Recent results in enumeration

Univ. of Minnesota seminar

October 6, 2017

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(joint work with J. Fulman, R. Guralnick, M. Ismail,
J. Kim, J. Lewis, K. O'Hara,
E. Rains, V. Reiner)

1. integer partitions - Rogers-Ramanujan identities
   (K. O'Hara)

2. enumeration over finite fields
   (J. Fulman, R. Guralnick, J. Lewis, V. Reiner)

3. integrals and posets
   (J. Kim)

4. basic hypergeometric series and orthogonal polynomials
   (M. Ismail, E. Rains)

I. Rogers-Ramanujan identities

$$1 + \frac{q^2}{1-q} + \frac{q^6}{(1-q)(1-q^2)} + \frac{q^{12}}{(1-q)(1-q^2)(1-q^3)} + \cdots$$

$$= \frac{1}{(1-q^2)(1-q^3)(1-q^7)(1-q^8)\cdots}$$

OR

$$\sum_{m=0}^{\infty} \frac{q^{m^2+m}}{(1-q)\cdots(1-q^m)} = \prod_{k \equiv 2,3 \bmod 5} (1-q^k)^{-1}.$$

History: Rogers 1894

Ramanujan 1913 (no proof)

Schur 1917

Baxter 1980 (statistical mechanics)

Proofs: q-series: Rogers, Ramanujan, Watson, Andrews

Bressoud, Sylvester

rep theory: Lepowsky-Wilson

symm functions: Stembridge, Warnaar

"semi"-combinatorial: Garsia-Milne

Generalizations to all moduli: Andrews-Gordon

Bressoud

Theorem (MacMahon)

partitions of n into parts $\equiv 2,3 \bmod 5$

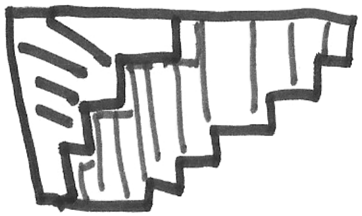
= # partitions of n into parts with difference ≥ 2 and no 1's.

EX $n=11$

83	11
722	9 2
3332	8 3
32222	7 4

KEY IDEA

double staircase



m parts λ^* left justified

q^{m^2+m} $\leq m$

Suppose that a part of size 3 is marked

$$\frac{1}{(1-q^2)(1-\underline{tq^3})(1-q^7)(1-q^8)\cdots} =$$

$$1 + q^2\frac{(1+tq+q^2)}{\underline{1-tq^3}} + \frac{q^6}{(1-q^2)}\frac{[3]_q}{\underline{1-tq^3}} +$$

$$\sum_{k=3}^{\infty} \frac{q^{k(k+1)}}{(1-q)(1-q^2)(1-\underline{tq^3})(1-q^4)\cdots(1-q^k)}$$

(O'Hara–S. 2014)

<u>Theorem</u> # partitions of n into parts $\equiv 2,3 \bmod 5$, with exactly k 3's

= # partitions λ of n with difference ≥ 2, no 1's,

if λ has ≥ 3 parts, λ^* has k 3's

if λ has 2 parts, λ^* has $3k$, $3k+1$, or $3k+2$ 1's

if λ has 1 part, then n [redacted] $= 3k+2, 3k+4, 3k$.

<u>EX</u> $n = 11$

$k=0$ 722 $\leftrightarrow$ 74 $\quad (\lambda^* = (32)^t = 221)$

$k=1$ 83 $\leftrightarrow$ 83 $\quad (\lambda^* = (41)^t = 2111)$

32222 $\leftrightarrow$ 92 $\quad (\lambda^* = (5)^t = 11111)$

$k=3$ 3332 $\leftrightarrow$ 11 $\quad (11 = 3 \cdot 3 + 2)$

ANOTHER ONE

$$1 + q^2 \frac{1+tq}{\underline{1-q^2}} + q^6 \frac{(q+q^2+t^2)}{(1-q^2)\underline{(1-tq^3)}}$$

$$+ \sum_{k=3}^{\infty} \frac{q^{k(k+1)}}{(1-q)(1-q^2)\underline{(1-tq^3)}(1-q^4)\cdots(1-q^k)}$$

$$= \frac{1}{(1-q^2)\underline{(1-tq^3)}(1-q^7)(1-q^8)\cdots}$$

Also

= # partitions λ of n with difference ≥ 2, no 1's

if λ has ≥ 3 parts, λ^* has k 3's

if λ has 2 parts, λ^* has $3k+1$, $3k+2$, or $3k-6$ 1's

if λ has 1 part, then $n = \text{ODD}$, $n \geq 3$, $k=1$.

