THE ROGERS-RAMANUJAN IDENTITIES AND CAUCHY'S IDENTITY

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Abstract. The Rogers-Ramanujan identities are investigated using the Cauchy identity for Schur functions.

1. Introduction

Two of Steve Milne's most noteworthy works are on the Rogers-Ramanujan identities (see [2])

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} = \frac{1}{(q^1;q^5)_\infty (q^4;q^5)_\infty}, \qquad \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q;q)_n} = \frac{1}{(q^2;q^5)_\infty (q^3;q^5)_\infty}. \tag{1}$$

With J. Lepowsky he proved (1) algebraically (see [6], [7]). The involution principle, with A. Garsia [3], gave an indirect bijection for MacMahon's combinatorial interpretation of the identities.

Stembridge [9] used symmetric function identities via Hall-Littlewood polynomials to prove and generalize the Rogers-Ramanujan identities. This was continued by Jouhet-Zeng [5] and S. Ole Warnaar [11]. A vast generalization to the Rogers-Ramanujan identities, corresponding to affine Lie algebras, was given in [4], [10]. Here the appropriate Hall-Littlewood polynomials are specializations of the Macdonald-Koornwinder polynomials.

The purpose of this note is explore a naive approach using the Cauchy identity for Schur functions. What would be required for a explicit bijective proof via the Cauchy identity is discussed in Section 2. Some related identities and a speculation are given in Sections 3 and 4, while Section 5 has two remarks.

All symmetric function facts can be found in Macdonald's book, [8].

2. A proposal for a bijection

MacMahon's combinatorial interpretation of (1) uses integer partitions.

Proposition 2.1. *The first Rogers-Ramanujan identity is equivalent to the following two sets of integer partitions being equinumerous for any n:*

(1) integer partitions of n into parts congruent to 1 or 4 modulo 5,
(2) integer partitions of n whose parts differ by at least 2.

There is no known direct bijection between these two finite sets of partitions. There is a similar statement for the second Rogers-Ramanujan identity, also with an unknown bijection.

The Cauchy identities for Schur functions $s_\lambda(x_1, \cdots, x_n, \cdots)$ provides a start for a Rogers-Ramanujan bijection. The Cauchy identity is

$$\sum_{\lambda} s_\lambda(x_1, \cdots, x_n, \cdots) s_\lambda(y_1, \cdots, y_m, \cdots) = \prod_{i,j} (1 - x_i y_j)^{-1}. \tag{2}$$

Moreover it is known that Robinson-Schensted-Knuth correspondence is a direct bijection for (2).

Choose

$$(x_1, \cdots, x_n, \cdots) = (1, q^5, q^{10}, q^{15}, \cdots), \quad (y_1, y_2) = (q^1, q^4)$$

so that the right side of (2) is the product side of the first Rogers-Ramanujan identity

$$\frac{1}{(q;q^5)_\infty (q^4;q^5)_\infty},$$

while the left side is restricted to partitions with at most two rows

$$\sum_{\lambda \text{ at most 2 rows}} s_\lambda(1, q^5, q^{10}, q^{15}, \cdots) s_\lambda(q^1, q^4). \tag{3}$$

Proposition 2.2. *The Robinson-Schensted-Knuth correspondence provides a direct bijection between*

(1) integer partitions of n into parts congruent to 1 or 4 modulo 5,
(2) pairs of column strict tableaux (P, Q) of the same shape with at most two rows, P having entries congruent to 0 modulo 5, Q having entries 1 or 4, whose sum of entries is n.

Proposition 2.2 offers some advantages and disadvantages for a bijection. On the plus side, it changes the problem to a problem on tableaux, for which there is a well developed machinery of bijections. These more refined objects may be easier to sort than integer partitions. Conversely, the simple answer required, partitions whose parts differ by at least two, may not be apparent from this detailed view.

Here are the terms in (3) corresponding to $\lambda = \varnothing, 1, 2$:

$$1 + \frac{q + q^4}{1 - q^5} + \frac{q^2 + q^5 + q^8}{(1 - q^5)(1 - q^{10})} \tag{4}$$

We want the Rogers-Ramanujan sum side

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} = 1 + \frac{q}{1-q} + \frac{q^4}{(1-q)(1-q^2)} + \cdots$$
$$= \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q^5;q^5)_n} \prod_{j=1}^{n} \sum_{p=0}^{4} q^{jp}.$$

Problem 1. *Can one choose (P, Q) whose generating function is*

$$F_n(q) = q^{n^2} \prod_{j=1}^{n} \sum_{p=0}^{4} q^{jp} / (q^5;q^5)_n?$$

Solving Problem 1 would give a Rogers-Ramanujan bijection.

