CHANGE OF BASE IN BAILEY PAIRS

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Abstract. Versions of Bailey's lemma which change the base from q to q^2 or q^3 are given. Iterates of these versions give many new versions of multisum Rogers-Ramanujan identities.

1. Introduction.

The Bailey chain is a well-known and frequently used technique in the theory of partitions. It arose from W. N. Bailey's realization [5] that the Rogers-Ramanujan identities could be derived from the simple observation that if $\{\alpha_0, \alpha_1, \ldots\}$ and $\{\delta_0, \delta_1, \ldots\}$ are sequences that satisfy

$$\beta_K = \sum_{r=0}^{k} \alpha_r u_{k-r} v_{k+r} \quad \text{and} \quad \gamma_k = \sum_{r=k}^{\infty} \delta_r u_{r-k} v_{r+k},$$

then

$$\sum_{k=0}^{\infty} \alpha_k \gamma_k = \sum_{k=0}^{\infty} \beta_k \delta_k,$$

provided all infinite sums converge uniformly. L. J. Slater used this idea to produce her list of 130 identities of the Rogers-Ramanujan type [11,12].

It was G. Andrews [3,4 section 3.4] who showed how to use Bailey's insight to generate infinite families of equivalent identities. Each of these families is called a Bailey chain, each identity arising from a Bailey pair which corresponds to a link in the chain. Bailey's lemma, stated at the end of this section and, in fact, little more than the observation given in the first paragraph, is the recipe for passing between adjacent links. A variation of the Bailey lemma was described in [1]. It extends the notion of a Bailey chain to a two-dimensional lattice. The purpose of this paper is to give other explicit versions of the Bailey lemma which change the base q.

This freedom to change the base creates new chains of identities. A wide variety of new Rogers-Ramanujan identities is the result. For example, iterating the change of base $q \to q^2$ yields Theorem 4.3 which, with $a = 1$, becomes

$$\sum_{s_1,\cdots,s_{k+1}} \frac{q^E(-q;q)_{2s_3}(-q^2;q^2)_{2s_4}\cdots(-q^{2^{k-2}};q^{2^{k-2}})_{2s_{k+1}}}{(q;q)_{s_1-s_2}(q^2;q^2)_{s_2-s_3}\cdots(q^{2^k};q^{2^k})_{s_{k+1}}} = \prod_{n\not\equiv 0,\pm 2 \pmod{4+2^k}} (1-q^n)^{-1}$$

where $E = s_1^2 + s_2^2 + s_2 + s_3 + 2s_4 + \cdots + 2^{k-2}s_{k+1}$.

The main theorems are given in §2. Appropriate limiting cases are stated in §3, and these are used in §4 to find several new multisum Rogers-Ramanujan identities. In §5, we show these techniques can be used to prove conjectures of Melzer [10] for the Fermionic forms of the supersymmetric analogues of Virasoro characters. Applications to basic hypergeometric series are given in §6. In §7, we show how to use these transformations to prove Stembridge's [13] identities of Rogers-Ramanujan type, and give a sample of other identities that arise from mixing base changes.

We shall need the definition of a Bailey pair, given below, and Bailey's lemma, which produces a new Bailey pair from a given Bailey pair. We use the standard notation found in [8].

Definition. *A pair of sequences $(\alpha_n(a,q), \beta_n(a,q))$ is called a Bailey pair with parameters (a,q) if*

$$\beta_n(a,q) = \sum_{r=0}^{n} \frac{\alpha_r(a,q)}{(q;q)_{n-r}(aq;q)_{n+r}}$$

for all $n \geq 0$.

Bailey's Lemma. *Suppose that $(\alpha_n(a,q), \beta_n(a,q))$ is a Bailey pair with parameters (a,q). Then $(\alpha'_n(a,q), \beta'_n(a,q))$ is another Bailey pair with with parameters (a,q), where*

$$\alpha'_n(a,q) = \frac{(\rho_1, \rho_2; q)_n}{(aq/\rho_1, aq/\rho_2; q)_n} \left(\frac{aq}{\rho_1\rho_2}\right)^n \alpha_n(a,q),$$

and

$$\beta'_n(a,q) = \sum_{k=0}^{n} \frac{(\rho_1, \rho_2; q)_k (aq/\rho_1\rho_2; q)_{n-k}}{(aq/\rho_1, aq/\rho_2; q)_n (q;q)_{n-k}} \left(\frac{aq}{\rho_1\rho_2}\right)^k \beta_k(a,q).$$

2. The main theorems.

In this section we state and prove versions of Bailey's lemma in which the base q changes from q to q^2 or q^3. Theorem 2.2 (Theorem 2.4) is the inverse of Theorem 2.1 (Theorem 2.3), and could be considered as changing q to $q^{1/2}$ $(q^{1/3})$.

Theorem 2.1. *Suppose that $(\alpha_n(a,q), \beta_n(a,q))$ is a Bailey pair with parameters (a,q). If*

$$\beta'_n(a,q) = \sum_{k=0}^{n} \frac{(-aq;q)_{2k}(B^2;q^2)_k (q^{-k}/B, Bq^{k+1};q)_{n-k}}{(-aq/B, B; q)_n (q^2;q^2)_{n-k}} B^{-k} q^{-\binom{k}{2}} \beta_k(a^2,q^2),$$

then $(\alpha'_n(a,q), \beta'_n(a,q))$ is a Bailey pair with parameters (a,q), where

$$\alpha'_r(a,q) = \frac{(-B;q)_r}{(-aq/B;q)_r} B^{-r} q^{-\binom{r}{2}} \alpha_r(a^2,q^2).$$

Proof. This follows routinely from the definition of a Bailey pair by interchanging summations and using Singh's quadratic transformation (III.21) and the q-analogue of the Pfaff-Saalschütz theorem (II.12) in [8]

$$(2.1) \qquad {}_4\phi_3\left(\begin{matrix} q^{-2m}, & C^2q^{2m}, & D, & Dq \\ & Cq, & Cq^2, & D^2 \end{matrix} \middle| q^2; q^2\right) = D^m \frac{(Cq/D, -q;q)_m (1-C)}{(C, -D; q)_m (1-Cq^{2m})}.$$

with $m = n-r$, $C = Bq^{-n+2r}$, and $D = -aq^{1+2r}$. $\square$

Bailey's lemma is its own inverse, as one could replace ρ_1 and ρ_2 by aq/ρ_1 and aq/ρ_2. Since Theorem 2.1 changes the base q, its inverse is distinct from Theorem 2.1: Theorem 2.2.

Theorem 2.2. *Suppose that $(\alpha_n(a,q),\beta_n(a,q))$ is a Bailey pair with parameters (a,q). If*

$$\gamma_n(a,q)=\sum_{k=0}^{n}\frac{(qa^2/B;q^2)_{2n-k}(-Bq;q^2)_k}{(-q^2a^2;q^2)_{2n}(a^4q^2/B^2;q^4)_n(q^4;q^4)_{n-k}}a^{2k}B^{-k}q^{k^2}\beta_k(a^2,q^2),$$

then $(\alpha'_n(a,q),\gamma_n(a,q))$ is a Bailey pair with parameters (a^4,q^4), where

$$\alpha'_r(a,q)=\frac{(-Bq;q^2)_r}{(-qa^2/B;q^2)_r}a^{2r}B^{-r}q^{r^2}\alpha_r(a^2,q^2).$$

Proof. This follows as in the proof of Theorem 2.1 using the q-analogue of the Pfaff-Saalschütz theorem

$$\begin{aligned}&{}_3\phi_2\left(\begin{matrix}q^{-2n+2r}, & -q^{-2n+2r}, & -Bq^{2r+1}\\ & a^2q^{4r+2}, & Bq^{1-4n+2r}/a^2\end{matrix}\,\middle|\, q^2;q^2\right)\\ &=\frac{(-a^2q^{2n+2r},-a^2q^{2r+1}/B;q)_{n-r}}{(a^2q^{4r+2},a^2q^{2n+1}/B;q)_{n-r}}.\end{aligned}$$