Theorem 6. *A t, w, v, x-refinement of the second Rogers-Ramanujan identity for part sizes* 2, 3, 7 *and* 8 *is*

$$
\begin{aligned}
&1+q^2\frac{(t+wq)}{1-tq^2}+q^6\frac{(w^2+vq+xq^2)}{(1-tq^2)(1-wq^3)}+q^{12}\frac{(1+q+v^2q^2+xvq^3+x^2q^4+q^5+q^6)}{(1-tq^2)(1-wq^3)(1-vq^7)}\\
&+q^{20}\frac{(x+xq+q^2+q^3+(1+x^3)q^4+(1+x)q^5+(1+x)q^6+q^7+q^8+q^9+q^{10})}{(1-tq^2)(1-wq^3)(1-xq^8)(1-vq^7)}\\
&+q^{30}\frac{(1+q+q^2+q^3+q^4+q^5+q^6)(1+q^4)}{(1-tq^2)(1-wq^3)(1-xq^8)(1-q^5)(1-vq^7)}\\
&+q^{42}\frac{(1+q+q^2+q^3+q^4+q^5+q^6)(1+q^4)}{(1-tq^2)(1-wq^3)(1-xq^8)(1-q^5)(1-q^6)(1-vq^7)}\\
&+q^{56}\frac{(1+q^4)}{(1-q)(1-tq^2)(1-wq^3)(1-xq^8)(1-q^5)(1-q^6)(1-vq^7)}+\\
&\sum_{m=8}^{\infty}\frac{q^{m(m+1)}}{(1-q)(1-tq^2)(1-wq^3)(1-q^4)(1-q^5)(1-q^6)(1-vq^7)(1-xq^8)\cdots(1-q^m}\\
&=\frac{1}{(1-tq^2)(1-wq^3)(1-vq^7)(1-xq^8)(1-q^{12})(1-q^{13})\cdots}.
\end{aligned}
$$

Theorem 7. *A t, w, v, x-refinement of the first Rogers-Ramanujan identity for part sizes* 1, 4, 6 *and* 9 *is*

$$
\begin{aligned}
&1 + q\frac{t}{1-tq} + q^4\frac{(w+vq^2)}{(1-tq)(1-wq^4)} + q^9\frac{(x+q^2+v^2q^3+q^5)}{(1-tq)(1-wq^4)(1-vq^6)} \\
&+ q^{16}\frac{(1+x^2q^2+q^3+xq^4+q^5+q^6+xq^7+q^8+q^{10})}{(1-tq)(1-wq^4)(1-vq^6)(1-xq^9)} \\
&+ q^{25}\frac{(1+q^2+q^3+q^4+q^5+q^6+q^7+q^8+q^{10})}{(1-tq)(1-wq^4)(1-vq^6)(1-xq^9)(1-q^5)} \\
&+ q^{36}\frac{(1+q^2+q^3+q^4+q^5+q^6+q^7+q^8+q^{10})}{(1-tq)(1-wq^4)(1-vq^6)(1-xq^9)(1-q^5)(1-q^6)} \\
&+ q^{49}\frac{(1+q^2+q^3+q^4+q^5+q^6+q^7+q^8+q^{10})}{(1-tq)(1-wq^4)(1-vq^6)(1-xq^9)(1-q^5)(1-q^6)(1-q^7)} \\
&+ q^{64}\frac{(1+q^2+q^3+q^4+q^5+q^6+q^7+q^8+q^{10})}{(1-tq)(1-wq^4)(1-vq^6)(1-xq^9)(1-q^5)(1-q^6)(1-q^7)(1-q^8)} \\
&+ \sum_{m=9}^{\infty}\frac{q^{m^2}}{(1-tq^1)(1-q^2)(1-q^3)(1-wq^4)(1-q^5)(1-vq^6)(1-q^7)(1-q^8)(1-xq^9)\cdots} \\
&= \frac{1}{(1-tq^1)(1-wq^4)(1-vq^6)(1-xq^9)(1-q^{11})(1-q^{14})\cdots}.
\end{aligned}
$$

