THE ARIKI–KOIKE ALGEBRAS AND q-APPELL FUNCTIONS

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Dedicated to Mourad Ismail on his 80th Birthday

Abstract. Two double sums originating from the Ariki–Koike algebras are evaluated and generalized using a transformation of F. H. Jackson. Two combinatorial approaches to these identities are given, using 2-cores and overpartitions.
Keywords: q-Appell functions, Partitions, Overpartitions, 2-core.

1. Introduction

The Ariki–Koike algebras, denoted by $\mathcal{H}_{\mathbb{C},v;Q_1,\ldots,Q_m}\big(G(m,1,n)\big)$, can be viewed as the Iwahori–Hecke algebras associated to the complex reflection groups $G(m,1,n) \cong S_n \ltimes (\mathbb{Z}/m\mathbb{Z})^n$, where $v, Q_i, i = 1,\ldots,m$ are parameters. They were introduced by Ariki and Koike [3] and independently by Broué and Malle [8]. In [2,4], Ariki and Mathas showed that the simple modules of the Ariki–Koike algebras (when the parameters are roots of unity) are labelled by the so-called Kleshchev multipartitions.

While studying enumerative aspects of the Ariki–Koike algebras and Kleshchev multipartitions [9], two double sum evaluations were found (see [9, Corollary 1.5]),

$$\sum_{r,s\geq 0} \frac{q^{r^2+s^2+r+s}(q^2;q^2)_{r+s+1}}{(q^2;q^2)_r(q^2;q^2)_r(q^2;q^2)_s(q^2;q^2)_{s+1}} = \frac{(-q^2;q^2)_\infty}{(q^2;q^2)_\infty} \tag{1}$$

and

$$\sum_{r,s\geq 0} \frac{q^{r^2+s^2+2s}(q^2;q^2)_{r+s}}{(q^2;q^2)_r(q^2;q^2)_r(q^2;q^2)_s(q^2;q^2)_s} + \sum_{r,s\geq 1} \frac{q^{r^2+s^2}(q^2;q^2)_{r+s-1}}{(q^2;q^2)_r(q^2;q^2)_{r-1}(q^2;q^2)_s(q^2;q^2)_{s-1}} = \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}. \tag{2}$$

Here and throughout this paper, we adopt the following *q-Pochhammer symbols*:

$$(a;q)_\infty := \prod_{j\geq 0}(1-aq^j),$$

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and

$$(a;q)_n := \frac{(a;q)_\infty}{(aq^n;q)_\infty} \text{ for any integer } n.$$

The first goal of this paper is to realize these two identities as special confluent q-Appell function evaluations. Moreover we generalize, in Theorem 3.1 and Theorem 3.2, each evaluation by adding an arbitrary parameter w. We need a transformation for q-Appell functions found by F. H. Jackson [15] in 1944.

The second goal of this paper is to give in Sections 5 and 6 an integer partition interpretation of the sum sides. When $w = 1$, this is accomplished in Section 5 using Kleshchev 2-multipartitions. These partitions are known to parametrize the simple modules for specialized Ariki–Koike algebras and are the original motivation for this work. In Section 6 an independent interpretation is given using overpartitions. This also proves the infinite product summation for general w. The combinatorial meaning of the double sum parameters is known but difficult.

The rest of this paper is organized as follows. In Section 2, we recall some identities on basic hypergeometric series, F. H. Jackson's transformation for q-Appell functions, and some basics for integer partitions. In Section 3, we prove the generalizations of the identities in (1) and (2). In Section 4, we also prove the non-negativity of the individual terms in the w-generalizations of the sum sides of (1) and (2). In Section 5, we combinatorially interpret the double sum sides, while the general w case is done in Section 6. We then conclude our paper providing some remarks in Section 7.

2. Preliminaries

2.1. **Some identities on basic hypergeometric series.**

The *q-binomial coefficients*, also known as the *Gaussian polynomials*, are given by

$$\begin{bmatrix} N \\ M \end{bmatrix} := \begin{bmatrix} N \\ M \end{bmatrix}_q := \begin{cases} \frac{(q;q)_N}{(q;q)_M (q;q)_{N-M}} & \text{if } 0 \le M \le N, \\ 0 & \text{otherwise.} \end{cases}$$

They satisfy the following recurrences [1, eqs. (3.3.3), (3.3.4)]:

$$\begin{bmatrix} N \\ M \end{bmatrix} = \begin{bmatrix} N-1 \\ M \end{bmatrix} + q^{N-M} \begin{bmatrix} N-1 \\ M-1 \end{bmatrix}, \tag{3}$$

$$\begin{bmatrix} N \\ M \end{bmatrix} = \begin{bmatrix} N-1 \\ M-1 \end{bmatrix} + q^{M} \begin{bmatrix} N-1 \\ M \end{bmatrix}. \tag{4}$$

Next, we list a set of identities for later use:

- The q-binomial theorem [1, eq. (3.3.6)]:

$$\sum_{n\ge 0} (-1)^n z^n q^{\binom{n}{2}} \begin{bmatrix} N \\ n \end{bmatrix} = (z;q)_N. \tag{5}$$

- The q-binomial theorem [1, eq. (2.2.6)]:

$$\sum_{n\ge 0} \frac{z^n q^{\binom{n}{2}}}{(q;q)_n} = (-z;q)_\infty. \tag{6}$$

- A special case of the q-Gauss sum [1, eq. (2.2.8)]:

$$\sum_{n\ge 0} \frac{z^n q^{n^2}}{(q;q)_n (zq;q)_n} = \frac{1}{(zq;q)_\infty}. \tag{7}$$

- Jacobi's triple product identity [1, eq. (2.2.10)]:

$$\sum_{n=-\infty}^{\infty} z^n q^{\binom{n}{2}} = (q, -z, -q/z; q)_\infty. \tag{8}$$

- A q-analog of the Chu-Vandermonde summation [1, eq. (3.3.10)]:

$$\sum_{k=0}^{h} \begin{bmatrix} n \\ k \end{bmatrix} \begin{bmatrix} m \\ h-k \end{bmatrix} q^{(n-k)(h-k)} = \begin{bmatrix} m+n \\ h \end{bmatrix}. \tag{9}$$

- A q-analog of the Vandermonde formula [12, eq. (1.5.2)]:

$$\sum_{j=0}^{\infty} \frac{(q^{-n};q)_j (b;q)_j}{(q;q)_j (c;q)_j} \left(\frac{cq^n}{b}\right)^j = \frac{(c/b;q)_n}{(c;q)_n}. \tag{10}$$

2.2. **F. H. Jackson's transformation.** In 1944 F. H. Jackson [15] gave a collection of transformations for various q-Appell functions, which are basic hypergeometric series in two variables. The one we use involves the following function (see [15, eq. (2)]).

Definition 1. *Let*

$$\Psi_1(a;b;c,c';x,y;\lambda) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(a;q)_{m+n}(b;q)_m}{(q;q)_m (q;q)_n (c;q)_m (c';q)_n} x^m y^n q^{\lambda n(n-1)}.$$

His transformation is [15, eq. (30)].

Proposition 2.1. *We have*

$$\Psi_1(a;b;c,c';x,y;\lambda) = \sum_{r=0}^{\infty} \frac{(a;q)_r (b;q)_r}{(q;q)_r (c;q)_r (c';q)_r} x^r y^r a^r q^{(1+\lambda)r(r-1)} \times \Phi(aq^r;bq^r;cq^r;x)\ {}_1\Phi_1(aq^r;c'q^r;yq^{2\lambda r};\lambda),$$

where

$$\Phi(A,B;C;X) = \sum_{m=0}^{\infty} \frac{(A;q)_m (B;q)_m}{(q;q)_m (C;q)_m} X^m,$$

$${}_1\Phi_1(A;C;Y;\lambda) = \sum_{n=0}^{\infty} \frac{(A;q)_n}{(q;q)_n (C;q)_n} Y^n q^{\lambda n(n-1)}.$$

The following special cases of the subsidiary functions in Proposition 2.1 will be used.

Proposition 2.2. *If $a = c'$, then*

$${}_1\Phi_1(aq^r;c'q^r;yq^r;1/2) = (-yq^r;q)_\infty.$$

If $a = c$, then

$$\lim_{b\to\infty} \Phi(aq^r,bq^r;cq^r;x/b) = (xq^r;q)_\infty.$$

Proof. These follow from the q-binomial theorem in (6),

$${}_1\Phi_1(aq^r;aq^r;yq^r;1/2) = \sum_{n=0}^{\infty} \frac{1}{(q;q)_n} (yq^r)^n q^{n(n-1)/2} = (-yq^r;q)_\infty,$$

and

$$\lim_{b\to\infty} \Phi(aq^r,bq^r;aq^r;x/b) = \sum_{m=0}^{\infty} \frac{(-1)^m}{(q;q)_m} (xq^r)^m q^{m(m-1)/2} = (xq^r;q)_\infty.$$

□

2.3. **Some basics on integer partitions.** An integer partition λ is a weakly decreasing sequence of positive integers. These positive integers are called the parts of λ. We denote by $|\lambda|$ the sum of all parts of λ and by $\ell(\lambda)$ the number of parts of λ. If $|\lambda| = n$, then λ is called a partition of n and denoted by $\lambda \vdash n$. It is a convention that the empty sequence $\varnothing$ is considered a partition of 0. We denote by $\mathcal{P}$ the set of all partitions.