□

Theorem 2.3. *Suppose that $(\alpha_n(a,q),\beta_n(a,q))$ is a Bailey pair with parameters (a,q). Then $(\alpha'_n(a,q),\beta'_n(a,q))$ is a Bailey pair with parameters (a^3,q^3), where*

$$\alpha'_r(a,q)=a^rq^{r^2}\alpha_r(a,q)$$

$$\beta'_n(a,q)=\frac{1}{(a^3q^3;q^3)_{2n}}\sum_{k=0}^{n}\frac{(aq;q)_{3n-k}a^kq^{k^2}}{(q^3;q^3)_{n-k}}\beta_k(a,q). \tag{T1}$$

Proof. This follows as in the proof of Theorem 2.1 again using Saalschütz's evaluation [8]

$${}_3\phi_2\left(\begin{matrix}q^{-n+r}, & \omega q^{-n+r}, & \omega^2q^{-n+r}\\ & aq^{2r+1}, & q^{r-3n}/a\end{matrix}\,\middle|\, q;q\right)=\frac{(a\omega q^{r+n+1},a\omega^2q^{r+n+1};q)_{n-r}}{(aq^{2r+1},aq^{2n+1};q)_{n-r}}.$$

where ω is a primitive cube root of 1. □

The inverse of Theorem 2.3 is Theorem 2.4.

Theorem 2.4. *Suppose that $(\alpha_n(a,q),\beta_n(a,q))$ is a Bailey pair with parameters (a,q). Then $(\alpha'_n(a,q),\beta'_n(a,q))$ is a Bailey pair with parameters (a,q), where*

$$\alpha'_r(a,q)=a^{-r}q^{-r^2}\alpha_r(a^3,q^3)$$

$$\begin{aligned}\beta'_n(a,q)=&\frac{1}{(aq;q)_{2n}}\sum_{k=0}^{n}\frac{(aq^{2n+1};q^{-1})_{3k}(a^3q^3;q^3)_{2(n-k)}}{(q^3;q^3)_k}\\ &\times(-1)^kq^{3\binom{k}{2}-n^2}a^{-n}\beta_{n-k}(a^3,q^3).\end{aligned}\tag{T2}$$

Proof. This follows as in the proof of Theorem 2.1 using the strange ${}_5\phi_4$ evaluation [9, (6.28)]

$$\sum_{k=0}^{m} \frac{(q^{-3m};q^3)_k (A^3;q^3)_{2k} q^{3k}}{(q^3,A^3;q^3)_k (Aq^{1-m};q)_{3k}} = \frac{q^{-3m^2/2+m/2}(-1)^m (q^3;q^3)_m (1-A)}{(q^{-1},A^{-1};q^{-1})_m (Aq^{1-m};q)_m (1-Aq^{2m})}$$

with $A = aq^{1+2r}$ and $m = n-r$. $\square$

There is a companion evaluation to (2.1), which implies a result closely related to Theorem 2.1

$$(2.2) \qquad {}_4\phi_3 \left(\begin{matrix} q^{-2m}, & C^2q^{2m}, & D, & Dq \\ & C, & Cq, & D^2q^2 \end{matrix} \middle| \, q^2;q^2 \right) = D^m \frac{(C/D,-q;q)_m}{(C,-Dq;q)_m}.$$

We use (2.2) with $m = n-r$, $C = Bq^{-n+2r}$, and $D = -aq^{2r}$ for the next theorem.

Theorem 2.5. *Suppose that $(\alpha_n(a,q), \beta_n(a,q))$ is a Bailey pair with parameters (a,q). If*

$$\beta'_n(a,q) = \sum_{k=0}^{n} \frac{(-a;q)_{2k}(B^2;q^2)_k (q^{-k+1}/B, Bq^k;q)_{n-k}}{(-aq/B,B;q)_n (q^2;q^2)_{n-k}} B^{-k} q^{k-\binom{k}{2}} \beta_k(a^2,q^2),$$

then $(\alpha'_n(a,q), \beta'_n(a,q))$ is a Bailey pair with parameters (a,q), where

$$\alpha'_r(a,q) = \frac{(-B;q)_r}{(-aq/B;q)_r} \frac{1+a}{1+aq^{2r}} B^{-r} q^{r-\binom{r}{2}} \alpha_r(a^2,q^2).$$

3. Limiting cases.

It is well-known [3,4] that Bailey's lemma implies the multisum versions of the Rogers-Ramanujan identities due to Andrews. In this section we record the appropriate limiting cases of Bailey's lemma and Theorems 2.1-2.4.

First we review [3,4] the limiting cases of Bailey's lemma which are used for the Andrews-Gordon identities. If $\rho_1, \rho_2 \to \infty$ in Bailey's Lemma, we have

$$(\text{S1}) \qquad \begin{aligned} \alpha'_r(a,q) &= a^r q^{r^2} \alpha_r(a,q), \\ \beta'_n(a,q) &= \sum_{k=0}^{n} \frac{a^k q^{k^2}}{(q;q)_{n-k}} \beta_k(a,q). \end{aligned}$$

Iterate (S1) k times to obtain

$$(3.1) \qquad \alpha_r^{(k)}(a,q) = a^{rk} q^{kr^2} \alpha_r(a,q).$$

If $n \to \infty$, we have

$$\begin{aligned} \beta_\infty^{(k)} &= \frac{1}{(q;q)_\infty} \sum_{s_1,\cdots,s_k \geq 0} \frac{a^{s_1+\cdots+s_k} q^{s_1^2+\cdots+s_k^2}}{(q;q)_{s_1-s_2}(q;q)_{s_2-s_3}\cdots(q;q)_{s_k}} \beta_{s_k}(a,q) \\ (3.2) \qquad &= \frac{1}{(q,aq;q)_\infty} \sum_{r=0}^{\infty} a^{rk} q^{kr^2} \alpha_r(a,q). \end{aligned}$$

If we choose the unit Bailey pair [3,4]

$$\text{(UBP)}\qquad \beta_n(a,q)=\begin{cases}1, & \text{if } n=0\\ 0, & \text{if } n>0,\end{cases}\qquad \alpha_n(a,q)=\frac{(a;q)_n}{(q;q)_n}\frac{(1-aq^{2n})}{(1-a)}(-1)^n q^{\binom{n}{2}}$$

and then put $a=1$, we obtain a Rogers-Ramanujan identity for modulus $2k+1$

$$\begin{aligned}\sum_{s_1,\cdots,s_{k-1}\geq 0}&\frac{q^{s_1^2+\cdots+s_{k-1}^2}}{(q;q)_{s_1-s_2}(q;q)_{s_2-s_3}\cdots(q;q)_{s_{k-1}}}\\ &=\frac{1}{(q;q)_\infty}\left(1+\sum_{r=1}^{\infty}q^{(k+1/2)r^2}(q^{-r/2}+q^{r/2})(-1)^r\right)\\ &=\frac{(q^{2k+1},q^k,q^{k+1};q^{2k+1})_\infty}{(q;q)_\infty}.\end{aligned}$$

There are five other choices for iterating Bailey's Lemma which each shift the modulus of the resulting theta-function by one: If we take $\rho_1\to\infty,\rho_2=-\sqrt{aq})$, then we get

$$\text{(S2)}\qquad \begin{aligned}\alpha'_r(a,q)&=a^{r/2}q^{r^2/2}\alpha_r(a,q),\\ \beta'_n(a,q)&=\sum_{k=0}^{n}\frac{(-\sqrt{aq};q)_k}{(q;q)_{n-k}(-\sqrt{aq};q)_n}a^{k/2}q^{k^2/2}\beta_k(a,q).\end{aligned}$$