Rational function identity

$$\frac{1}{1-q^3}\left(1+\frac{q^2(1+tq+q^2)}{1-tq^3}+\frac{q^6\,(1+q+q^2)}{(1-q^2)(1-tq^3)}\right)$$

$$=\frac{1}{1-tq^3}\left(1+\frac{q^2}{1-q}+\frac{q^6}{(1-q)(1-q^2)}\right)$$

Two variable version

$$\frac{1}{(1-q^2)(1-q^3)}\left(1+\frac{q^2(t+wq)}{1-tq^2}+\frac{q^6(w^2+q+q^2)}{(1-wq^3)(1-tq^2)}\right)$$

$$=$$

$$\frac{1}{(1-tq^2)(1-wq^3)}\left(1+\frac{q^2}{1-q}+\frac{q^6}{(1-q)(1-q^2)}\right)$$

KNOWN: **Positive** expansions for

$1, 4, 6, 9 \quad (1, 4 \bmod 5)$

$2, 3, 7, 8 \quad (2, 3 \bmod 5)$

PROBLEM RR-1 Find an algorithm which produces positive numerator polynomials for any subset S.

SPECULATION RR-2

① Is there a commutative algebra explanation for these rational function identities?

② Is there a polytope decomposition with enhanced Ehrhart functions here?

Non-uniqueness of decomposition $\approx$ Non-uniqueness of identity

7. RED HERRING ?

q- Fibonacci numbers

$$F_n(q) = \sum_{\lambda} q^{\|\lambda\|}$$

λ diff 2

largest part of $\lambda \leq n$

$$F_n(q) = F_{n-1}(q) + q^n F_{n-2}(q)$$

$F_n(1)$ golden ratio, $\sqrt{5}$, powers of 5

Is this 5 the 5 of R-R ?

II. S_n enumeration $\to$ $GL_n(\mathbb{F}_q)$ enumeration

Three questions

(A) $(12\cdots n) = t_1 t_2 \cdots t_{n-1}$ transpositions

n-cycle

(B) $Z_n(\pi) = \#$ fixed pts. of π

$$\lim_{n\to\infty} Z_n = ?$$

(C) $I_n = \#$ involutions in S_n

Asymptotics + Gen. Fct for I_n known

Factorizations

THM (Hurwitz 1891) # factorizations of an n-cycle in S_n into $n-1$ transpositions is

$$n^{n-2}.$$

THM (Deligne) W finite Coxeter group

c Coxeter element

factorizations of c as a minimal product of N reflections is

$$\frac{N!\, h^N}{|W|} \qquad h = \text{order of } c$$

W-Catalan numbers

<u>DEFN</u> W real reflection group, Coxeter system (W, S), Coxeter element c.

$$NC(W) = [e, c]$$

all possible partial products of reflections to get to c

<u>THM</u> (Bessis 2003)

$$\mathrm{Cat}(W) = |NC(W)| = \prod_{i=1}^{n} \frac{h + d_i}{d_i}$$

$d_1, \ldots, d_n$ degrees of the basic invariants

q-version of PROBLEM

n-cycle $\longrightarrow$ Singer cycle $c \in GL_n(\mathbb{F}_q)$

transposition $\longrightarrow$ reflection

Singer cycle c

Let $c \in \mathbb{F}_{q^n}^\times$ generate. Multiplication by c in $\mathbb{F}_{q^n} \cong \mathbb{F}_q^n$ gives a linear transformation

FACT c acts transitively on 1-dimensional subspaces

Reflection t

t fixes a hyperplane $\begin{cases} \text{transvections} & \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \\ \text{semi-simple} & \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \end{cases}$