A partition λ is called a strict partition if the parts of λ are distinct. We denote by $\mathcal{D}$ the set of all strict partitions. An overpartition is an ordered pair of partitions (θ_1, θ_2), where θ_1 is strict and θ_2 is arbitrary.

For $\lambda \vdash n$, its *Young diagram*, also known as the *Ferrers diagram*, is the graphical representation, which consists of n boxes (or dots) placed left justified in rows with λ_i boxes (or dots) in the i-th row. We denote the Young diagram of λ by Y_λ. In Figure 1, the Young diagram of $\lambda = (4, 2, 2, 1)$ is illustrated.

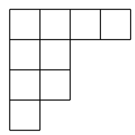

FIGURE 1. $Y_{(4,2,2,1)}$

For a partition λ, let $x = (i, j)$ denote the box in the i-th row and j-th column of the Young diagram Y_λ of λ. The hook of the box $x = (i, j)$ is the following set of boxes

$$H_x := \{(k, m) \mid k \geq i, m \geq j\}.$$

The size of H_x is called the hook length of x and denoted by h_x. In Figure 2, the number in each box indicates its hook length.

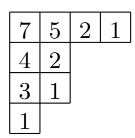

FIGURE 2. Hook lengths of $(4, 2, 2, 1)$

For a positive integer t, λ is called a t-core partition if none of the hooks have length divisible by t. As seen in Figure 2, the partition $(4, 2, 2, 1)$ is a 6-core partition.

There is a well-known algorithm for getting a t-core partition from an arbitrary partition λ [16]. The rim hook of a box x consists of the boxes of the rim between two ends of the hook of x. We successively remove rim hooks of length t until no hooks of length t remain. We call this resulting partition the t-core of λ and denote it by $\lambda_{t\text{-core}}$.

This algorithm can be described using an abacus diagram [16]. For a positive integer t, a t-abacus diagram is a diagram with infinitely many rows and t columns, in which each position is labelled by $0, 1, 2, \ldots$ from left to right and from bottom to top. The columns are called runners, and the i-runner is the column in which positions are labelled with integers congruent to i modulo t for $i = 1, 2, \ldots, t$. For a partition λ, we place a bead in each position labelled by the hook lengths of the boxes in the first column of λ in the t-abacus diagram, which is called the t-abacus of λ. A position with no bead is called a spacer.

In this paper we are concerned with 2-cores. It is well-known that if λ is a 2-core, λ has distinct parts which differ by one and smallest part 1. This implies the next proposition.

Proposition 2.3. *A partition λ is a 2-core partition if and only if $\lambda = \varnothing$ or $\lambda = (n, n-1, \dots, 1)$ for some $n \geq 1$. We have*

$$\sum_{n\geq 0} c_2(n)q^n = \sum_{n\geq 0} q^{\binom{n+1}{2}},$$

where $c_2(n)$ counts the number of 2-core partitions of n.

We shall describe the operations (see [16, p. 76]) on the 2-abacus of λ which correspond to removing rim hooks of size 2 from λ. These rim hooks are dominos.

(i) If a horizontal or vertical domino is removed, keeping the number of parts of λ fixed, then a bead $k > 2$ is reduced by 2, replaced by a bead $k - 2$. Thus a bead on a runner moves down by one to a spacer on the same runner.

(ii) If λ has at least 2 parts of size 1, removing this vertical domino corresponds to deleting the beads 1 and 2, and subtracting 2 from all other beads. This may be considered as moving the bead 2 to bead 0, eliminating the first row of beads 0 and 1, and moving the remaining beads on their runners down by one position.

(iii) Finally if the smallest part of λ is a 2, 1 is a spacer. Removing this smallest part as a horizontal domino removes the bead 2, and subtracts one from the remaining beads. So they switch runners.

Figures 3 illustrates the algorithm with $\lambda = (4, 2, 2, 1)$. Its 2-core can be easily constructed from the Young diagram or the abacus on the right side, namely $\lambda_{\text{2-core}} = (2, 1)$.

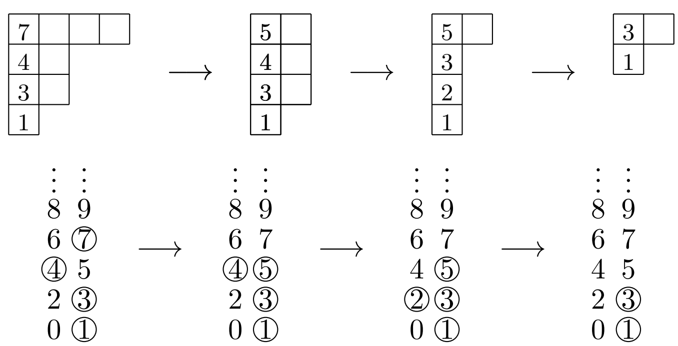

FIGURE 3. Algorithm for $(4, 2, 2, 1)$ and its 2-core

Proposition 2.4. *In the 2-abacus of λ, let s_i count the number of beads in the i-runner for $i = 1, 2$. Then $\lambda_{\text{2-core}} = (j, \dots, 2, 1)$ if and only if $s_2 = s_1 - j$ or $s_2 = s_1 + j + 1$*

Proof. First move all the beads down so that there are no spacers below beads.

If $s_1 \geq s_2$, then we eliminate the smallest s_2 beads in each runner and $s_1 - s_2 = j$ beads remain, namely $\{1, 3, \dots, 2j - 1\}$, for the 2-core and $\lambda_{\text{2-core}} = (j, \dots, 2, 1)$.

If $s_2 > s_1$, then we eliminate the smallest s_1 beads in each runner and $s_2 - s_1 = j + 1$ beads remain, namely $\{2, 4, \dots, 2j + 2\}$. Applying the final rule, the bead 2 is eliminated and the other beads switch runners and become $\{3, 5, \dots, 2j+1\}$. We then shift them down to $\{1, 3, \dots, 2j-1\}$. Again $\lambda_{\text{2-core}} = (j, \dots, 2, 1)$. □

Next, we define the 2-residue of $x = (i, j)$ by

$$Res(x) := i - j \pmod 2.$$

Figure 4 shows the 2-residues of $\lambda = (5, 4, 1)$.

0	1	0	1	0
1	0	1	0	
0				

FIGURE 4. 2-Residues of $(5, 4, 1)$

We are ready to define the 2-residue statistic $\omega(\lambda)$, which is also called the BG-rank by Berkovich and Garvan [5, eq. (1.2)].

Definition 2. *The 2-residue statistic ω (BG-rank) is*

$$\omega(\lambda) := |\{x \in Y_\lambda \mid Res(x) = 0\}| - |\{x \in Y_\lambda \mid Res(x) = 1\}|.$$

Since all rim hooks of even length are removed while performing the algorithm to get $\lambda_{\text{2-core}}$, we can easily see that

$$\omega(\lambda) = \omega(\lambda_{\text{2-core}}). \tag{11}$$

3. THE w-GENERALIZATIONS

In this section we give generalizations of the main two identities (1) and (2) which include a new parameter w. The choice of $w = 1$ in Theorems 3.1 and 3.2 gives (1) and (2).