When applied to the unit Bailey pair, (S2) has the effect of increasing the modulus by one instead of 2, in fact (S2)(S2)=(S1). Thus (S2) may be considered the square root of (S1). If we take $\rho_1\to\infty,\rho_2=-q^{1/2}$, then we get

$$\text{(S3)}\qquad \begin{aligned}\alpha'_r(a,q)&=\frac{(-q^{1/2};q)_r}{(-aq^{1/2};q)_r}\,a^r q^{r^2/2}\alpha_r(a,q),\\ \beta'_n(a,q)&=\sum_{k=0}^{n}\frac{(-q^{1/2};q)_k}{(q;q)_{n-k}(-aq^{1/2};q)_n}\,a^k q^{k^2/2}\beta_k(a,q).\end{aligned}$$

If we take $\rho_1\to\infty,\rho_2=-aq^{1/2}$, then we get

$$\text{(S4)}\qquad \begin{aligned}\alpha'_r(a,q)&=\frac{(-a\,q^{1/2};q)_r}{(-q^{1/2};q)_r}q^{r^2/2}\alpha_r(a,q),\\ \beta'_n(a,q)&=\sum_{k=0}^{n}\frac{(-a\,q^{1/2};q)_k}{(q;q)_{n-k}(-q^{1/2};q)_n}\,q^{k^2/2}\beta_k(a,q).\end{aligned}$$

If we take $\rho_1\to\infty,\rho_2=-a^{1/2}q$, then we get

$$\text{(S5)}\qquad \begin{aligned}\alpha'_r(a,q)&=\frac{(-a^{1/2}q;q)_r}{(-a^{1/2};q)_r}a^{r/2}q^{(r^2-r)/2}\alpha_r(a,q),\\ \beta'_n(a,q)&=\sum_{k=0}^{n}\frac{(-a^{1/2}q;q)_k}{(q;q)_{n-k}(-a^{1/2};q)_n}\,a^{k/2}q^{(k^2-k)/2}\beta_k(a,q).\end{aligned}$$

If we take $\rho_1 \to \infty, \rho_2 = -a^{1/2}$, then we get

(S6)
$$\begin{aligned} \alpha'_r(a,q) = &\ \frac{(-a^{1/2};q)_r}{(-a^{1/2}q;q)_r} a^{r/2} q^{(r^2+r)/2} \alpha_r(a,q), \\ \beta'_n(a,q) = &\ \sum_{k=0}^{n} \frac{(-a^{1/2};q)_k}{(q;q)_{n-k}(-a^{1/2}q;q)_n}\, a^{k/2} q^{(k^2+k)/2} \beta_k(a,q). \end{aligned}$$

We have that (S1) is the same as (S3)(S4), (S4)(S3), (S5)(S6), or (S6)(S5).

For Theorem 2.1, we have three possible choices of B, which change $\alpha_r(a,q)$ by a quadratic power of q, ($B \to \infty$, $B \to 0$, and $B^2 = aq$).

(D1)
$$\begin{aligned} \alpha'_r(a,q) &= \alpha_r(a^2,q^2), \\ \beta'_n(a,q) &= \sum_{k=0}^{n} \frac{(-aq;q)_{2k}}{(q^2;q^2)_{n-k}}\, q^{n-k} \beta_k(a^2,q^2), \end{aligned}$$

(D2)
$$\begin{aligned} \alpha'_r(a,q) &= a^{-r} q^{-r^2} \alpha_r(a^2,q^2), \\ \beta'_n(a,q) &= \sum_{k=0}^{n} \frac{(-aq;q)_{2k}}{(q^2;q^2)_{n-k}}\, q^{k^2+k-2kn-n} (-1)^{n-k} a^{-n} \beta_k(a^2,q^2), \end{aligned}$$

and

(D3)
$$\begin{aligned} \alpha'_r(a,q) &= a^{-r/2} q^{-r^2/2} \alpha_r(a^2,q^2), \\ \beta'_n(a,q) &= \sum_{k=0}^{n} \frac{(-aq;q)_{2k} (q^{-1/2-k}/\sqrt{a}, q^{k+3/2}\sqrt{a};q)_{n-k}}{(aq^{2k+1};q^2)_{n-k}(q^2;q^2)_{n-k}} q^{-\binom{k}{2}} (aq)^{-k/2} \beta_k(a^2,q^2). \end{aligned}$$

For Theorem 2.5 we record only the $B \to \infty$ case

(D4)
$$\begin{aligned} \alpha'_r(a,q) &= \frac{1+a}{1+aq^{2r}} q^r \alpha_r(a^2,q^2), \\ \beta'_n(a,q) &= \sum_{k=0}^{n} \frac{(-a;q)_{2k}}{(q^2;q^2)_{n-k}}\, q^k \beta_k(a^2,q^2). \end{aligned}$$

The corresponding cases $B \to 0$ and $B^2 = aq$ are labelled (D5) and (D6), respectively.

Because Theorem 2.3 and Theorem 2.4 have no parameters besides a, we label an application of these theorems by (T1) and (T2), respectively. We also do not state the analogous three possibilities for Theorem 2.2 in this paper.

4. Multisum Rogers-Ramanujan identities.

We have reviewed in §2 that iterating (S1) gives a multisum Rogers-Ramanujan identity. In this section and the next section, we consider other iterates of (S1)–(S6), (D1)–(D6), (T1)–(T2). We obtain Bressoud's multisum version for even modulus, and many new multisum identities.

Before considering the iterates, first we record a proposition which allows us to insert linear functions of the summation indices on the multisum side of Rogers-Ramanujan identities. We need it to change the restricted moduli in (3.2) from $\equiv 0, \pm k \mod 2k+1$ to $\equiv 0, \pm i \mod 2k+1$. It replaces the Bailey lattice [1] and is tailored to the choice of $a = 1$ in the unit Bailey pair.

Proposition 4.1. *If $(\alpha_n(q), \beta_n(q))$ is a Bailey pair with parameters $(1, q)$,*

$$\alpha_n(a,q) = \begin{cases} 1 \text{ for } & n = 0, \\ q^{An^2}(q^{(A-1)n} + q^{-(A-1)n})(-1)^n & \text{for } n > 0, \end{cases}$$

then $(\alpha'_n(q), \beta'_n(q))$ is Bailey pair with parameters $(1, q)$, where $\beta'_n(q) = q^n\beta_n(q)$, and

$$\alpha'_n(q) = \begin{cases} 1 \text{ for } & n = 0, \\ q^{An^2}(q^{An} + q^{-An})(-1)^n & \text{for } n > 0. \end{cases}$$

Proof. Proposition 4.1 is equivalent to

$$\sum_{s=-n}^{n} \begin{bmatrix} 2n \\ n-s \end{bmatrix}_q w^{s^2-s}(-1)^s = q^n \sum_{s=-n}^{n} \begin{bmatrix} 2n \\ n-s \end{bmatrix}_q w^{s^2-s}(-q)^s$$

where $w = q^A$. This is easy to verify by considering the s and $1-s$ terms on each side. $\square$

We now show how Proposition 4.1 may be used to insert linear factors into the exponent of q on the sum side of (3.2), thereby changing the excluded moduli on the product side. Suppose that we start at the (UBP) with $a = 1$,

$$\alpha_n^{(0)}(q) = q^{n^2/2}(q^{n/2} + q^{-n/2})(-1)^n, \quad \beta_n^{(0)}(q) = \delta_{0n}.$$

If we then apply (S1), to obtain a Bailey pair $(\alpha_n^{(1)}(q), \beta_n^{(1)}(q))$ we have $\alpha_n^{(1)}(q) = q^{3n^2/2}(q^{n/2} + q^{-n/2})(-1)^n$. We next apply Proposition 4.1 with $A = 3/2$ to obtain another Bailey pair

$$\alpha_n^{(2)}(q) = q^{3n^2/2}(q^{3n/2} + q^{-3n/2})(-1)^n, \quad \beta_n^{(2)}(q) = q^n\beta_n^{(1)}(q).$$

We could apply (S1) yet again followed by Proposition 4.1 with $A = 5/2$, to obtain

$$\alpha_n^{(4)}(q) = q^{5n^2/2}(q^{5n/2} + q^{-5n/2})(-1)^n, \quad \beta_n^{(4)}(q) = q^n\beta_n^{(3)}(q).$$