<u>THM</u> (LRS 2014) # factorizations of the Singer cycle c into n reflections is

$$(q^n-1)^{n-1}.$$

<u>THM</u> (LRS 2014) # factorizations of the Singer cycle $c = t_1 t_2 \cdots t_\ell$ with exactly m transvections ($\dim(t_i)=1$) is, with $(\det(t_1), \ldots, \det(t_\ell))$ fixed,

$$[n]_q^{\ell-1} \sum_{i=0}^{\min(m,\ell-n)} (-1)^i \binom{m}{i} \begin{bmatrix} \ell-i-1 \\ n-1 \end{bmatrix}_q$$

($\ell=n$ independent of det's)

THM (LRS 2014) The # of factorizations of the Singer cycle $c \in GL_n(\mathbb{F}_q)$ into ℓ reflections is

$$[n]_q^{\ell-1} \sum_{k=0}^{\ell-n} (-1)^k (q-1)^{\ell-k-1} \binom{\ell}{k} \begin{bmatrix} \ell-k-1 \\ n-1 \end{bmatrix}_q .$$

<u>THM</u> (Jackson 1988) The generating function for factorizations of an n-cycle into ℓ transpositions is

$$n^{n-2}\, x^{n-1} \prod_{k=0}^{n-1} \left(1 - nx\left(\tfrac{n-1}{2} - k\right)\right)^{-1}.$$

<u>THM</u> (LRS, 2014) The generating function for factorizations of the Singer cycle c into ℓ reflections is

$$(q^n-1)^{n-1}\, x^n \left(1 + x[n]_q\right)^{-1} \prod_{k=0}^{n-1} \left(1 + [n]_q x \left(1 - q^k - q^{k+1}\right)\right)^{-1}$$

PROBLEM SINGER-1 Is there an apriori reason for this simple rational function generating function?

The S_n-case is closely related to geometry Lyashko-Looijenga morphism (T. Douvropoulos thesis)

(B) Limits of fixed pt random variables

<u>Theorem</u> (Diaconis-Shahshahani 1994)

For $\pi \in S_n$, let $Z_n(\pi)$ = # fixed pts. of π

$$\lim_{n\to\infty} Z_n = P = \text{Poisson parameter } a=1$$

Recall Poisson

$$\text{Prob}(P=x) = e^{-a}\frac{a^x}{x!} \qquad x=0,1,2,\ldots$$

The orthogonal polynomials for this measure are Charlier $C_n(x,a)$.

q-world $\quad k!_q \leftarrow k!$

$$\sum_{k=0}^{\infty} \frac{a^k}{k!_q} \leftarrow e^a \longrightarrow \sum_{j=0}^{\infty} \frac{a^j q^{\binom{j}{2}}}{j!_q}.$$

$$\shortparallel$$

$$\prod_{j=0}^{\infty} \left(1-(1-q)q^j a\right)^{-1} = \left((1-q)a;q\right)_{\infty}^{-1}.$$

INFINITE PRODUCT

q-Charlier polynomials

$C_n(x;a;q)$ $\qquad \dfrac{a^x q^{\binom{x}{2}}}{(1-q)\cdots(1-q^x)}$ $\qquad x=0,1,2,\ldots$

Thm (Fulman–S. 2015)

For $g \in GL_n(\mathbb{F}_q)$, $Z_n(g) = $ # fixed vectors of g

$$\lim_{n \to \infty} Z_n = q\text{-Poisson from}$$

Al-Salam Carlitz polynomials

$U_n^{(a)}(x;p)$ measure $\dfrac{(ap;p)_\infty \, p^{k^2} \, a^k}{(p;p)_k (ap;p)_k}$

$p = 1/q$, $a = 1$

Moments

permutations $E(Z_n^j) = \sum_{k=1}^{j} S(j,k) = B_j$

if $n \geq j$

general linear group $E(Z_n^j) = \sum_{k=0}^{j} \begin{bmatrix} j \\ k \end{bmatrix}_q$

if $n \geq j$

rescaling $\cong \sum_{k=0}^{j} S_q(j,k)\, q^k$

FP problem I   Arratia–Tavare 1992

"The cycle structure of random permutations" have a general result for Poisson RV. Find the analogue for $GL_n(\mathbb{F}_q)$, $SO_n(q)$, etc. using the q-cycle index. Make thes above results clear, with different q-Poisson.

