. His hunch was right and the later work with Andrews and Wilson totally changed the subject.

It may be of interest to share with the reader how Dick's wise hunch paid off and resulted in [5]. Even and Gillis showed that the number of Derangements $D(n_1, n_2, \cdots, n_k)$ is

$$D(n_1, n_2, \cdots, n_k) = (-1)^{n_1+\cdots+n_k} \int_0^\infty e^{-x} \prod_{j=1}^{k} L_{m_j}(x) dx$$

where the L_n's are Laguerre polynomials. Dick knew that this can be proved using the MacMahon Master theorem. Dick also realized that the integral representations can be used to derive accurate asymptotic approximations. So the asymptotics were derived and Dick suggested that Mourad Ismail go to the Boca Raton combinatorics conference, talk about what they have, and ask the audience for help with references on this problem. Sure enough L. Bruce Richmond, who was a classmate of Mourad in graduate school, attended

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the lecture and had a long discussion with Mourad and gave him references to papers by Riordan and Kaplansky. It turned out that the symbolic approach used by Kaplansky and Riordan implied the Even and Gillis result if we write $n!$ as $\int_0^\infty x^n e^{-x} dx$. This led to finding integral representations of many other combinatorial numbers using moment representations of certain sequences. The details are in [5].

Dick started corresponding with Marcel-Paul Schützenberger and Dominique Foata about combinatorial issues. Then Foata and Konrad Jacobs organized a meeting at the mathematics Institute in Oberwolfach in 1977. Foata organized several others later on and they were all very successful, Dick raised many questions, and offered insightful comments at these meetings. In particular the Askey Scheme (see [29]) was born at one of these meetings. In fact it was Mike Hoare and Mizan Rahman who first thought of having a scheme of orthogonal polynomials organized in a tree like diagram. The advantage of this structure is that the polynomials at the lower branches can be obtained as special or limiting cases of the ones on higher branches. The Hoare-Rahman chart went only up to the Hahn polynomials. Dick then suggested that the Wilson polynomials [41] should be at the highest level and after his survey article with George Andrews [2] was completed the Askey Scheme took the final shape that we know nowadays. Jacques Labelle produced it as a poster and made available to the mathematical community at large. Dick had great communication skills and knew how to attract people to his favorite subject by showing them the relevance of the subject and its interaction with other fields.

Foata [20], and later with Strehl [22], used the exponential formula to give devastatingly short and incisive combinatorial proofs of generating functions for classical orthogonal polynomials. Another French school soon developed for general orthogonal polynomials. Flajolet had given a combinatorial interpretation of certain continued fractions [19] using lattice paths. This could be applied to the Jacobi continued fraction for the moment generating function

$$\sum_{n=0}^{\infty} \mu_n t^n, \quad \mu_n = \int_{-\infty}^{\infty} x^n d\mu(x).$$

The paths are weighted by the coefficients in the three term recurrence relation for $p_n(x)$. Viennot [39] and the Bordeaux school applied these ideas for general combinatorial results, and also specific classical sets. One may see that this approach will be fruitful from these examples: Hermite, Laguerre, and Charlier polynomials. Their moments are respectively

$$\begin{aligned} \mu_n &= \begin{cases} \prod_{j=0}^{n/2-1}(n-1-2j), \text{ if } n \text{ is even,} \\ 0 \text{ if } n \text{ is odd,} \end{cases} \\ &= \text{the number of perfect matchings on } \{1, 2, \cdots, n\}, \\ \mu_n &= n!, \text{ the number of permutations on } \{1, 2, \cdots, n\}, \\ \mu_n &= B_n, \text{ the } n\text{th Bell number, the number of set partitions on } \{1, 2, \cdots, n\}. \end{aligned}$$

Parameters are introduced by weighting the objects. Then any integral involving polynomials with an orthogonal polynomial measure may be reinterpreted as a weighted lattice path, and identities become equivalent to weighted bijections. The UQAM school was very active in this area, [21], [30]. Askey himself did not think much would come from the general moments, and he was surprised. Zeilberger [42] used moments of a set of q-Legendre polynomials to prove the refined alternating sign matrix conjecture.

In the spring of 1975 Dick organized a symposium on special functions and George Andrews gave a lecture [1] on q-series. He formulated a q-analogue of the Dyson conjecture.

It said that the constant term (CT) of a specific Laurent polynomial in several variables could be evaluated

$$CT \prod_{1\le i<j\le n} \left(\frac{x_i}{x_j};q\right)_{a_i} \left(\frac{qx_j}{x_i};q\right)_{a_j} = \begin{bmatrix} a_1+\cdots+a_n \\ a_1,\cdots,a_n \end{bmatrix}_q.$$

Dick soon was interested in beta functions and their generalizations, because they provided measures for the polynomials in the Askey Scheme. He found some multivariable discrete versions, but also searched for absolutely continuous versions. The constant terms could be written as an integral. His hope was that these integrals would lead to the key multivariable versions of Jacobi polynomials, and ultimately the Askey-Wilson polynomials.