We see that applying (S1) and Proposition 4.1 alternatively i times inserts $q^{s_{k-i}+\cdots+s_{k-1}}$ into the left side of (3.2), and changes the term $q^{-r/2} + q^{r/2}$ on the the right side to $(q^{-(i+1/2)r} + q^{(i+1/2)r})$. We now have the full form of the Andrews-Gordon identities,

$$\begin{aligned} (q;q)_\infty \beta_\infty^{(k)} &= \sum_{s_1,\cdots,s_{k-1}\geq 0} \frac{q^{s_1^2+\cdots+s_{k-1}^2+s_{k-i}+\cdots+s_{k-1}}}{(q;q)_{s_1-s_2}(q;q)_{s_2-s_3}\cdots(q;q)_{s_{k-1}}} \\ &= \frac{1}{(q;q)_\infty}\left(1 + \sum_{r=1}^{\infty} q^{(2k+1)r^2/2}(q^{-(i+1/2)r} + q^{(i+1/2)r})(-1)^r\right) \\ &= \frac{(q^{2k+1}, q^{k-i}, q^{k+i+1}; q^{2k+1})_\infty}{(q;q)_\infty}. \end{aligned}$$

Note that iterating (S1) k times corresponds to adding 2 to the base k times

$$1 \xrightarrow{(S1)} 3 \xrightarrow{(S1)} 5 \xrightarrow{(S1)} \cdots \xrightarrow{(S1)} 2k+1.$$

For Bressoud's [7] identities of modulus $2k$ we first double the base using (D1), then apply (S1) and Proposition 4.1 $i-1$ times, and finally (S1) $k-i$ times,

$$1 \xrightarrow{(D1)} 2 \xrightarrow{(S1)} 4 \xrightarrow{(S1)} \cdots \xrightarrow{(S1)} 2k,$$

$$\begin{aligned}(q;q)_\infty \beta_\infty^{(k)} &= \sum_{s_1,\cdots,s_{k-1}\geq 0} \frac{q^{s_1^2+\cdots+s_{k-1}^2+s_{k-i}+\cdots+s_{k-1}}}{(q;q)_{s_1-s_2}(q;q)_{s_2-s_3}\cdots(q;q)_{s_{k-2}-s_{k-1}}(q^2;q^2)_{s_{k-1}}} \\ &= \frac{1}{(q;q)_\infty}\left(1+\sum_{r=1}^\infty q^{kr^2}(q^{-ir}+q^{ir})(-1)^r\right) \\ &= \frac{(q^{2k},q^{k-i},q^{k+i};q^{2k})_\infty}{(q;q)_\infty}.\end{aligned}$$

One may also obtain the modulus $2k$ by using (S1) $k-1$ times and (S2) once with $a=1$,

$$1 \xrightarrow{(S1)} 3 \xrightarrow{(S1)} 5 \xrightarrow{(S1)} \cdots \xrightarrow{(S1)} 2k-1 \xrightarrow{(S2)} 2k.$$

By the same method we obtain the generalized Göllnitz-Gordon identities [1, (7.4.4)]

$$\begin{aligned}(q;q)_\infty \beta_\infty^{(k)} &= \sum_{s_1,\cdots,s_{k-1}\geq 0} \frac{q^{s_1^2/2+\cdots+s_k^2+s_{k-i}+\cdots+s_{k-1}}(-q^{1/2};q)_{s_1}}{(q;q)_{s_1-s_2}(q;q)_{s_2-s_3}\cdots(q;q)_{s_{k-1}}} \\ &= \frac{1}{(q;q)_\infty}\left(1+\sum_{r=1}^\infty q^{kr^2}(q^{-(i+1/2)r}+q^{(i+1/2)r})(-1)^r\right) \\ &= \frac{(-q;q)_\infty(q^{2k},q^{k-i-1/2},q^{k+i+1/2};q^{2k})_\infty}{(q;q)_\in fty}.\end{aligned}$$

The Bressoud and Göllnitz-Gordon identities may be "combined" if we apply (D1) once, (S1) $k-1$ times, and then (S2)

$$1 \xrightarrow{(D1)} 2 \xrightarrow{(S1)} 4 \xrightarrow{(S1)} \cdots \xrightarrow{(S1)} 2k \xrightarrow{(S2)} 2k+1.$$

Choosing $a=1$, we obtain

$$\begin{aligned}&\sum_{s_0,\cdots,s_{k-1}\geq 0} \frac{(-q^{1/2};q)_{s_0} q^{s_0^2/2+s_1^2+\cdots+s_{k-1}^2+s_i+\cdots+s_{k-1}}}{(q;q)_{s_0-s_1}\cdots(q;q)_{s_{k-2}-s_{k-1}}(q^2;q^2)_{s_{k-1}}} \\ &\qquad = \frac{(-q^{1/2};q)_\infty}{(q;q)_\infty}(q^{2k+1},q^{i+1/2},q^{2k-i+1/2};q^{2k+1})_\infty.\end{aligned}$$

Another modulus $2k$ identity may be found by applying (S2) first and then (S1) $k-1$ times,

$$1 \xrightarrow{(S2)} 2 \xrightarrow{(S1)} 4 \xrightarrow{(S1)} \cdots \xrightarrow{(S1)} 2k$$

with $a = 1$. The result is

$$
\begin{aligned}
(q;q)_\infty \beta_\infty^{(k)} &= \sum_{s_1,\cdots,s_{k-1}\geq 0} \frac{q^{s_1^2+\cdots+s_{k-1}^2}}{(q;q)_{s_1-s_2}\cdots(q;q)_{s_{k-2}-s_{k-1}}(q^{1/2};q^{1/2})_{2s_{k-1}}} \\
&= \frac{1}{(q;q)_\infty}\left(1+\sum_{r=0}^{\infty} q^{kr^2}(q^{-r/2}+q^{r/2})\right) \\
&= \frac{(q^{2k},-q^{k-1/2},q^{k+1/2};q^{2k})_\infty}{(q;q)_\infty}. \qquad (4.1)
\end{aligned}
$$

This form has an unusual linear perturbation: if we insert $q^{-(s_i+\cdots+s_{k-1})/2}$, the excluded congruence class does not change, rather the base changes! Proposition 4.1 does not apply because only one application of (S2) was used. We state this unusual result in a proposition.

Proposition 4.2. *If k and i are positive integers such that $1 \leq i \leq k$, then*

$$
\begin{aligned}
\sum_{s_1,\cdots,s_{k-1}\geq 0} & \frac{q^{s_1^2+\cdots+s_{k-1}^2-(s_i+\cdots+s_{k-1})/2}}{(q;q)_{s_1-s_2}\cdots(q;q)_{s_{k-2}-s_{k-1}}(q^{1/2};q^{1/2})_{2s_{k-1}}} \\
&= \frac{(q^{2i},-q^{i-1/2},-q^{i+1/2};q^{2i})_\infty}{(q;q)_\infty}.
\end{aligned}
$$

Proof. This follows immediately from (4.1) and the limiting case of the q-Vandermonde identity (II.7) in [8]

$$
\sum_{s=0}^{n} \frac{q^{s^2-s/2}}{(q;q)_{n-s}(q^{1/2};q^{1/2})_{2s}} = \frac{1}{(q^{1/2};q^{1/2})_{2n}}. \qquad \square
$$

If we apply (D1) toward the end, the doubling of the modulus is more pronounced. For example

$$
1 \xrightarrow{(S1)} 3 \xrightarrow{(S1)} \cdots \xrightarrow{(S1)} 2k-1 \xrightarrow{(D1)} 4k-2 \xrightarrow{(S1)} 4k
$$

gives for $k \geq 2$, $1 \leq i \leq k$,

$$
\begin{aligned}
\sum_{s_1,\cdots,s_k\geq 0} & \frac{(-q;q)_{2s_2} q^{s_1^2+2s_3^2+\cdots+2s_k^2+s_1-s_2+2(s_{i+1}+\cdots+s_k)}}{(q^2;q^2)_{s_1-s_2}\cdots(q^2;q^2)_{s_{k-1}-s_{k-2}}(q^2;q^2)_{s_k}} \\
&= \frac{(q^{4k},q^{2i-1},q^{4k-2i+1};q^{4k})_\infty}{(q;q)_\infty}.
\end{aligned}
$$