For example the $n = 1$ term is

$$\frac{q^1 + q^2 + q^3 + q^4 + q^5}{1 - q^5} = \frac{q}{1 - q}.$$

If one takes the two terms in (4) from $\lambda = 1$, and the three terms in (4) from $\lambda = 2$ with 1 chosen from $1/(1 - q^{10})$, the numerator factors are $q^1 + q^4 + q^2 + q^5 + q^8$. This is missing a single term q^3 because $q^3 + q^8/(1 - q^5) = q^3/(1 - q^5)$. This term appears in $\lambda = 3$, with $P = (0, 0, 0)$ and $Q = (1, 1, 1)$.

3. Formulas

For clarity here are the explicit generating functions of the Schur functions as products. These follow from the principle specialization of Schur functions, the hook-content formula.

Proposition 3.1. *Let* $\lambda=(a+b,a)$. *Then*

$$s_\lambda(1,q^5,q^{10},q^{15},\cdots)=\frac{q^{5a}}{(q^5;q^5)_a(q^5;q^5)_b(q^{5(b+2)};q^5)_a},$$
$$s_\lambda(q^1,q^4)=q^{5a+b}\sum_{k=0}^{b}q^{3k},$$

There is weighted version using two new parameters x and y.

Theorem 3.2. *Choosing* $y_1=xq^1$, $y_2=yq^4$,

$$\frac{1}{(xq;q^5)_\infty(yq^4;q^5)_\infty}=\sum_{a,b\geq 0}\frac{q^{5a}}{(q^5;q^5)_a(q^5;q^5)_b(q^{5(b+2)};q^5)_a}x^ay^aq^{5a+b}\sum_{k=0}^{b}x^{b-k}y^kq^{3k}$$

Theorem 3.2 independently follows from the finite identity

$$\sum_{a=0}^{M}\begin{bmatrix}N\\a\end{bmatrix}_q(q^a-q^{N-a})=\frac{(q;q)_N}{(q;q)_M(q;q)_{N-M-1}}$$

for $0\leq M\leq N-1$.

Finally, a simple subclass of (P,Q) has a product formula. A proof of a more general result is given in Theorem 4.1.

Proposition 3.3.

$$\sum_{\lambda \text{ at most 1 row}}s_\lambda(1,q^5,q^{10},q^{15},\cdots)s_\lambda(q^1,q^4)=\frac{1}{1-q^3}\left(\frac{1}{(q^1;q^5)_\infty}-\frac{q^3}{(q^4;q^5)_\infty}\right)\tag{5}$$

4. Rogers-Ramanujan mod $2k+3$

The same steps as in §2 can be done for higher moduli $2k+3$, the integer partitions whose parts avoid $\pm i$ and 0 mod $2k+3$, $1\leq i\leq 2k+2$. Set

$$\begin{aligned}(x_1,\cdots,x_n,\cdots)&=(1,q^{2k+3},q^{2(2k+3)},q^{3(2k+3)},\cdots)\\(y_1,\cdots,y_{2k})&=(q^1,\cdots,q^{2k+2}),\text{ with } q^i \text{ and } q^{2k+3-i}\text{ deleted}\end{aligned}\tag{6}$$

so that

$$\sum_{\lambda\text{ at most 2k rows}}s_\lambda(x_1,\cdots,x_n,\cdots)s_\lambda(y_1,\cdots,y_{2k})=\prod_{\substack{j=1\\ j\not\equiv\pm i,0 \mod 2k+3}}^{\infty}(1-q^j)^{-1}.$$

The product side of the Rogers-Ramanujan identities (1) are the $k=1$ and $i=2,1$ special cases.

There is always a version of the subclass formula (5) as a sum of infinite products.

Theorem 4.1. *Let* $k\geq 1$, $1\leq i\leq 2k+2$ *and*

$$(y_1,\cdots,y_{2k})=(q^1,\cdots,q^{2k+2}),\ \textit{with } q^i \textit{ and } q^{2k+3-i}\textit{ deleted.}$$

Then

$$\sum_{\lambda \text{ at most 1 row}} s_\lambda(1, q^{2k+3}, q^{2(2k+3)}, q^{3(2k+3)}, \cdots) s_\lambda(y_1, \cdots, y_{2k})$$
$$= \sum_{\substack{p=1\\ p\neq i, 2k+3-i}}^{2k+2} \frac{A_p}{(q^p; q^{2k+3})_\infty}, \quad \textit{where} \quad A_p = \prod_{\substack{j=1\\ j\neq p,i,2k+3-i}}^{2k+2} \frac{1}{1-q^{-p+j}}. \tag{7}$$

Proof. If $\lambda = N$ has a single part we have

$$s_\lambda(1, q^{2k+3}, q^{2(2k+3)}, q^{3(2k+3)}, \cdots) = \frac{1}{(q^{2k+3}; q^{2k+3})_N},$$
$$s_\lambda(y_1, \cdots, y_{2k}) = \text{ the coefficient of } t^N \text{ in } \prod_{\substack{j=1\\ j\neq i,2k+3-i}}^{2k+2} (1-tq^j)^{-1}. \tag{8}$$