The $a_i = \infty$ case of the Andrews conjecture followed from the type A_{n-1} Macdonald identities. The area expanded very quickly under the leadership of Askey, Andrews, Ian Macdonald, and others. The versions of the Andrews conjecture on root systems were done simultaneously by Askey's student Morris [34] and Macdonald [33]. This whole circle of ideas also involved evaluation of Selberg and Aomoto type integrals and the development of multivariate special functions and their q-analogues, for example the Macdonald and Koornwinder polynomials. The Macdonald conjectures were proven by Opdam [35] and Cherednik [9] using algebraic techniques, see [25] for more historical information. Kadell [28] did the $n = 4$ case of the Andrews conjecture, and Zeilberger and Bressoud [8] combinatorially proved the general Andrews conjecture. Bressoud and Goulden [7] extended the results to other terms. Garvan [23], Habsieger [24], and Zeilberger [43] proved the Macdonald conjecture for some exceptional root systems, and Askey's student Cooper [10] continued in this direction. Johnson's thesis [27] was on classical combinatorial questions. Some of these topics are discussed in [3].

Stanton attended Askey's Special Functions course, first day August 27, 1974. Jim Wilson was also in this class. Askey knew that ultraspherical polynomials were the spherical functions $\phi(g)$ for a Gelfand pair (G, H) on the orthogonal groups. This immediately implied that they satisfied a product formula

$$\phi(g_1)\phi(g_2) = \int_H \phi(g_1 h g_2).$$

Moreover this was the first term of the ultraspherical addition theorem. He also knew that the 2-point homogeneous spaces had been classified by Wang [40], and their spherical functions were special Jacobi polynomials. But you did not need continuous groups for the formalism to apply. Dunkl in 1975 [15] had found an addition theorem for Krawtchouk polynomials, using representation theory and a finite Gelfand pair.

This connected to classical coding theory via the Hamming scheme of N-tuples of 0's and 1's. It was known that Krawtchouk polynomials were central to coding theory, for example Lloyd's theorem [32] says that if perfect e-codes exist, then a certain Krawtchouk polynomial must have integral zeros.

Delsarte [11] had generalized Lloyd's result to P- and Q- polynomial association schemes. In these combinatorial objects, orthogonal polynomials $p_k(\lambda)$ appear as eigenvalues of commuting sets of matrices. (See [6] or the survey paper [14] for applications to generalized codes and designs.) Moreover they automatically give a combinatorial interpretation to the linearization coefficients c_{ij}^k,

$$p_i(\lambda)p_j(\lambda) = \sum_k c_{ij}^k p_k(\lambda).$$

Not only are the coefficients c_{ij}^k positive, but they also count triangles in a graph.

Askey believed there should be a q-analogue of the Hamming scheme which involved graphs and groups. For the binary case, the graph of the hypercube, his intuition was correct. This follows from the fact that the hyperoctahedral group acts on the cube, and it is the Weyl group of types B_N and C_N. Thus there is a natural q-analogue by taking a group G of Lie type B_N or C_N over a finite field $\mathbb{F}_q$, and considering cosets G/P, see [36]. Some properties of the classical Krawtchouk polynomials were not retained, such as symmetry. But the group theoretic techniques that Dunkl used did work here, and also in the type A_N case, see [16], [17]. Delsarte [12], [13] also found other association schemes of matrices over finite fields. Thus the combinatorics of these q-analogues came from classical groups over a finite field.

Suppose that $p_n(x),\ 0 \le n \le N$, is a finite set of orthogonal polynomials whose measure is purely discrete

$$\sum_{j=0}^{N} p_n(x_j) p_m(x_j) w_j = 0, \quad 0 \le m, n \le N, \quad m \ne n.$$

This may interpreted as a row orthogonality of a finite matrix, and, upon rescaling, there is another orthogonality: by columns. It is not guaranteed that $p_n(x_j)$ is a polynomial of degree j in some function θ_n. This is the hypothesis in a Q-polynomial association scheme. Askey knew classical examples, for example the Hahn and dual Hahn polynomials. He also knew that the $3j$ and $6j$ coefficients in angular momentum theory satisfied the above orthogonality, and had explicit formulas as single sums. This led to Racah and q-Racah polynomials [4], which have four parameters, and sit at the top of the discrete part of the Askey Scheme.

It was something of a surprise when the Askey Scheme appeared in association schemes. All of the known infinite families [6] of orthogonal polynomials in P- and Q- polynomial association schemes were special or limiting cases of the q-Racah polynomials. The surprise was Leonard's theorem [31], which, when informally stated, said that the orthogonal polynomials for a P- and Q- polynomial association schemes must be special or limiting cases of q-Racah polynomials. Terwilliger and his school have developed algebraic approaches [37], [38], [26], Leonard pairs and the Terwilliger algebra, to work on the classification of these schemes.

Last but not least we come to the Andrews, Askey, Roy monumental work [3]. As we said earlier, Askey's original plan for writing a book needed to be modified to include q-series and the new polynomials of Wilson, Askey-Wilson, $\cdots$. Originally it was supposed to be Andrews and Askey but as the time went by, it became clear that they needed help. Rangan Roy was ideal for the job and he was attending Askey's seminars. Roy basically turned the Andrews-Askey notes to a well-written book. It contains some combinatorics and pointers to where things can be found, see [3, §6.9]. It contains a beautiful introduction to q-series and mentions many combinatorial problems, Chapters 11 and 12.

A referee reminded us of the following story about Askey. The first time Schützenberger met I. Gelfand, the discussion turned to "combinatorialist" Askey, according to Gelfand. When this pronouncement got back to Dick, he was amused!

In summary, although Dick himself did not write many papers in combinatorics, he was very influential in three areas of its development. His contacts with Foata and the French school invigorated enumeration and weighted bijections. The Askey Scheme is central to P- and Q-polynomial association schemes. His promotion of multidimensional beta integrals, constant term identities, and root systems contributed to Macdonald's development of the

Macdonald symmetric functions. Dick had this canny ability to ask the right people the right questions at the right time.

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