We may also use (D3) or (D2) instead of (D1). We give two examples using (D3). The (D3) version of Bressoud's even modulus theorem is

$$
1 \xrightarrow{(D3)} 1 \xrightarrow{(S1)} 3 \xrightarrow{(S1)} \cdots \xrightarrow{(S1)} 2k-1.
$$

If we use $\sqrt{a}=\pm 1$ we have

$$\sum_{s_1,\cdots,s_{k-1}\geq 0}\frac{(\pm q^{-1/2},\pm q^{3/2};q)_{s_{k-1}}q^{s_1^2+\cdots+s_{k-1}^2}}{(q;q)_{s_1-s_2}\cdots(q;q)_{s_{k-2}-s_{k-1}}(q;q)_{2s_{k-1}}}$$
$$=\frac{(q^{2k-1},\pm q^{k-3/2},\pm q^{k+1/2};q^{2k-1})_\infty}{(q;q)_\infty}.$$

A more unusual identity occurs from

$$1\xrightarrow{(S1)}3\xrightarrow{(S1)}\cdots\xrightarrow{(S1)}2k-1\xrightarrow{(D3)}4k-3\xrightarrow{(S1)}4k-1,$$

where $k\geq 2$, $1\leq i\leq k$,

$$\sum_{s_1,\cdots,s_k\geq 0}\frac{(-q;q)_{2s_2}(q^{-1/2-s_2},q^{s_2+3/2};q)_{s_1-s_2}q^{s_1^2-s_2^2/2+2s_3^2+\cdots+2s_k^2+2(s_{i+1}+\cdots+s_k)}}{(q^{2s_2+1};q^2)_{s_1-s_2}(q^2;q^2)_{s_1-s_2}\cdots(q^2;q^2)_{s_{k-1}-s_{k-2}}(q^2;q^2)_{s_k}}$$
$$=\frac{(q^{4k-1},q^{2i-3/2},q^{4k-2i+1/2};q^{4k-1})_\infty}{(q;q)_\infty}.$$

Let's take an example which includes modulus 5 and uses (D1) k times:

$$1\xrightarrow{(D1)}2\xrightarrow{(D1)}\cdots\xrightarrow{(D1)}2^k\xrightarrow{(S1)}2^k+2\xrightarrow{(S1)}2^k+4.$$

$$\alpha_r^{(k+2)}(a,q)=a^{2r}q^{2r^2}\alpha_r(a^{2^k},q^{2^k}).$$

$$\beta_n^{(k+2)}(a,q)=\sum_{s_1,\cdots,s_{k+2}\geq 0}\frac{a^{s_1+s_2}q^E}{(q;q)_{n-s_1}(q;q)_{s_1-s_2}}$$
$$\times\prod_{i=3}^{k+2}\frac{(-a^{2^{i-3}}q^{2^{i-3}};q^{2^{i-3}})_{2s_i}}{(q^{2^{i-2}};q^{2^{i-2}})_{s_{i-1}-s_i}}\beta_{s_{k+2}}(a^{2^k},q^{2^k}).$$

where

$$E=s_1^2+s_2^2+s_2+s_3+2s_4+\cdots+2^{k-2}s_{k+1}-2^{k-1}s_{k+2}.$$

Choosing the unit Bailey pair (UBP), and letting $n\to\infty$, we have the following theorem.

Theorem 4.3. *For any non-negative integer k,*

$$\frac{1}{(aq;q)_\infty}\sum_{r=0}^{\infty}\frac{a^{2r}q^{2r^2+2^k\binom{r}{2}}(1-a^{2^k}q^{r2^{k+1}})(a^{2^k};q^{2^k})_r(-1)^r}{(1-a^{2^k})(q^{2^k};q^{2^k})_r}$$
$$=\sum_{s_1,\cdots,s_{k+1}\geq 0}\frac{a^{s_1+s_2}q^E}{(q;q)_{s_1-s_2}}\prod_{i=3}^{k+2}\frac{(-a^{2^{i-3}}q^{2^{i-3}};q^{2^{i-3}})_{2s_i}}{(q^{2^{i-2}};q^{2^{i-2}})_{s_{i-1}-s_i}}$$

where

$$E=s_1^2+s_2^2+s_2+s_3+\cdots+2^{k-2}s_{k+1}\quad s_{k+2}=0.$$

The case $a=1$ of Theorem 4.3 is a Rogers-Ramanujan identity on base 2^k+4.

Corollary 4.4. *For any non-negative integers k and j with $1 \le j \le k$, the generating function for partitions with part sizes not congruent to 0 or $\pm(2+2^{j-1})$ mod 2^k+4 is*

$$\sum_{s_1,\cdots,s_{k+1}\ge 0} \frac{q^E}{(q;q)_{s_1-s_2}} \prod_{i=3}^{k+2} \frac{(-q^{2^{i-3}};q^{2^{i-3}})_{2s_i}}{(q^{2^{i-2}};q^{2^{i-2}})_{s_{i-1}-s_i}} = \frac{(q^{2^k+4},q^{2^{j-1}+2},q^{2^k-2^{j-1}+2};q^{2^k+4})_\infty}{(q;q)_\infty}$$

where

$$E = s_1^2+s_2^2+s_2+s_3+\cdots+2^{k-2}s_{k+1}-2^{j-1}s_{j+1}, \quad s_{k+2}=0.$$

Moreover the same statement holds for part sizes not congruent to 0 or $\pm 2 \mod 2^k+4$, and not congruent to 0 or $\pm 1 \mod 2^k+4$, if the term $-2^{j-1}s_{j+1}$ in E is replaced by 0 or s_1, respectively.

Proof. The case $E = s_1^2+s_2^2+s_2+s_3+\cdots+2^{k-2}s_{k+1}$ follows immediately from Theorem 4.3 with $a=1$. We need to insert the appropriate linear factors via Proposition 4.1 for the other excluded congruence classes.

To insert q^{s_1}, note that after applying (D1) k times and then (S1) once, we have

$$\alpha_n^{(k+1)}(1,q) = q^{n^2}q^{2^{k-1}n^2}(q^{2^{k-1}n}+q^{-2^{k-1}n})(-1)^n.$$

We apply Proposition 4.1 with $A = 2^{k-1}+1$ which changes $\alpha_n^{(k+1)}(1,q)$ to

$$q^{(2^{k-1}+1)n^2}(q^{(2^{k-1}+1)n}+q^{-(2^{k-1}+1)n})(-1)^n,$$

then the final application of (S1) gives

$$\alpha_n^{(k+2)}(1,q) = q^{(2^{k-1}+2)n^2}(q^{(2^{k-1}+1)n}+q^{-(2^{k-1}+1)n})(-1)^n.$$

which excludes the classes $0,\pm 1$ by the Jacobi triple product formula.

For the stated values of j, we use Proposition 4.1 in reverse to insert a linear term after $k-j+1$ iterations of (D1). The term $q^{-2^{j-1}s_{j+1}}$ appears because we use $j-1$ iterations of (D1) after $q^{-s_{j+1}}$ has been inserted. $\square$

Note that for $k=0$ Corollary 4.4 becomes the usual Rogers-Ramanujan identities for modulus 5. Thus we have embedded the odd modulus 5 into an infinite family of even moduli theorems. Moreover the number of summations for the moduli 2^k+4 is $k+1$, compared to $2^{k-1}+1$ for the known even moduli theorems.

It is natural to ask if there exist other linear perturbations of E in Corollary 4.4 which will give the missing excluded congruence classes. For example, if $k=3$, the classes $0,\pm 5 \mod 12$ do not appear. However no such perturbation was found for this case.