Theorem 3.1. *We have*

$$\sum_{r,s\geq 0} \frac{q^{r^2+s^2+r+s}(wq^2;q^2)_{r+s+1}w^r}{(wq^2;q^2)_r(q^2;q^2)_r(q^2;q^2)_s(wq^2;q^2)_{s+1}} = \frac{(-q^2;q^2)_\infty}{(wq^2;q^2)_\infty}.$$

Proof. The double sum, once q^2 is replaced by q, is a confluent limit of Jackson's

$$\lim_{b\to\infty} \Psi_1(a;b;c,c';x/b,y;\lambda),$$

with the choices

$$a = c' = wq^2, \quad c = wq, \quad x = -c = -wq, \quad y = q, \quad \lambda = 1/2.$$

Because $a = c'$, Proposition 2.2 shows

$${}_1\Phi_1(aq^r;c'q^r;yq^r;1/2) = (-yq^r;q)_\infty.$$

For the factor $\lim_{b\to\infty} \Phi(aq^r, bq^r; cq^r; x/b)$ we must work a bit harder and use $a = cq$.

$$\begin{aligned}
&\lim_{b\to\infty} \Phi(aq^r, bq^r; cq^r; x/b) \\
&= \sum_{p=0}^{\infty} \frac{(aq^r;q)_p}{(q;q)_p(cq^r;q)_p} q^{\binom{p}{2}}(-1)^p(xq^r)^p \\
&= \sum_{p=0}^{\infty} \frac{(1-cq^{r+p})}{(1-cq^r)(q;q)_p} q^{\binom{p}{2}}(-1)^p(xq^r)^p && \text{(because } a = cq\text{)} \\
&= \frac{1}{(1-cq^r)} \sum_{p=0}^{\infty} \frac{q^{\binom{p}{2}}}{(q;q)_p}(-1)^p(xq^r)^p - \frac{cq^r}{(1-cq^r)} \sum_{p=0}^{\infty} \frac{q^{\binom{p}{2}}}{(q;q)_p}(-1)^p(xq^{r+1})^p \\
&= \frac{1}{1-cq^r}((xq^r;q)_\infty - cq^r(xq^{r+1};q)_\infty) \\
&= \frac{1}{1-cq^r}(-cq^{r+1};q)_\infty. && \text{(because } x = -c\text{)}
\end{aligned}$$

Putting these two pieces into Proposition 2.1, we have an r-sum which is

$$\begin{aligned}
&\sum_{r=0}^{\infty}\frac{(cq;q)_r q^{\binom{r}{2}}(-1)^r}{(q;q)_r(c;q)_r(cq;q)_r}(-c)^r q^r (cq)^r q^{3\binom{r}{2}}(-q^{r+1};q)_\infty\frac{1}{1-cq^r}(-cq^{r+1};q)_\infty\\
=&(-q;q)_\infty(-cq;q)_\infty\sum_{r=0}^{\infty}\frac{q^{2r+4\binom{r}{2}}c^{2r}}{(q;q)_r(c;q)_{r+1}}\frac{1}{(-q;q)_r(-cq;q)_r}\\
=&\frac{(-q;q)_\infty(-cq;q)_\infty}{1-c}\sum_{r=0}^{\infty}\frac{c^{2r}q^{2r^2}}{(q^2;q^2)_r(c^2q^2;q^2)_r}\\
=&\frac{(-q;q)_\infty(-cq;q)_\infty}{1-c}\frac{1}{(c^2q^2;q^2)_\infty} \qquad \text{(by (7))}\\
=&\frac{(-q;q)_\infty}{(c;q)_\infty}=\frac{(-q;q)_\infty}{(wq;q)_\infty}.
\end{aligned}$$

With q replaced by q^2, we complete the proof. □

Theorem 3.2. *We have*

$$\begin{aligned}
\sum_{r,s=0}^{\infty}\frac{w^s q^{r^2+s^2+2s}(wq^2;q^2)_{r+s}}{(wq^2;q^2)_r(q^2;q^2)_r(wq^2;q^2)_s(q^2;q^2)_s}&+\sum_{r,s=1}^{\infty}\frac{w^s q^{r^2+s^2}(wq^2;q^2)_{r+s-1}}{(wq^2;q^2)_r(q^2;q^2)_{r-1}(wq^2;q^2)_s(q^2;q^2)_{s-1}}\\
&=\frac{(-q;q^2)_\infty}{(wq^2;q^2)_\infty}.
\end{aligned}$$

Proof. The first double sum can be obtained once q is replaced by q^2 in a confluent limit of Jackson's

$$\lim_{b\to\infty}\Psi_1(a;b;c,c';x/b,y;\lambda),$$

with the choices

$$a=c=c'=wq^2,\quad x=-q,\quad y=wq^3,\quad \lambda=1/2.$$

This time we may apply both parts of Proposition 2.2 to obtain

$$\begin{aligned}
&(-q;q^2)_\infty(-wq^3;q^2)_\infty\sum_{r=0}^{\infty}\frac{q^{4r^2+2r}w^{2r}}{(q^2;q^2)_r(wq^2;q^2)_r(-q;q^2)_r(-wq^3;q^2)_r}\\
&=(-q;q^2)_\infty(-wq^3;q^2)_\infty\sum_{r=0}^{\infty}\frac{q^{(2r)^2+2r}w^{2r}}{(-q;-q)_{2r}(wq^2;-q)_{2r}}.
\end{aligned}$$

Again, the second double sum can be obtained once q is replaced by q^2 in a confluent limit of Jackson's

$$\frac{wq^2}{1-wq^2}\lim_{b\to\infty}\Psi_1(a;b;c,c';x/b,y;\lambda),$$

with the choices

$$a=c=c'=wq^4,\quad x=-q^3,\quad y=wq^3,\quad \lambda=1/2.$$

As in the first double sum, the resulting r-sum becomes

$$\begin{aligned}
&\frac{wq^2}{1-wq^2}(-q^3;q^2)_\infty(-wq^3;q^2)_\infty\sum_{r=0}^{\infty}\frac{q^{4r^2+6r}w^{2r}}{(q^2;q^2)_r(wq^4;q^2)_r(-q^3;q^2)_r(-wq^3;q^2)_r}\\
=&(-q;q^2)_\infty(-wq^3;q^2)_\infty\sum_{r=0}^{\infty}\frac{q^{(2r+1)^2+(2r+1)}w^{2r+1}}{(-q;-q)_{2r+1}(wq^2;-q)_{2r+1}}.
\end{aligned}$$

So the sum of these two double sums is

$$(-q;q^2)_\infty(-wq^3;q^2)_\infty \sum_{r=0}^{\infty} \frac{q^{r^2+r}w^r}{(-q;-q)_r(wq^2;-q)_r}$$
$$= (-q;q^2)_\infty(-wq^3;q^2)_\infty \frac{1}{(wq^2;-q)_\infty} \qquad \text{(by (7))}$$
$$= \frac{(-q;q^2)_\infty}{(wq^2;q^2)_\infty}$$

as required. □

4. Non-Negativity

The individual terms of the series in (1) and (2) are generating functions for a set of multi-partitions. Thus they have non-negative coefficients as a formal power series in q. (This will be discussed and proven in Section 5.) One may ask if the individual terms of the series in Theorem 3.1 and Theorem 3.2 have non-negative coefficients as a formal power series in q and w. In this section we verify in Theorem 4.6 that this does hold.

For Theorem 3.1 we need a definition.

Definition 3. *For nonnegative integers r and s, let*

$$g(r,s,w) := \frac{(wq;q)_{r+s+1}}{(wq;q)_r(q;q)_r(q;q)_s(wq;q)_{s+1}}.$$

Note that

$$g(r,s,1) = \frac{1}{(q;q)_r(q;q)_s} \begin{bmatrix} r+s+1 \\ s+1 \end{bmatrix}$$

has non-negative coefficients as a power series in q. However, a natural w-version of the q-binomial coefficients $(wq;q)_{r+s+1}/((q;q)_r(wq;q)_{s+1})$ is not a polynomial function and does not have non-negative coefficients. The extra factors in the denominator which depend on w will force $g(r,s,w)$ to have non-negative coefficients.

Lemma 4.1. *If $r, s \geq 1$, then*

$$g(r,s,w) = \frac{g(r,s-1,w)}{1-q^s} + wq^{s+1}\frac{g(r-1,s,w)}{1-wq^r}.$$

Proof. This Pascal-type relation follows from Definition 3. □

Proposition 4.2. *As a formal power series in q and w, $g(r,s,w)$ has non-negative coefficients.*

Proof. Because Lemma 4.1 recursively preserves non-negativity, we must only check the initial conditions. If $r = 0$, the numerator factor cancels in $g(0,s,w)$, so it has non-negative coefficients. If $r \geq 1$ and $s = 0$,

$$g(r,0,w) = \frac{1-wq^{r+1}}{1-wq}\frac{1}{(q;q)_r} = \frac{1}{(q;q)_r} + \frac{wq}{1-wq}\frac{1}{(q;q)_{r-1}}$$

has non-negative coefficients. □

For the two double sums in Theorem 3.2, we need the following definitions.

Definition 4. *For nonnegative integers r and s, let*

$$h(r,s,w) := \frac{(wq;q)_{r+s}}{(wq;q)_r(q;q)_r(q;q)_s(wq;q)_s},$$

and for positive integers r and s, let

$$hh(r,s,w) := \frac{(wq;q)_{r+s-1}}{(wq;q)_r(q;q)_{r-1}(q;q)_{s-1}(wq;q)_s}.$$

The Pascal-type relations for $h(r,s,w)$ and $hh(r,s,w)$ in the following lemmas follow easily from the Definition 4.