(c) $I_n = \#$ involutions $\pi \in S_n$.

KNOWN: $$\sum_{n=0}^{\infty} I_n \frac{x^n}{n!} = e^{x + x^2/2}$$

$$\frac{I_n}{n!} = \frac{e^{-1/4}\, n^{-n/2}}{2\sqrt{\pi n}}\, e^{n/2 + \sqrt{n}} \left(1 + O\left(\frac{1}{n^{1/5}}\right)\right)$$

? from Pasechnik: # $GL_n(\mathbb{F}_q)$ involutions

Robinson (2015): n even, close to $2^{n^2/2}$.

$q=2$: $\frac{3}{4} \leq \lim_{n\to\infty} \frac{\#\text{invol}}{2^{n^2/2}} \leq 2+\epsilon$

<u>Theorem</u> (Fulman - Guralnick - S. 2016)

If q is even,

$$\lim_{\substack{n\to\infty \\ n \text{ even}}} \frac{\mathrm{inv}_{GL}(n,q)}{q^{n^2/2}} = \frac{1}{2}\left[\left(-\frac{1}{\sqrt{q}};\frac{1}{q}\right)_\infty + \left(\frac{1}{\sqrt{q}};\frac{1}{q}\right)_\infty\right]$$

$$= \prod_{i=1}^{\infty} \frac{(1+q^{5-8i})(1+q^{3-8i})(1-q^{-8i})}{(1-q^{-2i})}$$

≈ 1.6793 if $q=2$.

Also generating functions are infinite products

$$\sum_{n=0}^{\infty} \frac{u^n q^{n/2} q^{\binom{n}{2}}}{|GL_n(\mathbb{F}_q)|} \, inv_{GL}(n,q) =$$

$$\frac{1}{1-u^2} \frac{\left(-\frac{u}{\sqrt{q}} ; \frac{1}{q}\right)_\infty^2}{\left(u^2/q ; 1/q\right)_\infty} .$$

Three other results, parity n, q.

Results for

$U_n(q)$

$Sp_{2n}(q)$ asymptotics + generating function

$O^+_{2n}(q)$

$O^-_{2n}(q)$

PROBLEM INV-I Show these results are immediate from a q-cycle index generating function.

$$S_n : \quad \exp\left(\sum_{i=1}^{\infty} \frac{t_i}{i} x^i\right)$$

Related work of Taylor-Vinroot 2017.

3 Posets

P poset on $\{x_1, x_2, \ldots, x_n\}$ (labelled)

<u>DEFN</u> The order polytope of P

$$\mathcal{O}(P) = \left\{ (x_1, \ldots, x_n) : 0 \le x_i \le 1, \ x_i \le x_j \text{ if } x_i \le_P x_j \right\}$$

<u>FACT</u> $\displaystyle \mathrm{Vol}(\mathcal{O}(P)) = \frac{\#\text{linear extensions of } P}{n!}$

$$\parallel$$

$$\int_{\mathcal{O}(P)} 1 \, dx_1 \, dx_2 \cdots dx_n$$

q-version (Kim-S. 2016)

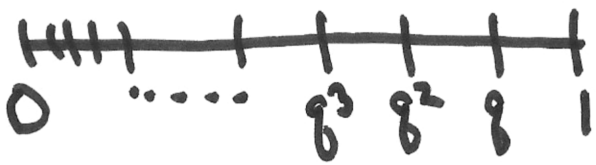
<u>Theorem</u> $$\mathrm{Vol}_q(\mathcal{O}(P)) = \sum_{\pi \in \mathcal{L}(P)} q^{\mathrm{maj}(\pi)} \Big/ n!_q .$$

$$\parallel$$

$$\int_{\mathcal{O}(P)} 1 \, d_q x_1 \, d_q x_2 \cdots d_q x_n$$

<u>DEFN</u> $$\int_a^b f(x)\, d_q x = (1-q) \sum_{k=0}^{\infty} \left[f(bq^k)\, bq^k - f(aq^k)\, aq^k \right]$$