By partial fractions on t we see that the A_p satisfy

$$\prod_{\substack{j=1\\ j\neq i,2k+3-i}}^{2k+2} (1-tq^j)^{-1} = \sum_{\substack{p=1\\ p\neq i,2k+3-i}}^{2k+2} A_p(1-tq^p)^{-1},$$

so that

$$s_\lambda(y_1, \cdots, y_{2k}) = \sum_{\substack{p=1\\ p\neq i,2k+3-i}}^{2k+2} A_p q^{pN}.$$

We then use

$$\sum_{N=0}^{\infty} \frac{q^{pN}}{(q^{2k+3}; q^{2k+3})_N} = \frac{1}{(q^p; q^{2k+3})_\infty}.$$

to complete the proof. ∎

Note that Theorem 4.1 for $k = 1$ and $i = 2$ is (5).

Speculation 1. *The* 1 *row and at most* $2k$ *rows cases are sums of products in the Rogers-Ramanujan infinite product. Perhaps this works for any number of rows* $R \leq 2k$. *Is*

$$\sum_{\lambda \text{ at most } R \text{ rows}} s_\lambda(x_1, \cdots, x_n, \cdots) s_\lambda(y_1, \cdots, y_{2k})$$

a sum of a product of R *infinite products, each of the form,*

$$1/(q^j; q^{2k+3})_\infty, \quad j \not\equiv \pm i, 0 \mod 2k+3,$$

with coefficients which are rational functions in q*?*

Note that Speculation 1 holds for $R = 1$ and $R = 2k$.

5. Other symmetric function Cauchy identities

Michael Schlosser has pointed out that the dual Cauchy identity

$$\sum_\lambda s_\lambda(x_1, \cdots, x_n) s_{\lambda'}(y_1, \cdots, y_m) = \prod_{i=1}^{n}\prod_{j=1}^{m}(1 + x_i y_j) \tag{9}$$

can be similarly used with $m = 2$ and

$$(x_1, x_2, \cdots, x_n) = (1, q^3, \cdots, q^{3(n-1)}), \quad (y_1, y_2) = (-q, -q^2)$$

to obtain the Borwein product $(q^1;q^3)_n(q^2,q^3)_n$. Again column strict tableaux could be used to approach that problem, see [1].

There are other Cauchy identities. If x and y are arbitrary sets of variables, the Macdonald polynomials satisfy

$$\sum_{\lambda} P_\lambda(x;q,t)Q_\lambda(y;q,t) = \prod_{i,j} \frac{(tx_iy_j;q)_\infty}{(x_iy_j;q)_\infty}. \tag{10}$$

Special cases of this identity, restricted by rows, have been extensively used by Rains and S. Ole Warnaar [10].

References

[1] G. Andrews. On a conjecture of Peter Borwein. Symbolic computation in combinatorics Δ_1 (Ithaca, NY, 1993) J. Symbolic Comput. 20 (1995), no. 5-6, 487-501.

[2] G. Andrews. The Theory of Partitions. Cambridge University Press, Cambridge, 1998. xvi+255 pp.

[3] A. Garsia and S. Milne. A Rogers-Ramanujan bijection. J. Combin. Theory Ser. A **31** (1981), no. 3, 289-339.

[4] M. Griffin, K. Ono, and S. Ole Warnaar. A framework of Rogers-Ramanujan identities and their arithmetic properties. Duke Math. J. **165** (2016), no. 8, 1475-1527.

[5] F. Jouhet and J. Zeng. New identities for Hall-Littlewood polynomials and applications. Ramanujan J. **10** (2005), no. 1, 89-112.

[6] J. Lepowsky and S. Milne. Lie algebraic approaches to classical partition identities. Adv. in Math. **29** (1978), no. 1, 15-59.

[7] J. Lepowsky and S. Milne. Lie algebras and classical partition identities. Proc. Nat. Acad. Sci. U.S.A. **75** (1978), no. 2, 578-579.

[8] I. G. Macdonald, Symmetric functions and Hall polynomials. Second edition. With contribution by A. V. Zelevinsky and a foreword by Richard Stanley. The Clarendon Press, Oxford University Press, New York, 2015. xii+475 pp.

[9] J. Stembridge. Hall-Littlewood functions, plane partitions, and the Rogers-Ramanujan identities. Trans. Amer. Math. Soc. **319** (1990), no. 2, p. 469-498.

[10] E. Rains and S. Ole Warnaar. Bounded Littlewood identities Mem. Amer. Math. Soc. **270** (2021), no. 1317, vii+115 pp.

[11] S. Ole Warnaar. Hall-Littlewood functions and the A_2 Rogers-Ramanujan identities. Adv. Math. **200** (2006), no. 2, 40-434.

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