Lemma 4.3. *For $r,s \geq 1$,*

$$h(r,s,w) = \frac{h(r,s-1,w)}{1-q^s} + wq^s\frac{h(r-1,s,w)}{1-wq^r},$$

with the initial conditions at $r=0$ or $s=0$ are

$$h(0,t,w) = h(t,0,w) = \frac{1}{(q;q)_t} \text{ for } t \geq 0,$$

which has non-negative coefficients.

Lemma 4.4. *For $r,s \geq 2$,*

$$hh(r,s,w) = \frac{hh(r,s-1,w)}{1-q^{s-1}} + wq^s\frac{hh(r-1,s,w)}{1-wq^r}$$

with the initial conditions at $r=1$ or $s=1$ is given by

$$hh(t,1,w) = hh(1,t,w) = \frac{1}{(1-wq)(q;q)_{t-1}} \text{ for } t \geq 1,$$

which has non-negative coefficients.

The non-negativity properties of $h(r,s,w)$ and $hh(r,s,w)$ follow from Lemmas 4.3 and 4.4

Proposition 4.5. *As a formal power series in q and w, both $h(r,s,w)$ and $hh(r,s,w)$ have non-negative coefficients.*

By Propositions 4.2 and 4.5, we obtain the following non-negativity.

Theorem 4.6. *The r,s terms in Theorems 3.1 and 3.2 have non-negative coefficients as a formal power series in q and w.*

5. Kleshchev 2-Multipartitions and 2-cores

In this section, we prove in Theorem 5.7 that the generating function for a set of 2-multipartitions, denoted $\Lambda_0^{1,2}$, is given by the double sum side of Theorem 3.1. An analogous statement holds for Theorem 3.2 using $\Lambda_0^{2,2}$ and $\Lambda_1^{2,2}$, but we do not give the details here. We also collect some facts about strict partitions and 2-core partitions.

Definition 5. *For $a=1,2$, we define $\Lambda^{a,2}$ to be the set of pairs of strict partitions $\left(\lambda^{(1)},\lambda^{(2)}\right)$ satisfying*

$$\lambda_1^{(1)} \leq \ell(\lambda^{(2)}) + (2-a).$$

Such a pair of partitions in $\Lambda^{a,2}$ is called a Kleshchev 2-multipartition. We put

$$\omega(\lambda^{(1)},\lambda^{(2)}) := \omega(\lambda^{(1)}) + (-1)^a\omega(\lambda^{(2)}),$$

where the 2-residue statistic ω for $\lambda^{(1)}$ and $\lambda^{(2)}$ is defined in Definition 2.

5.1. **Strict partitions and 2-core partitions Δ_j.** For a positive integer j, we denote by Δ_j the partition with parts exactly $1, 2, \dots, j$, namely

$$\Delta_j := (j, \dots, 2, 1).$$

Also, we define

$$\Delta_0 := \varnothing.$$

We begin with a fundamental result about strict partitions and their 2-cores.

Theorem 5.1. *For an integer $j \geq 0$,*

$$\sum_{\substack{\lambda \in \mathcal{D} \\ \lambda_{2\text{-}core} = \Delta_j}} q^{|\lambda|} = \frac{q^{\binom{j+1}{2}}}{(q^2;q^2)_\infty}. \tag{12}$$

Proof. Suppose that λ is strict. In the Young diagram of λ, starting from the bottom row to top, shift one box to the left, two boxes to the left, etc. We then obtain the shifted Young diagram of λ. For example, the shifted Young diagram of the partition $(6, 5, 3, 1)$ is illustrated in Figure 5.

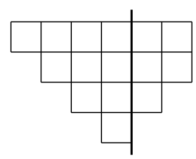

FIGURE 5. The shifted Young diagram of $(6, 5, 3, 1)$

The diagram to the right of the vertical bar in the middle is called the shifted shape of λ. This shifted shape is an ordinary partition.

In the 2-abacus of λ, let s_1 and s_2 be the numbers of beads in the 1-runner and the 2-runner, respectively. The bead numbers β_i for λ, the first column hook lengths, have the opposite parity of the parts in the shifted shape of λ which are $\lambda_i - (\ell(\lambda) - i + 1)$,

$$\beta_i = \lambda_i + (\ell(\lambda) - i) = \lambda_i - (\ell(\lambda) - i + 1) + 2(\ell(\lambda) - i) - 1.$$

So, the shifted shape has s_1 even parts (0 allowed) and s_2 odd parts. For example if $\lambda = (6, 5, 3, 1)$, then $\beta = (9, 7, 4, 1)$ and the shifted shape is $(2, 2, 1, 0)$.

Thus, λ can be decomposed into a triple $(\Delta_{\ell(\lambda)}, \mu, \nu)$, where μ and ν are partitions consisting of even parts and odd parts from the shifted shape of λ. Since

$$s_1 + s_2 = \ell(\lambda),$$

it follows that

$$\begin{aligned}\sum_{\lambda \in \mathcal{D}} q^{|\lambda|} &= \sum_{s_1, s_2 \geq 0} q^{|\Delta_{s_1+s_2}|} \sum_{\ell(\mu)=s_1} q^{|\mu|} \sum_{\ell(\nu)=s_2} q^{|\nu|} \\ &= \sum_{s_1, s_2 \geq 0} \frac{q^{\binom{s_1+s_2+1}{2}+s_2}}{(q^2;q^2)_{s_1}(q^2;q^2)_{s_2}}. \end{aligned} \tag{13}$$

By Proposition 2.4, we know that $\lambda_{\text{2-core}} = (j, \dots, 1)$ corresponds to $s_2 = s_1 - j$ or $s_2 = s_1 + j + 1$.

Case 1: $(s_1, s_2) = (s + j, s)$. Because

$$\binom{2s+j+1}{2} + s = \binom{j+1}{2} + 2s(s+j+1)$$

the generating function from (13) becomes

$$q^{\binom{j+1}{2}} \sum_{s\geq 0} \frac{q^{2s(s+j+1)}}{(q^2;q^2)_s (q^2;q^2)_{s+j}}. \tag{14}$$

Case 2: $(s_1, s_2) = (s, s + j + 1)$. Similarly the generating function in this case is

$$q^{\binom{j+1}{2}} \sum_{s\geq 0} \frac{q^{2(s+1)(s+j+1)}}{(q^2;q^2)_s (q^2;q^2)_{s+j+1}}. \tag{15}$$

Therefore,

$$\begin{aligned}
\sum_{\substack{\lambda\in\mathcal{D} \\ \lambda_{\text{2-core}}=\Delta_j}} q^{|\lambda|} &= q^{\binom{j+1}{2}} \left(\sum_{s\geq 0} \frac{q^{2s(s+j+1)}}{(q^2;q^2)_s (q^2;q^2)_{s+j}} + \sum_{s\geq 0} \frac{q^{2(s+1)(s+j+1)}}{(q^2;q^2)_s (q^2;q^2)_{s+j+1}} \right) \\
&= q^{\binom{j+1}{2}} \sum_{s\geq 0} \frac{q^{2s(s+j+1)}}{(q^2;q^2)_s (q^2;q^2)_{s+j+1}} \\
&= \frac{q^{\binom{j+1}{2}}}{(q^2;q^2)_\infty}, \qquad \text{(by (7) with } z \to q^{j+1})
\end{aligned}$$

which completes the proof. □

In the following theorem, we give the generating function for strict partitions λ with 2-residue weight $\omega(\lambda)$.

Theorem 5.2. *We have*

$$\sum_{\lambda\in\mathcal{D}} x^{\omega(\lambda)} q^{|\lambda|} = (-xq;q^4)_\infty (-q^3/x;q^4)_\infty (-q^2;q^2)_\infty.$$

Proof. Note that

$$\omega(\lambda) = \omega(\lambda_{\text{2-core}}).$$

Also,

$$\omega(\Delta_j) = (-1)^{j+1} \left\lceil \frac{j}{2} \right\rceil.$$

Thus, by Theorem 5.1

$$\begin{aligned}
\sum_{\lambda\in\mathcal{D}} x^{\omega(\lambda)} q^{|\lambda|} &= \frac{1}{(q^2;q^2)_\infty} \sum_{j\geq 0} x^{(-1)^{j+1}\lceil j/2\rceil} q^{\binom{j+1}{2}} \\
&= \frac{1}{(q^2;q^2)_\infty} \sum_{j=-\infty}^{\infty} x^j q^{j(2j-1)} \\
&= \frac{(-xq;q^4)_\infty (-q^3/x;q^4)_\infty (q^4;q^4)_\infty}{(q^2;q^2)_\infty} \qquad \text{(by (8))} \\
&= (-xq;q^4)_\infty (-q^3/x;q^4)_\infty (-q^2;q^2)_\infty,
\end{aligned}$$

as desired. □

5.2. **A finite version of Theorem 5.1.** We now consider a finite version of Theorem 5.1 by restricting the largest part. This is already discussed by Berkovich and Uncu in [6]. We give another proof using 2-cores.