$a=0 \quad b=1$



Beta function $\displaystyle\int_0^1 x^n (1-x)^m \, dx = \frac{n!\, m!}{(n+m+1)!}$

q-beta function

$$\int_0^1 x^n (xq;q)_m \, d_q x = \frac{n!_q \, m!_q}{(n+m+1)!_q}$$

chain with $n+m+1$ points

$$(xq;q)_m = (1-xq)(1-xq^2)\cdots(1-xq^m)$$

Selberg's integral 1944

$$\int_0^1 \int_0^1 \cdots \int_0^1 \prod_{i=1}^{n} x_i^{\alpha-1} (1-x_i)^{\beta-1} \prod_{1 \le i < j \le n} |x_i - x_j|^{2\gamma} \, dx_1 \cdots dx_n$$

$$= \prod_{j=1}^{n} \frac{\Gamma(\alpha+(j-1)\gamma)\,\Gamma(\beta+(j-1)\gamma)\,\Gamma(1+j\gamma)}{\Gamma(\alpha+\beta+(n+j-2)\gamma)\,\Gamma(1+\gamma)}$$

Diaconis, Stanley

Askey-Selberg integral 1980

$$\int_0^1 \cdots \int_0^1 \prod_{i=1}^{n} x_i^{\alpha-1} (q x_i; q)_{\beta-1} \prod_{1 \le i < j \le n} x_j^{2m-1} \left(q^{1-m} x_i/x_j ; q\right)_{2m-1}$$

$$\Delta(x_1, \ldots, x_n)\, d_q x_1 \cdots d_q x_n$$

$$= q^{\alpha m \binom{n}{2} + 2m^2 \binom{n}{3}} \quad \times$$

$$\prod_{j=1}^{n} \frac{\Gamma_q(\alpha + (j-1)m)\, \Gamma_q(\beta + (j-1)m)\, \Gamma_q(1 + jm)}{\Gamma_q(\alpha + \beta + (n+j-2)m)\, \Gamma_q(1+m)}$$

Theorem There is an explicit poset $P(\alpha,\beta,m,n)$ for α,β non-negative integers such that

$\mathrm{Vol}_q(P(\alpha,\beta,m,n))$ is given by the Askey-Selberg integral.

Poset Prob I Combinatorially find the predicted # linear ext. (q-case!)

NOTE: There is a q-Ehrhart function which is a polynomial in $[n]_q$ whose leading coefficient is

$$\mathrm{Vol}_q(\mathcal{O}(P)).$$

Another q-beta function

Askey-Wilson integral

$$\frac{(q;q)_\infty}{2\pi}\int_0^\pi \frac{(e^{2i\theta}, e^{-2i\theta};q)_\infty \, d\theta}{(ae^{i\theta}, ae^{-i\theta}, be^{i\theta}, be^{-i\theta}, ce^{i\theta}, ce^{-i\theta}, de^{i\theta}, de^{-i\theta};q)_\infty}$$

$$= \frac{(abcd;q)_\infty}{(ab, ac, ad, bc, bd, cd;q)_\infty}.$$

POSET Prob II Is there a Vol_q

$$\int_0^1 \cdots \int_0^1 1_q \, dx_1 \ldots dx_n \quad \text{OR}$$

$$\int_{\mathcal{O}(P)} 1_q \, dx_1 \ldots dx_n .$$

4 Orthogonal polynomials

$$\int_{-\infty}^{\infty} p_n(x)\, p_m(x)\, d\mu(x) = 0 \qquad n \neq m \qquad \deg p_n = n$$