5. The Melzer conjectures.

Melzer [10] conjectured Rogers-Ramanujan multisum representations for some closely related infinite products. As Berkovich, McCoy, and Orrick [6] have pointed out, these can be transformed into known Rogers-Ramanujan generalizations, using basic hypergeometric transformations to change each pair of adjacent indices into a single index of summation. In this section we shall prove the most general forms of these conjectures using the methods of §4.

Theorem 5.1. *For $i = 1, 2, \ldots, k$, we have that*

$$\begin{aligned}
&\sum_{n=0}^{\infty}(-1)^n a^{kn} q^{kn^2+(k-i+1/2)n} \frac{(-q^{1/2};q)_n(-aq^{n+3/2};q)_\infty}{(q;q)_n(aq^{n+1};q)_\infty} \\
&\qquad \times \left(1+aq^{n+1/2}-(1+q^{n+1/2})a^i q^{(i-1/2)(2n+1)}\right) \\
&= \sum_{s_1,\ldots,s_{k-1}=0}^{\infty} \frac{a^{s_1+s_2+\cdots+s_{k-1}}(-q^{1/2};q)_{s_1} q^{s_1^2/2+s_2^2+\cdots+s_{k-1}^2+s_i+\cdots+s_{k-1}}}{(q;q)_{s_1-s_2}(q;q)_{s_2-s_3}\cdots(q;q)_{s_{k-1}}} \\
&= \sum_{s_1,\ldots,s_{2k-2}=0}^{\infty} \frac{a^{s_1+s_3+\cdots+s_{2k-3}} q^{(s_1^2+s_2^2+\cdots+s_{2k-2}^2)/2+s_{2i-1}+s_{2i+1}+\cdots+s_{2k-3}}}{(q;q)_{s_1-s_2}(q;q)_{s_2-s_3}\cdots(q;q)_{s_{2k-2}}}.
\end{aligned}$$

There is a companion identity for which Melzer [10, (2.10)] had only conjectured the $a = 1$ case.

Theorem 5.2. *For $i = 1, 2, \ldots, k$, we have that*

$$\begin{aligned}
&\frac{(-a^{1/2}q;q)_\infty}{(aq;q)_\infty}\sum_{n=0}^{\infty}(-1)^n a^{(k-1/2)n} q^{kn^2+(k-i)n}(1-a^i q^{(2n+1)i})\frac{(aq;q)_n}{(q;q)_n} \\
&= (-a^{1/2}q;q)_\infty \sum_{s_1,\ldots,s_{k-1}=0}^{\infty} \frac{a^{s_1+s_2+\cdots+s_{k-1}} q^{s_1^2+s_2^2+\cdots+s_{k-1}^2+s_i+s_{i+1}+\cdots+s_{k-1}}}{(q;q)_{s_1-s_2}(q;q)_{s_2-s_3}\cdots(q;q)_{s_{k-1}}(-a^{1/2}q;q)_{s_{k-1}}} \\
&= \sum_{s_1,\ldots,s_{2k-2}=0}^{\infty} \frac{a^{(s_1+s_2+\cdots+s_{2k-2})/2} q^{(s_1^2+s_2^2+\cdots+s_{2k-2}^2)/2+s_{2i}+s_{2i+2}+\cdots+s_{2k-2}+S/2}}{(q;q)_{s_1-s_2}(q;q)_{s_2-s_3}\cdots(q;q)_{s_{2k-2}}},
\end{aligned}$$

where $S = s_1 - s_2 + s_3 - s_4 + \cdots + s_{2k-1}$.

To prove these identities, we need one more Bailey lemma, the one that sits behind the Bailey lattice and enables us to change the parameter a to a/q. It was first stated and is proven in [1], lemma 1.2.

Proposition 5.3. *Let $(\alpha_n(aq,q), \beta_n(aq,q))$ be a Bailey pair with parameters (aq,q). If*

$$\begin{aligned}
\beta_n'(a,q) &= \sum_{k=0}^{n} \frac{(\rho_1,\rho_2;q)_k(aq/\rho_1\rho_2;q)_{n-k}(aq/\rho_1\rho_2)^k}{(q;q)_{n-k}(aq/\rho_1,aq/\rho_2;q)_n}\beta_n(aq,q) \\
\alpha_n'(a,q) &= (1-aq)\left(\frac{aq}{\rho_1\rho_2}\right)^n \frac{(\rho_1,\rho_2;q)_n}{(aq/\rho_1,aq/\rho_2;q)_n} \\
&\qquad \times \left(\frac{\alpha_n(aq,q)}{1-aq^{2n+1}} - aq^{2n-1}\frac{\alpha_{n-1}(aq,q)}{1-aq^{2n-1}}\right),
\end{aligned}$$

where $\alpha_{-1}(a,q) = 0$, then $(\alpha_n'(a,q), \beta_n'(a,q))$ is a Bailey pair with parameters (a,q).

The proofs of Theorems 5.1 and 5.2 rely on three special cases of this proposition. In the first, we let ρ_1 and ρ_2 approach infinity:

$$\text{(L1)}\qquad \begin{aligned}
\beta_n'(a,q) &= \sum_{k=0}^{n} \frac{a^k q^{k^2}}{(q;q)_{n-k}}\beta_k(aq,q), \\
\alpha_n'(a,q) &= (1-aq)a^n q^{n^2}\left(\frac{\alpha_n(aq,q)}{1-aq^{2n+1}} - aq^{2n-1}\frac{\alpha_{n-1}(aq,q)}{1-aq^{2n-1}}\right).
\end{aligned}$$

In the second, we let ρ_1 approach infinity and set $\rho_2 = -q^{1/2}$:
(L2)

$$\beta'_n(a,q) = \sum_{k=0}^{n} \frac{(-q^{1/2};q)_k a^k q^{k^2/2}}{(q;q)_{n-k}(-aq^{1/2};q)_n} \beta_k(aq,q),$$

$$\alpha'_n(a,q) = (1-aq)a^n q^{n^2/2} \frac{(-q^{1/2};q)_n}{(-aq^{1/2};q)_n} \left(\frac{\alpha_n(aq,q)}{1-aq^{2n+1}} - aq^{2n-1}\frac{\alpha_{n-1}(aq,q)}{1-aq^{2n-1}} \right).$$

In the third, we let ρ_1 approach infinity and set $\rho_2 = -a^{1/2}q$:
(L3)

$$\beta'_n(a,q) = \sum_{k=0}^{n} \frac{(-a^{1/2}q;q)_k a^{k/2} q^{(k^2-k)/2}}{(q;q)_{n-k}(-a^{1/2};q)_n} \beta_k(aq,q),$$

$$\alpha'_n(a,q) = (1-aq)a^{n/2} q^{(n^2-n)/2} \frac{(-a^{1/2}q;q)_n}{(-a^{1/2};q)_n} \left(\frac{\alpha_n(aq,q)}{1-aq^{2n+1}} - aq^{2n-1}\frac{\alpha_{n-1}(aq,q)}{1-aq^{2n-1}} \right).$$

We will use the fact that (L1) is the same as (L2) followed by (S4) which is also (L3) followed by (S6), a fact that is easily verified by observing their effect on $\alpha_n(aq,q)$.

To get the multisum in the second line of Theorem 5.1, we start with $\beta_k^{(0)}(aq,q)$ from the unit Bailey pair (UBP). If $i \geq 3$, then we apply (S1) $k-i+1$ times. We then apply (L1) once which changes the parameter aq to a, then apply (S1) $i-3$ times, and finally apply (S3). This yields

$$\beta_\infty^{(k)}(a,q) = \frac{1}{(q,-aq^{1/2};q)_\infty} \sum_{s_1,\dots,s_{k-1}=0}^{\infty} \frac{a^{s_1+\cdots+s_{k-1}}(-q^{1/2};q)_{s_1} q^{s_1^2/2+\cdots+s_{k-1}^2+s_i+\cdots+s_{k-1}}}{(q;q)_{s_1-s_2}\cdots(q;q)_{s_{k-1}}}.$$

If $i=2$, we apply (S1) $k-1$ times followed by (L2), and if $i=1$ we apply (S1) $k-1$ times followed by (S3). For purposes of illustration, we assume that $i \geq 3$; the other cases follow similarly.