Theorem 5.3. *For an integer $j \geq 0$,*

$$\sum_{\substack{\lambda\in\mathcal{D},\ \lambda_1\leq N\\ \lambda_{2\text{-}core}=\Delta_j}} q^{|\lambda|} = q^{\binom{j+1}{2}} \begin{bmatrix} N \\ \lfloor (N-j)/2\rfloor \end{bmatrix}_{q^2}. \tag{16}$$

Proof. Let λ be a strict partition such that $\lambda_{\text{2-core}} = \Delta_j$ and $\lambda_1 \leq N$. We copy the proof of Theorem 5.1 with the added restriction on the largest part of λ.

Case 1: $(s_1, s_2) = (s+j, s)$. The shifted Young diagram of λ has $s_1 = s+j$ even parts (0 allowed) and $s_2 = s$ odd parts, each part at most $N-(2s+j)$. The generating function for each parity of parts is given by a q^2-binomial coefficient corresponding to partitions inside a rectangle. Hence the generating function in this case is

$$\sum_{s\geq 0} q^{\binom{2s+j+1}{2}+s} \begin{bmatrix} \lfloor (N-j-1)/2\rfloor \\ s \end{bmatrix}_{q^2} \begin{bmatrix} \lfloor (N+j)/2\rfloor \\ s+j \end{bmatrix}_{q^2}. \tag{17}$$

Case 2: $(s_1, s_2) = (s, s+j+1)$. Similarly we obtain the following generating function

$$\sum_{s\geq 0} q^{\binom{2s+j+2}{2}+s+j+1} \begin{bmatrix} \lfloor (N-j-1)/2\rfloor \\ s \end{bmatrix}_{q^2} \begin{bmatrix} \lfloor (N+j)/2\rfloor \\ s+j+1 \end{bmatrix}_{q^2}. \tag{18}$$

By combining the two cases in (17) and (18),

$$\begin{aligned}
& q^{\binom{j+1}{2}} \sum_{s\geq 0} q^{2s(s+j+1)} \left(\begin{bmatrix} \lfloor (N+j)/2\rfloor \\ s+j \end{bmatrix}_{q^2} + q^{2(s+j+1)} \begin{bmatrix} \lfloor (N+j)/2\rfloor \\ s+j+1 \end{bmatrix}_{q^2} \right) \begin{bmatrix} \lfloor (N-j-1)/2\rfloor \\ s \end{bmatrix}_{q^2} \\
&= q^{\binom{j+1}{2}} \sum_{s\geq 0} q^{2s(s+j+1)} \begin{bmatrix} \lfloor (N+j)/2\rfloor + 1 \\ s+j+1 \end{bmatrix}_{q^2} \begin{bmatrix} \lfloor (N-j-1)/2\rfloor \\ s \end{bmatrix}_{q^2} && \text{(by (4))} \\
&= q^{\binom{j+1}{2}} \begin{bmatrix} \lfloor (N+j+1)/2\rfloor + \lfloor (N-j)/2\rfloor \\ \lfloor (N-j)/2\rfloor \end{bmatrix}_{q^2} && \text{(by (9))} \\
&= q^{\binom{j+1}{2}} \begin{bmatrix} N \\ \lfloor (N-j)/2\rfloor \end{bmatrix}_{q^2},
\end{aligned}$$

as desired. ☐

5.3. ***w*-Versions for Theorems 5.1 and 5.3.** To get w-versions for Theorems 5.1 and 5.3, we define a new statistic as follows. Recall that in the 2-abacus of λ, s_i counts the number of beads in the i-runner for $i = 1, 2$. Define

$$\beta(\lambda) := \begin{cases} s_1 + \frac{\text{largest even part in the shifted shape}}{2} & \text{if } s_1 \geq s_2, \\[2ex] s_2 + \frac{\text{largest odd part in the shifted shape }+1}{2} & \text{if } s_1 < s_2. \end{cases}$$

Theorem 5.4. *For an integer $j \geq 0$,*

$$\sum_{\substack{\lambda\in\mathcal{D}\\ \lambda_{2\text{-}core}=\Delta_j}} w^{\beta(\lambda)} q^{|\lambda|} = \frac{w^j q^{\binom{j+1}{2}}}{(wq^2;q^2)_\infty}. \tag{19}$$

Proof. For a strict partition λ, let s_i be the number of beads in the i-runner of the 2-abacus of λ for $i = 1, 2$. As seen in the proof of Theorem 5.1, λ can be decomposed into the staircase partition $\Delta_{s_1+s_2}$ and its shifted shape of s_1 even parts (0 allowed) and s_2 odd parts.

It follows from the definition of $\beta(\lambda)$ and the decomposition of λ seen in (13) that the generating function with weight $\beta(\lambda)$ is

$$\sum_{\lambda\in\mathcal{D}} w^{\beta(\lambda)} q^{|\lambda|} = \sum_{s_1\geq s_2\geq 0} \frac{w^{s_1} q^{\binom{s_1+s_2+1}{2}+s_2}}{(wq^2;q^2)_{s_1}(q^2;q^2)_{s_2}} + \sum_{s_2>s_1\geq 0} \frac{w^{s_2} q^{\binom{s_1+s_2+1}{2}+s_2}}{(q^2;q^2)_{s_1}(wq^2;q^2)_{s_2}}. \tag{20}$$

As done in the proof of Theorem 5.1, we divide into two cases.

Case 1: $(s_1, s_2) = (s + j, s)$. By (20),

$$\sum_{\substack{\lambda\in\mathcal{D}\\ s_1-s_2=j}} w^{\beta(\lambda)} q^{|\lambda|} = q^{\binom{j+1}{2}} \sum_{s\geq 0} \frac{w^{s+j} q^{2s(s+j+1)}}{(wq^2;q^2)_{s+j}(q^2;q^2)_s}. \tag{21}$$

Case 2: $(s_1, s_2) = (s, s + j + 1)$. By (20),

$$\sum_{\substack{\lambda\in\mathcal{D}\\ s_1-s_2=-j-1}} w^{\beta(\lambda)} q^{|\lambda|} = q^{\binom{j+1}{2}} \sum_{s\geq 0} \frac{w^{s+j+1} q^{2(s+1)(s+j+1)}}{(q^2;q^2)_s(wq^2;q^2)_{s+j+1}}. \tag{22}$$

Thus, by combining (21) and (22), we get

$$\begin{aligned}
\sum_{\substack{\lambda\in\mathcal{D}\\ \lambda_{2\text{-core}}=\Delta_j}} w^{\beta(\lambda)} q^{|\lambda|} &= q^{\binom{j+1}{2}} \left(\sum_{s\geq 0} \frac{w^{s+j} q^{2s(s+j+1)}}{(q^2;q^2)_s(wq^2;q^2)_{s+j}} + \sum_{s\geq 0} \frac{w^{s+j+1} q^{2(s+1)(s+j+1)}}{(q^2;q^2)_s(wq^2;q^2)_{s+j+1}} \right) \\
&= q^{\binom{j+1}{2}} \sum_{s\geq 0} \frac{w^{s+j} q^{2s(s+j+1)}}{(q^2;q^2)_s(wq^2;q^2)_{s+j+1}} \\
&= \frac{w^j q^{\binom{j+1}{2}}}{(wq^2;q^2)_\infty}. \qquad \text{(by (7) with } z \to wq^{2(j+1)})
\end{aligned}$$

□

Next, we consider the finite version of Theorem 5.4.

Theorem 5.5. *For an integer $j \geq 0$,*

$$\sum_{\substack{\lambda\in\mathcal{D},\lambda_1\leq N\\ \lambda_{2\text{-core}}=\Delta_j}} w^{\beta(\lambda)} q^{|\lambda|} = w^j q^{\binom{j+1}{2}} \sum_{t=0}^{\lfloor (N-j)/2\rfloor} w^t q^{2t} \begin{bmatrix} \lfloor (N+j-1)/2\rfloor + t \\ t \end{bmatrix}_{q^2}. \tag{23}$$

Proof. As done in the proof of Theorem 5.3, we need to get a finite version for the generating function of the shifted shape. We divide into two cases. Then the w-weighted generating function for each case follows from the proofs of Theorems 5.3 and 5.4.

Case 1: $(s_1, s_2) = (s + j, s)$. The generating function is

$$w^j q^{\binom{j+1}{2}} \sum_{s\geq 0} w^s q^{2s(s+j+1)} \begin{bmatrix} \lfloor (N-j-1)/2\rfloor \\ s \end{bmatrix}_{q^2} \sum_{i=0}^{\lfloor (N+j)/2\rfloor-(s+j)} w^i q^{2i} \begin{bmatrix} s+j+i-1 \\ i \end{bmatrix}_{q^2}, \tag{24}$$

where the coefficient of w^i in the inner sum generates even parts in the shifted shape of λ with its largest even part equal to $2i$.