<u>FACT</u> $\{p_n(x)\}_{n=0}^{\infty}$ OP for a positive measure iff

$$p_{n+1}(x) = (x - b_n)\, p_n(x) - \lambda_n\, p_{n-1}(x)$$

b_n real, $\lambda_n > 0$

Viennot - Flajolet : combinatorial model for OP, moments

weighted Favard paths, Motzkin paths

Classical OP

Jacobi $P_n^{(\alpha,\beta)}(x)$ $\quad (1-x)^\alpha (1+x)^\beta$ $\quad$ 2

Laguerre $L_n^\alpha(x)$ $\quad$ ultraspherical $C_n^\alpha(x)$ $\quad$ 1

$x^\alpha e^{-x}\,dx$ $\quad$ $(1-x^2)^\alpha\,dx$

Hermite $H_n(x)$ $\quad$ 0

$e^{-x^2/2}\,dx$

q-cases

4 Askey-Wilson $P_n(x;a,b,c,d\,|\,q)$

3 many

2 many

1 many

0

Basic hypergeometric series

$$p_n(x;a,b,c,d\,|\,q) = {}_4\phi_3\left(\begin{matrix} q^{-n}, abcd\,q^{n-1}, ae^{i\theta}, ae^{-i\theta} \\ ab, ac, ad \end{matrix} \,\middle|\, q;q\right)$$

$x = \cos\theta$

Includes Macdonald symmetric functions in 2 variables.

$$= \sum_{k=0}^{n} \frac{(q^{-n}, abcdq^{n-1}, ae^{-i\theta}, ae^{i\theta}; q)_k\, q^k}{(q, ab, ac, ad; q)_k}$$

measure in θ is on $[0,\pi]$

$$\frac{(e^{2i\theta}, e^{-2i\theta}; q)_\infty\, d\theta}{(ae^{i\theta}, ae^{-i\theta}, be^{i\theta}, be^{-i\theta}, ce^{i\theta}, ce^{-i\theta}, de^{i\theta}, de^{-i\theta}; q)_\infty}$$

Very well poised series

DEFN ${}_8W_7\left(A; a_0, a_1, a_2, a_3, a_4; q, z\right) =$

$$\sum_{k=0}^{\infty} \frac{1-Aq^{2k}}{1-A} \frac{(A, a_0, a_1, a_2, a_3, a_4; q)_k}{\left(q, \frac{qA}{a_0}, \frac{qA}{a_1}, \frac{qA}{a_2}, \frac{qA}{a_3}, \frac{qA}{a_4}; q\right)_k} z^k$$

The Askey-Wilson polynomials are a special case

${}_4\phi_3 = {}_8\phi_7$ transformation.

Theorem (Ismail-Rains-S. 2012?) Let

$$p_n(x, \vec{a}, \vec{b}) = \frac{z^n}{(1/z^2; q)_n} \frac{\prod_{i=1}^{m+4} (a_i/z; q)_n}{\prod_{i=1}^{m} (b_i/z; q)_n}$$

$${}_{2m+8}W_{2m+7}\left(q^{-n} z^2, q^{-n}, a_1 z, \dots, a_{m+4} z, \frac{q^{1-n} z}{b_1}, \dots, \frac{q^{1-n} z}{b_m} \,\Big|\, q; z\right)$$

$$z = \frac{b_1 \cdots b_m}{a_1 \cdots a_{m+4}} q^{2-n}$$

($m=0$ is Askey-Wilson)

then

$$\int_{-1}^{1} p_n(x,\vec{a},\vec{b})\,\pi(x)\,w(\vec{x},\vec{a},\vec{b})\,dx = 0$$

for any polynomial $\pi(x)$ of degree $\le n-1$,

$$w(x,\vec{a},\vec{b}) = \frac{\left(e^{2i\theta}, e^{-2i\theta}; q\right)_\infty \prod_{i=1}^{m} \left(b_i e^{i\theta}, b_i e^{-i\theta}; q\right)_\infty}{\prod_{i=1}^{m+4} \left(a_i e^{i\theta}, a_i e^{-i\theta}; q\right)_\infty}$$