We now apply the same sequence of transformations to $\alpha_n^{(0)}(aq,q)$:

$$\alpha_n^{(k-i+1)}(aq,q) = (-1)^n a^{(k-i+1)n} q^{(k-i+3/2)n^2+(k-i+1/2)n} \frac{(aq;q)_n}{(q;q)_n} \frac{(1-aq^{2n+1})}{(1-aq)},$$

$$\alpha_n^{(k)}(a,q) = (1-aq)a^{(i-1)n} q^{(i-3/2)n^2} \frac{(-q^{1/2};q)_n}{(-aq^{1/2};q)_n} \times \left(\frac{\alpha_n^{(k-i+1)}(aq,q)}{1-aq^{2n+1}} - aq^{2n-1}\frac{\alpha_{n-1}^{(k-i+1)}(aq,q)}{1-aq^{2n-1}} \right).$$

It follows that

$$\begin{aligned}
\beta_{\infty}^{(k)}(a,q) &= \frac{1}{(q,aq;q)_\infty}\sum_{n=0}^{\infty}(1-aq)a^{(i-1)n}q^{(i-3/2)n^2}\frac{(-q^{1/2};q)_n}{(-aq^{1/2};q)_n} \\
&\quad\times\left(\frac{\alpha_n^{(k-i+1)}(aq,q)}{1-aq^{2n+1}} - aq^{2n-1}\frac{\alpha_{n-1}^{(k-i+1)}(aq,q)}{1-aq^{2n-1}}\right) \\
&= \frac{1}{(q,aq;q)_\infty}\sum_{n=0}^{\infty}\frac{(1-aq)}{(1-aq^{2n+1})}a^{(i-1)n}q^{(i-3/2)n^2}\alpha_n^{(k-i+1)}(aq,q)\frac{(-q^{1/2};q)_n}{(-aq^{1/2};q)_n} \\
&\quad\times\left(1-a^iq^{(2n+1)(i-1/2)}\frac{1+q^{n+1/2}}{1+aq^{n+1/2}}\right) \\
&= \frac{1}{(q,aq;q)_\infty}\sum_{n=0}^{\infty}(-1)^na^{kn}q^{kn^2+(k-i+1/2)n}\frac{(aq,-q^{1/2};q)_n}{(q;q)_n(-aq^{1/2};q)_{n+1}} \\
&\quad\times\left(1+aq^{n+1/2}-a^iq^{(2n+1)(i-1/2)}(1+q^{n+1/2})\right).
\end{aligned}$$

To get the last multisum of Theorem 5.1, we again start with $\beta_n^{(0)}(aq,q)$ ¿from (UBP), we apply the pair of transformations (S3) followed by (S4) a total of $k-i+1$ times, then apply (L2), then apply the pair (S4) followed by (S3) a total of $i-2$ times. If $i=1$, then we just apply (S3)(S4) $k-1$ times, followed by (S3). Since (S3)(S4) = (S4)(S3) = (S1) and (L2)(S4) = (L1), this is equivalent to the sequence of transformations used to obtain the first two sums. □

A special case of this theorem is Melzer's conjecture (2.6) [10], a Fermionic form of the supersymmetric analogue $\hat{\chi}_{1,2k-2i-1}^{(2,4k)}$, $0\le i\le k-1$ of a Virasoro character:

$$\begin{aligned}
(q;q)_\infty\beta_\infty^{(2k-1)} &= \sum_{s_1,\cdots,s_{2k-2}\ge 0}\frac{q^{s_1^2/2+\cdots+s_{2k-2}^2/2+s_{2k-2i-1}+\cdots+s_{2k-5}+s_{2k-3}}}{(q;q)_{s_1-s_2}(q;q)_{s_2-s_3}\cdots(q;q)_{s_{2k-2}}} \\
&= \frac{(-q^{1/2};q)_\infty}{(q;q)_\infty}\left(1+\sum_{r=1}^{\infty}q^{kr^2}(q^{-(i+1/2)r}+q^{(i+1/2)r})(-1)^r\right) \\
&= \frac{(-q^{1/2};q)_\infty(q^{2k},q^{k-i-1/2},q^{k+i+1/2};q^{2k})_\infty}{(q;q)_\infty}.
\end{aligned}$$

Theorem 5.2 is proven similarly. We again start with the unit Bailey pair. To get the summation in the third line, we apply (S2) $2k-2i+1$ times, then (L3), then the pair (S6)(S5) $i-2$ times, and finally (S6). This is equivalent to (S2) followed by (S1) $k-i$ times followed by (L1) followed by (S1) $i-2$ times, which can be used to obtain the summations in the first and second lines.

With $a=1$ in Theorem 5.2, we get the even case of Melzer's (2.6), $\hat{\chi}_{1,2i}^{(2,4k)}$:

$$\begin{aligned}
&\sum_{s_1,\cdots,s_{2k-2}\ge 0}\frac{q^{s_1^2/2+\cdots+s_{2k-2}^2/2+s_{2i}+\cdots+s_{2k-4}+s_{2k-2}+S/2}}{(q;q)_{s_1-s_2}(q;q)_{s_2-s_3}\cdots(q;q)_{s_{2k-2}}} \\
&\qquad = \frac{(-q;q)_\infty}{(q;q)_\infty}(q^{2k},q^i,q^{2k-i};q^{2k})_\infty.
\end{aligned}$$

where $S=(s_1-s_2+s_3-s_4+\cdots+s_{2k-3}-s_{2k-2})$.

Melzer also conjectured [10, (2.3)] alternative forms for $\hat{\chi}_{1,2}^{(2,4k)}$ and $\hat{\chi}_{1,2k}^{(2,4k)}$. These follow easily in the same way.

6. Basic hypergeometric transformations.

It is well-known [3,4] that using Bailey's lemma twice with the unit Bailey pair gives the terminating version of the balanced ${}_4\phi_3$ to the very-well poised ${}_8\phi_7$ transformation. This transformation is a key one in the theory of basic hypergeometric series. In this section we record the analogous transformations obtained from Theorem 2.1-2.4 and Bailey's lemma. They should be the most important bibasic transformations.

First if we use Bailey's lemma, Theorem 2.1, and the unit Bailey pair we obtain a transformation of a balanced ${}_5\phi_4$ to the "mixed" very-well poised series

$$\begin{aligned}\frac{1}{(aq,q;q)_n}\sum_{r=0}^{n}&\frac{(q^{-n},-B;q)_r}{(aq^{n+1},-aq/B;q)_r}\frac{(\rho_1,\rho_2,a^2;q^2)_r}{(a^2q^2/\rho_1,a^2q^2/\rho_2,q^2;q^2)_r}\frac{1-a^2q^{4r}}{1-a^2}\left(\frac{a^2q^{n+2}}{B\rho_1\rho_2}\right)^r\\
&=\frac{(q^{1-n}B;q)_{2n}(-B)^nq^{\binom{n}{2}}}{(-aq/B,B;q)_n(q^2;q^2)_n}\\
&\qquad\times{}_5\phi_4\left(\begin{matrix}q^{-2n}, & B^2, & a^2q^2/\rho_1\rho_2, & -aq, & -aq^2\\ & a^2q^2/\rho_1, & a^2q^2/\rho_2, & Bq^{1-n}, & Bq^{2-n}\end{matrix}\,\middle|\, q^2;q^2\right).\end{aligned}$$

This is closely related to [8, (3.10.3)].