Case 2: $(s_1, s_2) = (s, s+j+1)$. The generating function is

$$w^j q^{\binom{j+1}{2}} \sum_{s\geq 0} w^s q^{2s(s+j+1)} \begin{bmatrix} \lfloor (N-j-1)/2 \rfloor \\ s \end{bmatrix}_{q^2} \sum_{i=1}^{\lfloor (N+j)/2 \rfloor - (s+j)} w^i q^{2i+2(s+j)} \begin{bmatrix} s+j+i-1 \\ i-1 \end{bmatrix}_{q^2}, \tag{25}$$

where the coefficient of w^i in the inner sum generates odd parts in the shifted shape of λ with its largest odd part equal to $2i-1$.

By (3), for $i \geq 1$,

$$\begin{bmatrix} s+j+i-1 \\ i \end{bmatrix}_{q^2} + q^{2(s+j)} \begin{bmatrix} s+j+i-1 \\ i-1 \end{bmatrix}_{q^2} = \begin{bmatrix} s+j+i \\ i \end{bmatrix}_{q^2}.$$

So, by combining the two cases in (24) and (25),

$$\begin{aligned} & w^j q^{\binom{j+1}{2}} \sum_{s\geq 0} w^s q^{2s(s+j+1)} \begin{bmatrix} \lfloor (N-j-1)/2 \rfloor \\ s \end{bmatrix}_{q^2} \sum_{i=0}^{\lfloor (N+j)/2 \rfloor - (s+j)} w^i q^{2i} \begin{bmatrix} s+j+i \\ i \end{bmatrix}_{q^2} \\ &= w^j q^{\binom{j+1}{2}} \sum_{t=0}^{\lfloor (N-j)/2 \rfloor} w^t q^{2t} \sum_{s\geq 0} q^{2s(s+j)} \begin{bmatrix} \lfloor (N-j-1)/2 \rfloor \\ s \end{bmatrix}_{q^2} \begin{bmatrix} t+j \\ t-s \end{bmatrix}_{q^2} \\ &= w^j q^{\binom{j+1}{2}} \sum_{t=0}^{\lfloor (N-j)/2 \rfloor} w^t q^{2t} \begin{bmatrix} \lfloor (N+j-1)/2 \rfloor + t \\ t \end{bmatrix}_{q^2} \qquad \text{(by (9))} \end{aligned}$$

as desired. □

5.4. Kleshchev 2-multipartitions in $\Lambda_0^{1,2}$. We use the techniques of the previous subsections to find the generating function for some Kleshchev 2-multipartitions (see Theorem 5.6 and Theorem 5.7).

Theorem 5.6. *We have*

$$\sum_{\lambda \in \Lambda_0^{1,2}} q^{|\lambda|} = \sum_{j\geq 0} \left(\sum_{s\geq 0} \frac{q^{2s(s+j+1)+2\binom{j+1}{2}}}{(q^2;q^2)_s (q^2;q^2)_{s+j}} \begin{bmatrix} 2s+j+1 \\ s \end{bmatrix}_{q^2} + \sum_{s\geq 0} \frac{q^{2(s+1)(s+j+1)+2\binom{j+1}{2}}}{(q^2;q^2)_s (q^2;q^2)_{s+j+1}} \begin{bmatrix} 2s+j+2 \\ s+1 \end{bmatrix}_{q^2} \right). \tag{26}$$

Proof. Let $\lambda = (\lambda^{(1)}, \lambda^{(2)}) \in \Lambda_0^{1,2}$. By Definition 5,

$$\omega(\lambda) = \omega(\lambda^{(1)}) - \omega(\lambda^{(2)}).$$

Since $\omega(\mu) = \omega(\mu_{2\text{-core}})$ for any partition μ,

$$\omega(\lambda) = \omega(\lambda^{(1)}_{2\text{-core}}) - \omega(\lambda^{(2)}_{2\text{-core}}).$$

Hence $\omega(\lambda) = 0$ implies that

$$\lambda^{(1)}_{2\text{-core}} = \lambda^{(2)}_{2\text{-core}}.$$

We now fix the 2-core of $\lambda^{(2)}$ to be Δ_j. For $i = 1, 2$, let

$$s_i := \#\text{ beads in the } i\text{-runner in the 2-abacus for } \lambda^{(2)}.$$

Then

$$s_1 + s_2 = \ell(\lambda^{(2)})$$

with

$$\lambda_1^{(1)} \le \ell(\lambda^{(2)}) + 1 = s_1 + s_2 + 1.$$

Thus,

$$\sum_{\lambda \in \Lambda_0^{1,2}} q^{|\lambda|} = \sum_{j \ge 0} \sum_{\lambda^{(2)}_{\text{2-core}} = \Delta_j} q^{|\lambda^{(2)}|} \sum_{\lambda^{(1)}_{\text{2-core}} = \Delta_j} q^{|\lambda^{(1)}|}. \tag{27}$$

Note that $\lambda^{(2)}$ has its 2-core equal to Δ_j when $(s_1, s_2) = (s+j, s)$ or $(s, s+j+1)$ for some $s \ge 0$.

By (14) and (16),

$$\sum_{s \ge 0} \sum_{\substack{\lambda^{(2)}_{\text{2-core}} = \Delta_j \\ (s_1,s_2)=(s+j,s)}} q^{|\lambda^{(2)}|} \sum_{\lambda^{(1)}_{\text{2-core}} = \Delta_j} q^{|\lambda^{(1)}|} = \sum_{s \ge 0} \frac{q^{2s(s+j+1)+2\binom{j+1}{2}}}{(q^2;q^2)_s (q^2;q^2)_{s+j}} \begin{bmatrix} 2s+j+1 \\ s \end{bmatrix}_{q^2}, \tag{28}$$

and by (15) and (16),

$$\sum_{s \ge 0} \sum_{\substack{\lambda^{(2)}_{\text{2-core}} = \Delta_j \\ (s_1,s_2)=(s,s+j+1)}} q^{|\lambda^{(2)}|} \sum_{\lambda^{(1)}_{\text{2-core}} = \Delta_j} q^{|\lambda^{(1)}|} = \sum_{s \ge 0} \frac{q^{2(s+1)(s+j+1)+2\binom{j+1}{2}}}{(q^2;q^2)_s (q^2;q^2)_{s+j+1}} \begin{bmatrix} 2s+j+2 \\ s+1 \end{bmatrix}_{q^2}. \tag{29}$$

By plugging (28) and (29) in (27), we complete the proof. □

Finally, to obtain the left hand side of the equation in Theorem 3.1 when $w = 1$, these two double sums collapse into a single double sum.

Theorem 5.7. *We have*

$$\sum_{\lambda \in \Lambda_0^{1,2}} q^{|\lambda|} = \sum_{r,s \ge 0} \frac{q^{r^2+s^2+r+s} (q^2;q^2)_{r+s+1}}{(q^2;q^2)_r (q^2;q^2)_r (q^2;q^2)_s (q^2;q^2)_{s+1}}.$$

Proof. First we rewrite the double sum side of Theorem 5.7 as two double sums, splitting the $r \le s$ and $r > s$ cases, and then obtaining the two double sums in Theorem 5.6. □

5.5. **Kleshchev 2-multipartitions in $\Lambda_0^{2,2} \cup \Lambda_1^{2,2}$.** Analogous results for $\Lambda_0^{2,2} \cup \Lambda_1^{2,2}$ can be obtained in similar ways to the ones used for $\Lambda_0^{1,2}$. We only state the results omitting the details.

Theorem 5.8. *We have*

$$\begin{aligned}
&\sum_{\lambda \in \Lambda_0^{2,2} \cup \Lambda_1^{2,2}} q^{|\lambda|} \\
&= \sum_{j \ge 0} \left(\sum_{s \ge 0} \frac{q^{2s(s+j+1)+j^2}}{(q^2;q^2)_s (q^2;q^2)_{s+j}} \begin{bmatrix} 2s+j \\ s \end{bmatrix}_{q^2} + \sum_{s \ge 0} \frac{q^{2(s+1)(s+j+1)+j^2}}{(q^2;q^2)_s (q^2;q^2)_{s+j+1}} \begin{bmatrix} 2s+j+1 \\ s+j \end{bmatrix}_{q^2} \right) \\
&+ \sum_{j \ge 0} \left(\sum_{s \ge 0} \frac{q^{2s(s+j+1)+(j+1)^2}}{(q^2;q^2)_s (q^2;q^2)_{s+j}} \begin{bmatrix} 2s+j \\ s+j+1 \end{bmatrix}_{q^2} + \sum_{s \ge 0} \frac{q^{2(s+1)(s+j+1)+(j+1)^2}}{(q^2;q^2)_s (q^2;q^2)_{s+j+1}} \begin{bmatrix} 2s+j+1 \\ s+j+1 \end{bmatrix}_{q^2} \right).
\end{aligned}$$

To obtain the left hand side of the equation in Theorem 3.2 when $w = 1$, these four double sums collapse into two double sums.