$x = \cos\theta$

SPECIAL CASE: $U_n(x) = p_n(x, (t_1, \ldots, t_5), q^{n-1} t_1 \cdots t_5)$

$(m=1)$

$$\cdot \left(t_1 \cdots t_5 z q^{n-1}, t_1 \cdots t_5 q^{n-1}/z ; q\right)_n$$

poly of degree n

Christoph Koutschan (2013) Explicit 3-term relation

$$U_{n+1} = A_{n+1}\left(1 - q^{2n-2} T/z\right)\left(1 - q^{2n-2} T z\right)\left(1 - \frac{q^{2n-3} T}{z}\right)\left(1 - q^{2n-3} T z\right)$$

$$\times \; U_{n-1} \quad +$$

$$\left(B_n \left(1 - t_1/z\right)\left(1 - t_1 z\right) + C_n\right) U_n$$

Proposition 9.1. *The polynomial $U_n(x) = U_n(x_n; \mathbf{t}|q)$ satisfies the following 3-term recurrence relation*

$$
\begin{aligned}
(9.1)\quad & U_{n+1} - A_{n+1}(1-q^{2n-2}Tz)(1-q^{2n-2}T/z)(1-q^{2n-3}Tz)(1-q^{2n-3}T/z)U_{n-1} \\
& \qquad - \left(B_n(1-t_1/z)(1-t_1z) + C_n)\right) U_n = 0,
\end{aligned}
$$

where

$$
\begin{aligned}
A_n = & \prod_{i=1}^{4}\prod_{j=i+1}^{5}(1-t_it_jq^{n-2})(1-q^{n-1})(-q^8)/\prod_{i=1}^{5}(1-e_5q^{n-2}/t_i)\frac{N}{D}, \text{ where} \\
N = & (q^{10} - e_5^4q^{8n} + e_5^2(q^{3n+5} - q^{5n+5}) - e_4q^{2n+8} + e_5^2e_4(q^{6n+3} - q^{5n+3}) + e_5e_3q^{3n+6} \\
& + e_5^3e_1q^{6n+2} - e_5^2e_2q^{5n+4} + e_5e_1(q^{3n+7} - q^{2n+7})) \\
D = & (q^{18} - e_5^4q^{8n} + e_5^2(q^{3n+10} - q^{5n+8}) - e_4q^{2n+14} + e_5^2e_4(q^{6n+5} - q^{5n+6}) \\
& + e_5e_3q^{3n+11} + e_5^3e_1q^{6n+4} - e_5^2e_2q^{5n+7} + e_5e_1(q^{3n+12} - q^{2n+13}));
\end{aligned}
$$

$$
\begin{aligned}
C_n = & \prod_{j=2}^{5}(1-q^nt_1t_j)\frac{(1-q^{2n-1}t_2t_3t_4t_5)(1-q^{2n}t_2t_3t_4t_5)}{t_1(1-q^{n-1}t_2t_3t_4t_5)} \\
& - A_{n+1}\frac{(1-t_1^2t_2t_3t_4t_5q^{2n-2})(1-t_1^2t_2t_3t_4t_5q^{2n-3})(1-t_2t_3t_4t_5q^{n-2})t_1}{\prod_{j=2}^{5}(1-q^{n-1}t_1t_j)}.
\end{aligned}
$$

$$
\begin{aligned}
B_n = & \Big((1-t_1t_2q^n)\prod_{j=3}^{5}(1-q^nt_2t_j)\frac{(1-q^{2n-1}t_1t_3t_4t_5)(1-q^{2n}t_1t_3t_4t_5)}{(1-q^{n-1}t_1t_3t_4t_5)t_2} \\
& - A_{n+1}\frac{(1-t_1t_2^2t_3t_4t_5q^{2n-2})(1-t_1t_2^2t_3t_4t_5q^{2n-3})(1-t_1t_3t_4t_5q^{n-2})t_2}{\prod_{j=3}^{5}(1-q^{n-1}t_2t_j)(1-t_1t_2q^{n-1})} - C_n\Big) \\
& /(1-t_1/t_2)(1-t_1t_2).
\end{aligned}
$$

and e_i is the elementary symmetric function of t_1, t_2, t_3, t_4, t_5 of degree i.

Type R_{II} polynomials (Ismail-Masson 1995)

$$P_n(x) = (x-c_n)P_{n-1}(x) - \lambda_n(x-a_n)(x-b_n)P_{n-2}(x)$$

<u>OP prob I</u> Find a Viennot-Flajolet theory for R_{II} polynomials, continued fractions.

! THANK YOU !