If we first use Theorem 2.1 and Bailey's lemma, and the unit Bailey pair we obtain another transformation of a special balanced ${}_5\phi_4$ to the "mixed" very-well poised series

$$\begin{aligned}&\sum_{r=0}^{n}\frac{(q^{-n},-B,\rho_1,\rho_2;q)_r}{(aq^{n+1},-aq/B,aq/\rho_1,aq/\rho_2;q)_r}\frac{1-a^2q^{4r}}{1-a^2}\left(\frac{aq^{n+1}}{B\rho_1\rho_2}\right)^r\\
&=\frac{(aq,aq/\rho_1\rho_2;q)_n}{(aq/\rho_1,aq/\rho_2;q)_n}\,{}_5\phi_4\left(\begin{matrix}q^{-n}, & Bq, & \rho_1 & \rho_2, & 1/B\\ & -aq/B, & B, & \rho_1\rho_2q^{-n}/a, & -q\end{matrix}\,\middle|\, q;q\right).\end{aligned}$$

The choice of Bailey's lemma followed by Theorem 2.4 gives yet another transformation for a special balanced ${}_6\phi_5$

$$\begin{aligned}&\frac{(q^3;q^3)_n(-a)^nq^{3n/2-n^2/2}}{(q;q)_n(1-aq^{2n+1})(aq^{2-n};q)_{2n-1}}\\
&\quad\times\sum_{r=0}^{n}\frac{(q^{-n};q)_r}{(aq^{n+1};q)_r}\frac{(\rho_1,\rho_2,a^3;q^3)_r}{(a^3q^3/\rho_1,a^3q^3/\rho_2,q^3;q^3)_r}\frac{1-a^3q^{6r}}{1-a^3}\left(\frac{a^2q^{n+2}}{\rho_1\rho_2}\right)^r\\
&={}_6\phi_5\left(\begin{matrix}q^{-3n}, & a^3q^3/\rho_1\rho_2, & a^{3/2}q^{3/2} & -a^{3/2}q^{3/2}, & a^{3/2}q^3 & -a^{3/2}q^3\\ & a^3q^3/\rho_1, & a^3q^3/\rho_2, & aq^{2-n}, & aq^{3-n}, & aq^{4-n}\end{matrix}\,\middle|\, q^3;q^3\right).\end{aligned}$$

As our final example we take Theorem 2.2 followed by Theorem 2.3 to obtain

$$\begin{aligned}&\frac{1}{(a^{12}q^{12};q^{12})_n}\sum_{r=0}^{n}\frac{(q^{-12n};q^{12})_r}{(a^{12}q^{12n+12};q^{12})_r}\frac{(-Bq,a^2;q^2)_r}{(-qa^2/B,q^2;q^2)_r}\frac{1-a^2q^{4r}}{1-a^2}\left(\frac{a^6q^{12n+5}}{B}\right)^r\\
&=\frac{(a^4q^4;q^4)_{3n}}{(a^{12}q^{12};q^{12})_{2n}}\\
&\quad\times{}_5\phi_4\left(\begin{matrix}q^{-4n}, & \omega q^{-4n}, & \omega^2q^{-4n} & qa^2/B, & q^3a^2/B\\ & q^{-12n}/a^4, & a^4q^2/B^2, & -q^2a^2, & -q^4a^2,\end{matrix}\,\middle|\, q^4;q^4\right).\end{aligned}$$

where ω is a primitive cube root of 1.

7. Conclusions.

It is clear that Theorems 2.1–2.4 and Bailey's lemma may be iterated in many different ways. It is possible to use them to prove all sixteen families of multisum identities given by Stembridge in [13]. For example, to prove (I14), we start with the unit Bailey pair with $a = 1$, iterate (S1) followed by Proposition 4.1 $k-1$ times, apply (S1) one more time, and then apply (L1) with $a = 1/q$ to get

$$\begin{aligned}
(q;q)_\infty \sum_{s_1,\dots,s_k\ge 0} & \frac{q^{s_1^2+\cdots+s_k^2-s_1+s_2+\cdots+s_k}}{(q;q)_{s_1-s_2}\cdots(q;q)_{s_{k-1}}} \\
&= 2+\sum_{r=1}^{\infty}(q^{(k+3/2)r^2}(q^r+q^{-r})(q^{(k-1/2)r}+q^{-(k-1/2)r})(-1)^r \\
&= (q,q^{2k+2},q^{2k+3};q^{2k+3})_\infty + (q^3,q^{2k},q^{2k+3};q^{2k+3})_\infty .
\end{aligned}$$

We subtract

$$(q;q)_\infty \sum_{s_1,\dots,s_k\ge 0} \frac{q^{s_1^2+\cdots+s_k^2+s_1+s_2+\cdots+s_k}}{(q;q)_{s_1-s_2}\cdots(q;q)_{s_{k-1}}} = (q,q^{2k+2},q^{2k+3};q^{2k+3})_\infty$$

¿from each side to get Stembridge's (I14):

$$(q;q)_\infty \sum_{s_1,\dots,s_k\ge 0} \frac{q^{s_1^2+\cdots+s_k^2-s_1+s_2+\cdots+s_k}(1-q^{2s_1})}{(q;q)_{s_1-s_2}\cdots(q;q)_{s_{k-1}}} = (q^3,q^{2k},q^{2k+3};q^{2k+3})_\infty .$$

For other Rogers-Ramanujan identities, one could consider the monoid generated by the symbols (S1)–(S6), (D1)–(D6), (T1), and (L1)–(L3) subject to the relations

$$\begin{aligned}
(S1) &= (S2)(S2) = (S3)(S4) = (S4)(S3) = (S5)(S6) = (S6)(S5),\\
(L1) &= (L2)(S4) = (L3)(S6),\\
(D1) &= (D2)(S2),\\
(D3) &= (S2)(D2),\\
(D2)(S2) &= (S2)(S2)(D2).
\end{aligned}$$

The number of different representations for a given identity is the number of words representing a given word.

We state here a few of the identities which may be obtained from such words. If we take (D1)(T1)(S1) the result is

$$\sum_{s_1,s_2\ge 0} \frac{q^{3s_1^2+s_2^2+s_2}(q;q)_{3s_1-s_2}}{(q^3;q^3)_{2s_1}(q^3;q^3)_{s_1-s_2}(q^2;q^2)_{s_2}} = \frac{(q^{10},q^4,q^6;q^{10})_\infty}{(q^3;q^3)_\infty}. \tag{7.1}$$

For (S1)(T1)(D1)(S1) we have

$$\sum_{s_1,s_2\ge 0} \frac{q^{3s_1^2+2s_2^2}(-q^3;q^6)_{s_1}(q^2;q^2)_{3s_1-s_2}}{(q^6;q^6)_{2s_1}(q^6;q^6)_{s_1-s_2}(q^2;q^2)_{s_2}} = \frac{(q^{16},q^7,q^9;q^{16})_\infty}{(q^3,q^9,q^{12};q^{12})_\infty}. \tag{7.2}$$

For (S1)(T2)(S1) we have

$$\sum_{s_1,s_2\geq 0} \frac{q^{3s_1^2+2s_2^2}(-q^3;q^6)_{s_1}(q^2;q^2)_{3s_1-s_2}}{(q^6;q^6)_{2s_1}(q^6;q^6)_{s_1-s_2}(q^2;q^2)_{s_2}} = \frac{(q^{16},q^7,q^9;q^{16})_\infty}{(q^3,q^9,q^{12};q^{12})_\infty}. \tag{7.3}$$

For (S1)(T1)(T1)(S1) we have

$$\sum_{s_1,s_2,s_3\geq 0} \frac{q^{9s_1^2+3s_2^2+s_3^2}(q^3;q^3)_{3s_1-s_2}(q;q)_{3s_2-s_3}}{(q^9;q^9)_{2s_1}(q^9;q^9)_{s_1-s_2}(q^3;q^3)_{2s_2}(q^3;q^3)_{s_2-s_3}(q;q)_{s_3}} = \frac{(q^{29},q^{14},q^{15};q^{29})_\infty}{(q^9;q^9)_\infty}. \tag{7.4}$$

Finally we remark that the new multisums should lead to new combinatorial interpretations of theorems such as Theorem 4.3.

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