Theorem 5.9. *We have*

$$\sum_{\lambda\in\Lambda_0^{2,2}\cup\Lambda_1^{2,2}} q^{|\lambda|} = \sum_{r,s\geq 0}\frac{q^{r^2+s^2+2s}(q^2;q^2)_{r+s}}{(q^2;q^2)_r(q^2;q^2)_r(q^2;q^2)_s(q^2;q^2)_s}+\sum_{r,s\geq 1}\frac{q^{r^2+s^2}(q^2;q^2)_{r+s-1}}{(q^2;q^2)_r(q^2;q^2)_{r-1}(q^2;q^2)_s(q^2;q^2)_{s-1}}.$$

6. Overpartitions

The right side of Theorem 3.1 is a generating function for overpartitions, once q^2 has been replaced by q. In this section we reprove Theorem 3.1 and find the combinatorial meaning of the summation parameters r and s. We give in Propositions 6.2 and 6.3 explicit combinatorial ways to compute r and s from an overpartition (θ_1, θ_2). Thus we give a combinatorial proof of Theorem 3.1. The key is a hidden auxiliary parameter p which comes from a Durfee rectangle decomposition.

This proof of Theorem 3.1 rewrites the double sum as a triple sum, and then evaluates each of the three sums.

Proof. To start the proof, we apply the q-Vandermonde theorem (10) to change the r, s term of Theorem 3.1 into a sum over p.

$$\frac{q^{\binom{r}{2}+\binom{s}{2}+r+s}(wq;q)_{r+s+1}w^r}{(wq;q)_r(q;q)_r(q;q)_s(wq;q)_{s+1}} = \sum_{p=0}^{\min(r,s+1)}\frac{q^{\binom{s+1}{2}}}{(q;q)_s}\begin{bmatrix}s+1\\p\end{bmatrix}_q\frac{q^{p^2}}{(q;q)_{r-p}}\frac{w^{p+r}}{(wq;q)_p}q^{\binom{r+1}{2}}. \tag{30}$$

The left side of Theorem 3.1 is therefore a triple sum over r, s, p. Each sum may be evaluated as an infinite product.

The s-sum is, after using the Pascal relation in (4) and replacing $s = t + p$, and $s = u + p - 1$,

$$\begin{aligned}\sum_{s\geq 0}\frac{q^{\binom{s+1}{2}}}{(q;q)_s}\begin{bmatrix}s+1\\p\end{bmatrix}_q =&\sum_{s\geq 0}\frac{q^{\binom{s+1}{2}}}{(q;q)_s}\left(q^p\begin{bmatrix}s\\p\end{bmatrix}_q+\begin{bmatrix}s\\p-1\end{bmatrix}_q\right)\\ =&\frac{1}{(q;q)_p}\sum_{t\geq 0}\frac{q^{\binom{t+p+1}{2}+p}}{(q;q)_t}+\frac{1}{(q;q)_{p-1}}\sum_{u\geq 0}\frac{q^{\binom{u+p}{2}}}{(q;q)_u}\\ =&\frac{q^{\binom{p}{2}+2p}}{(q;q)_p}(-q^{p+1};q)_\infty+\frac{q^{\binom{p}{2}}}{(q;q)_{p-1}}(-q^p;q)_\infty\\ =&\frac{q^{\binom{p}{2}}}{(q;q)_p}(-q^{p+1};q)_\infty=\frac{q^{\binom{p}{2}}}{(q^2;q^2)_p}(-q;q)_\infty.\end{aligned}\tag{31}$$

The r-sum is, upon replacing $r = t + p$,

$$\begin{aligned}\sum_{r=p}^{\infty}\frac{q^{\binom{r+1}{2}}}{(q;q)_{r-p}}w^r =&q^{\binom{p+1}{2}}\sum_{t=0}^{\infty}\frac{q^{\binom{t+1}{2}}}{(q;q)_t}w^{t+p}q^{tp}\\ =&w^pq^{\binom{p+1}{2}}(-wq^{p+1};q)_\infty.\end{aligned}\tag{32}$$

The remaining p-sum now is

$$\begin{aligned}(-q;q)_\infty\sum_{p=0}^{\infty}\frac{(-wq^{p+1};q)_\infty}{(q^2;q^2)_p(wq;q)_p}q^{2p^2}w^{2p} =&(-q;q)_\infty(-wq;q)_\infty\sum_{p=0}^{\infty}\frac{q^{2p^2}w^{2p}}{(q^2;q^2)_p(w^2q^2;q^2)_p}\\ =&(-q;q)_\infty\frac{(-wq;q)_\infty}{(w^2q^2;q^2)_\infty}=\frac{(-q;q)_\infty}{(wq;q)_\infty}\end{aligned}\tag{33}$$

as desired. □

To interpret this proof combinatorially, and find the meaning of the summation parameters r and s in Theorem 3.1, we consider the equations (33), (32) and (31) in reverse order. Note that each equation is manifestly positive so that bijections prove each one.

First consider (33), which concerns the arbitrary partition θ_2 in the overpartition (θ_1, θ_2). Here each part of θ_2 is weighted by w. The statement that

$$\frac{1}{(wq;q)_\infty} = \frac{(-wq;q)_\infty}{(w^2q^2;q^2)_\infty}$$

is considering the parts of θ_2 by even and odd multiplicity. Namely, if the multiplicity of a part is odd, we separate one copy to make a subpartition γ^o. Then the remaining parts form a subpartition γ^e, each part of which has an even multiplicity. For example, if $\theta_2 = (7,5,5,4,4,4,2,2,2,2,1)$, then $\gamma^o = (7,4,1)$ and $\gamma^e = (5,5,4,4,2,2,2,2)$.

The p-sum expansion

$$\frac{1}{(w^2q^2;q^2)_\infty} = \sum_{p=0}^{\infty} \frac{q^{2p^2} w^{2p}}{(q^2;q^2)_p (w^2q^2;q^2)_p}$$

in (33) is the Durfee rectangle decomposition of the subpartition γ^e of θ_2 consisting of its parts with even multiplicity.

Definition 6. *Let γ^e be the subpartition of θ_2 resulting from deleting one copy of each part with odd multiplicity. Let the Durfee rectangle of γ^e have size $2p \times p$.*

(1) Let Θ_1 be the partition of the column lengths in γ^e to the right of the Durfee rectangle, so Θ_1 has parts from the set $2, 4, \ldots, 2p$.

(2) Let Θ_2 be the parts of γ^e below the Durfee rectangle, so the parts of Θ_2 have even multiplicity and are from the set $1, 2, \ldots, p$.

Note that the generating function of Θ_1 is $1/(q^2;q^2)_p$ while Θ_2's is $1/(w^2q^2;q^2)_p$. In the above example, $\gamma^e = (5,5,4,4,2,2,2,2)$, so the corresponding Durfee rectangle has size 4×2, i.e., $p = 2$. Hence, $\Theta_1 = (4,4,2)$ and $\Theta_2 = (2,2,2,2)$. This is illustrated in Figure 6, where the Durfee rectangle is indicated with thicker sides.

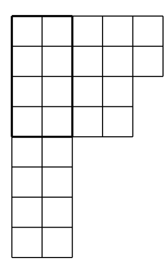

FIGURE 6. $\gamma^e = (5,5,4,4,2,2,2,2)$

Proposition 6.1. *Let (θ_1, θ_2) be an overpartition. Let γ^e be the subpartition of θ_2 resulting from deleting one copy of each part with odd multiplicity. The summation parameter p is the size of the $2p \times p$ Durfee rectangle for γ^e.*

Note that $p = 0$ when $\gamma^o = \theta_2$ and $\gamma^e = \varnothing$.

Next we consider (32) to interpret r. Note that

$$\frac{(-wq;q)_\infty}{(w^2q^2;q^2)_p} = \frac{(-wq^{p+1};q)_\infty}{(wq;q)_p}.$$

The numerator factor $(-wq^{p+1};q)_\infty$ is the generating function for the parts of γ^o which are at least $p+1$. Note that the number of parts of γ^o is the same as the number of parts of θ_2 with odd multiplicity.

Proposition 6.2. *Let (θ_1,θ_2) be an overpartition, and let p be given by Proposition 6.1. Let t be the number of part sizes of θ_2, each size at least $p+1$, which have odd multiplicity, or equivalently the number of parts of γ^o of size at least $p+1$. The summation parameter r in Theorem 3.1 corresponding to (θ_1,θ_2) is $p+t$.*

Finally we consider (31) to interpret s. The final term in (31)

$$\frac{(-q;q)_\infty}{(q^2;q^2)_p}$$

is the generating function for $\theta_1\cup\Theta_1$, which is the partition consisting of all parts of θ_1 and Θ_1. We can split each part of Θ_1 into two equal parts, to obtain a new partition $\tilde{\Theta}_1$, whose parts are from the set $1,2,\ldots,p$ and have even multiplicity. In the previous example, $\Theta_1=(4,4,2)$, so $\tilde{\Theta}_1=(2,2,2,2,1,1)$. The generating function for $\theta_1\cup\tilde{\Theta}_1$ is

$$\frac{(-q^{p+1};q)_\infty}{(q;q)_p}.$$

This means the parts $1,2,\ldots,p$ in $\theta_1\cup\tilde{\Theta}_1$ have arbitrary multiplicity, while the other part sizes, which must come from θ_1, must be distinct.

We must consider the two terms in (31) as subsets of the possible $\theta_1\cup\tilde{\Theta}_1$. In the first term of (31)

$$\frac{q^{2p}}{(q;q)_p}(-q^{p+1};q)_\infty$$

the part p has multiplicity at least 2 in $\theta_1\cup\tilde{\Theta}_1$. This occurs exactly when $2p$ is part of Θ_1.

In the second term

$$\frac{1}{(q;q)_{p-1}}(-q^{p};q)_\infty$$

the part p has multiplicity 0 or 1 in $\theta_1\cup\tilde{\Theta}_1$. This occurs exactly when $2p$ is not a part of Θ_1.

In the example above, since $p=2$ and $\Theta_1=(4,4,2)$, $\theta_1\cup\tilde{\Theta}_1$ has the part p with multiplicity at least 2. So, this example corresponds to the first case.

Proposition 6.3. *Let (θ_1,θ_2) be an overpartition, let Θ_1 be given by Definition 6, and let p be given by Proposition 6.1.*

(1) If $p=0$, then the summation parameter s in Theorem 3.1 corresponding to (θ_1,θ_2) is the number of parts of θ_1.
(2) If $p\geq 1$ and $2p$ is a part of Θ_1, then the summation parameter s in Theorem 3.1 corresponding to (θ_1,θ_2) is $t+p$, where t is the number of parts of θ_1 of size at least $p+1$.
(3) If $p\geq 1$ and $2p$ is not a part of Θ_1, then the summation parameter s in Theorem 3.1 corresponding to (θ_1,θ_2) is $u+p-1$, where u is the number of parts of θ_1 of size at least p.

Theorem 6.4. *Let (θ_1,θ_2) be an overpartition. The corresponding parameters r and s in Theorem 3.1 for (θ_1,θ_2) are given by Propositions 6.2 and 6.3.*

Theorem 6.4 identifies the (r,s)-subset of overpartitions (θ_1,θ_2) with weight

$$q^{|\theta_1|+|\theta_2|}w^{\ell(\theta_2)}.$$

This provides an explicit version of Theorem 4.6.

7. Remarks

We do not know an explicit bijection between $\Lambda_0^{1,2}(2n)$ and overpartitions of n. For $\left(\lambda^{(1)}, \lambda^{(2)}\right) \in \Lambda_0^{1,2}$ the parameters r and s are defined by the sizes of the runners in the 2-abacus of $\lambda^{(2)}$. For overpartitions (θ_1, θ_2), the r and s variables are hidden by the additional parameter p. However, the right side of Theorem 3.1 is clear from the overpartition definition, and is not clear from the definition of $\Lambda_0^{1,2}$. We could not find a version of Theorem 3.1 which included a parameter in the numerator infinite product.

Although we proved Theorems 5.1 and 5.3 using generating function manipulations, we can construct explicit bijections based on our proofs using generalized Durfee rectangles. Thus, our proofs can be considered combinatorial. We also note that Huang, Senger, Wear and Wu proved Theorem 5.3 combinatorially in [14], and their proof is essentially the same as ours. In [11], Fu and Tang proved Theorem 5.1 combinatorially using the idea of Vandervelde on balanced partitions from [17]. Recently, Dhar and Mukhopadhyay proved Theorem 5.3 combinatorially in [10] by employing the idea of Fu and Tang from [11].

The 2-core/quotient generating function in (17) and (18) has a connection to a cumulative crank generating function. In [9], we have the following theorem:

Theorem 7.1. *Let $M(m,n)$ be the number partitions of n with crank m. For any integer j,*

$$\sum_{m\geq j}\sum_{n\geq 0} M(m,n)q^n = \sum_{n\geq 0}\frac{q^{(n+1)(n+j)}}{(q)_n(q)_{n+j}}.$$

Since $M(m,n) = M(-m,n)$,

$$\begin{aligned}
&q^{\binom{j+1}{2}}\left(\sum_{s\geq 0}\frac{q^{2(s+1)(s-j)}}{(q^2;q^2)_s(q^2;q^2)_{s-j}} + \sum_{s\geq 0}\frac{q^{2(s+1)(s+j+1)}}{(q^2;q^2)_s(q^2;q^2)_{s+j+1}}\right)\\
&= q^{\binom{j+1}{2}}\left(\sum_{m\leq j}\sum_{n\geq 0} M(m,n)q^{2n} + \sum_{m\geq j+1}\sum_{n\geq 0} M(m,n)q^{2n}\right)\\
&= \frac{q^{\binom{j+1}{2}}}{(q^2;q^2)_\infty}.
\end{aligned}$$

In [7], Bessenrodt and Olsson defined 4-bar-core/quotient partitions for strict partitions to study characters in a fixed 2-block of the covering groups of the symmetric groups. Their 4-bar-core and quotient partition generating function is given by

$$(-xq;q^4)_\infty(-q^3/x;q^4)_\infty(-q^2;q^2)_\infty.$$

Thus, it follows from Theorem 5.2 that their 4-bar-quotient partition generating function matches the generating function for 2-quotient partitions.

References

[1] G. E. Andrews, The Theory of Partitions, Addison–Wesley, Reading, MA, 1976; reissued: Cambridge University Press, Cambridge, 1998.

[2] S. Ariki, *On the decomposition numbers of the Hecke algebra of $G(m,1,n)$*, J. Math. Kyoto Univ. **36** (1996), no. 4, 789–808.

[3] S. Ariki and K. Koike, *A Hecke algebra of $(\mathbb{Z}/r\mathbb{Z})\wr\mathfrak{S}_n$ and construction of irreducible representations*, Adv. Math. **106** (1994), no. 2, 216–243.

[4] S. Ariki and A. Mathas, *The number of simple modules of the Hecke algebras of type $G(r,1,n)$*, Math. Z. **233** (2000), no. 3, 601–623.
[5] A. Berkovich and F. Garvan, *The BG-rank of a partition and its applications*, Adv. Appl. Math. **40** (2008), 377–400.
[6] A. Berkovich and A. Uncu, *On partitions with fixed number of even-indexed and odd-indexed odd part*, J. Number Theory **167** (2016), 7–30.
[7] C. Bessenrodt and J. B. Olsson, *The 2-Blocks of the Covering Groups of the Symmetric Groups*, Adv. Math. **129** (1997), 261–300.
[8] M. Broué and G. Malle, *Zyklotomische Heckealgebren*, Astérisque **212** (1993), 119–189.
[9] S. Chern, Z. Li, D. Stanton, T. Xue and A. J. Yee, *The Ariki–Koike algebras and Rogers–Ramanujan type partitions*, JACO, to appear.
[10] A. Dhar and A. Mukhopadhyay, *Combinatorial proof of an identity of Berkovich and Uncu*, preprint.
[11] S. Fu and D Tang, *On certain unimodal sequences and strict partitions,* Discrete Math. **343** (2020), no. 2, 111650, 8 pp.
[12] G. Gasper and M. Rahman, Basic Hypergeometric Series. Encyclopedia of Mathematics and its Applications, 35. Cambridge University Press, Cambridge, 1990. xx+287 pp
[13] B. Hopkins, J. Sellers, A. J. Yee, *Combinatorial perspectives on partition crank and mex*, Electronic J. Combin. **29** (2022), no. 2, Paper No. 2.11.
[14] J. Huang, A. Senger, P. Wear and T. Wu, *Partition statistics equidistributed with the number of hook difference one cells*, 2014, arXiv preprint arXiv:1405.0072v1.
[15] F. H. Jackson, *Basic double hypergeometric functions. II*, Quart. J. Math. Oxford Ser. **15** (1944), 49–61.
[16] G. James and A. Kerber, The Representation Theory of the Symmetric Group, Encyclopedia of Mathematics and its Applications, vol. 16, Addison-Wesley Publishing Co., Reading, Mass., 1981,
[17] S. Vandervelde, *Balanced partitions*, Ramanujan J. **23** (2010), 297–306.
[18] E. M. Wright, *An enumerative proof of an identity of Jacobi*, J. London Math. Soc. **40** (1965) 55–57.

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