MOMENTS OF ORTHOGONAL POLYNOMIALS AND COMBINATORICS

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Abstract. This paper is a survey on combinatorics of moments of orthogonal polynomials and linearization coefficients. This area was started by the seminal work of Flajolet followed by Viennot at the beginning of the 1980's. Over the last 30 years, several tools were conceived to extract the combinatorics and compute these moments. A survey of these techniques is presented, with applications to polynomials in the Askey scheme.

1. Introduction

In this article we will consider polynomials $P_n(x)$ in one variable x with coefficients in a commutative ring $\mathbb{K}$ indexed by a non-negative integer n. In general $\mathbb{K}$ will be $\mathbb{C}$ or $\mathbb{R}$. Sometimes these polynomials will depend on formal variables a, b, c, d and q.

Given a sequence $\{\mu_n\}_{n\ge0}$ of elements of $\mathbb{K}$, there exists a unique linear functional $\mathcal{L}$ on the space of polynomials, $\mathcal{L}:\mathbb{K}[x]\to\mathbb{K}$ such that $\mu_n=\mathcal{L}(x^n)$. The sequence $\{\mu_n\}_{n\ge0}$ is called the *moment sequence* of this linear functional.

Definition 1.1. A sequence $\{P_n(x)\}_{n\ge0}$ is called an *orthogonal polynomial sequence* (OPS) with respect to a linear functional $\mathcal{L}$ if

(1) $P_n(x)$ is a polynomial of degree n for each $n\ge0$,
(2) $\mathcal{L}(P_n(x)P_m(x))=K_n\delta_{n,m}$ for all $n,m\ge0$, where $K_n\ne0$.

In this case we also call $P_n(x)$ an *orthogonal polynomial*.

Note that if $\mathcal{L}$ is a linear function for an OPS, then any linear functional $\mathcal{L}'$ which is obtained by multiplying a nonzero constant to $\mathcal{L}$ is also a linear functional for the OPS. Conversely, if $\mathcal{L}$ and $\mathcal{L}'$ are linear functionals for an OPS, then $\mathcal{L}=c\cdot\mathcal{L}'$ for a nonzero constant c. Thus, there is a unique linear functional $\mathcal{L}$ for an OPS such that $\mathcal{L}(1)=1$. In this case we say that $\mathcal{L}$ and its corresponding moments are *normalized*. Several books were written on the subject [2, 5, 31, 27]. Here we focus on the moments of these polynomials and their relation to combinatorics started by Flajolet [12] and Viennot [33].

The *Hankel determinant* of the moment sequence $\{\mu_n\}_{n\ge0}$ is

$$\Delta_n=\begin{vmatrix}\mu_0&\mu_1&\cdots&\mu_n\\ \mu_1&\mu_2&\cdots&\mu_{n+1}\\ \vdots&\vdots&\ddots&\vdots\\ \mu_n&\mu_{n+1}&\cdots&\mu_{2n}\end{vmatrix}.$$

Theorem 1.2. [31] *A linear functional $\mathcal{L}$ has an OPS if and only if $\Delta_n\ne0$ for all $n\ge0$. Moreover, an OPS is uniquely determined up to a constant multiple. Indeed, the polynomials*

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$P_n(x)$ in monic form, are given explicitly by :

$$P_n(x) = \frac{1}{\Delta_{n-1}} \begin{vmatrix} \mu_0 & \mu_1 & \cdots & \mu_n \\ \mu_1 & \mu_2 & \cdots & \mu_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{n-1} & \mu_n & \cdots & \mu_{2n-1} \\ 1 & x & \cdots & x^n \end{vmatrix}.$$

One of the most important theorems in the classical theory of orthogonal polynomials is the three-term recurrence relation.

Theorem 1.3. *Any monic OPS $\{P_n(x)\}_{n\ge 0}$ satisfies a three-term recurrence relation*

$$P_{n+1}(x) = (x - b_n)P_n(x) - \lambda_n P_{n-1}(x),$$

for $n \ge 1$, for some sequences $b_0, b_1, \dots$ and $\lambda_1, \lambda_2, \dots$, where $\lambda_n \ne 0$.

For non-monic orthogonal polynomials we can always rescale them to get monic ones. The three-term recurrence relation and the moments can be easily obtained as follows.

Proposition 1.4. *Suppose that $P_n(x)$ is an orthogonal polynomial with a three-term recurrence relation*

$$P_{n+1}(x) = (A_n x - B_n)P_n(x) - C_n P_{n-1}(x).$$

Then the rescaled orthogonal polynomial $\widetilde{P}_n(x) = a_n P_n(tx)$ satisfies

$$\widetilde{P}_{n+1}(x) = (\widetilde{A}_n x - \widetilde{B}_n)\widetilde{P}_n(x) - \widetilde{C}_n \widetilde{P}_{n-1}(x),$$

where

$$\widetilde{A}_n = \frac{a_{n+1}}{a_n} t A_n, \qquad \widetilde{B}_n = \frac{a_{n+1}}{a_n} B_n, \qquad \widetilde{C}_n = \frac{a_{n+1}}{a_{n-1}} C_n.$$

Moreover, if $\{\mu_n\}_{n\ge 0}$ and $\{\widetilde{\mu}_n\}_{n\ge 0}$ are moment sequences of $P_n(x)$ and $\widetilde{P}_n(x)$, respectively, then there exists a constant c such that for all $n \ge 0$,

$$\mu_n = c \cdot t^n \widetilde{\mu}_n.$$

The orthogonality of orthogonal polynomials with real coefficients is defined by a measure $w(x)$, that is a functional $\mathcal{L} : \mathbb{R}[x] \to \mathbb{R}$ defined by the Stieltjes integral

$$\mathcal{L}(P(x)) = \int_{-\infty}^{\infty} P(x) \mathrm{d}w(x).$$

The most general classical orthogonal polynomials are called the Askey-Wilson polynomials [3]. All of the other classical polynomials can be obtained as a specialization or a limit of these polynomials. The Askey-Wilson polynomials depend upon five parameters: a, b, c, d, and q and are defined by basic hypergeometric series. We use the usual notation

$$(a;q)_k = \prod_{i=0}^{k-1}(1 - aq^i), \quad (a_1, \dots, a_m; q)_k = \prod_{i=1}^{m}(a_i;q)_k.$$

Definition 1.5. The *Askey-Wilson polynomials* are defined by

$$p_n(x;a,b,c,d|q) = \frac{(ab,ac,ad;q)_n}{a^n} \sum_{k=0}^{n} \frac{(q^{-n}, abcdq^{n-1}, az, a/z;q)_k}{(q,ab,ac,ad;q)_k} q^k \tag{1}$$

with $x = \frac{z+z^{-1}}{2}$.

These functions are polynomials in x of degree n due to the relation

$$(az, a/z; q)_k = \prod_{j=0}^{k-1} (1 - 2axq^j + a^2q^{2j}).$$

A representing measure for the Askey-Wilson polynomials may be given for $0 < q < 1$ and $\max\{|a|, |b|, |c|, |d|\} < 1$. The orthogonality relation is

$$\int_0^\pi p_n(\cos\theta; a, b, c, d|q) p_m(\cos\theta; a, b, c, d|q) w(\cos\theta; a, b, c, d|q) d\theta = 0, \qquad n \neq m,$$

where $x = \cos\theta$ and the measure is given by

$$w(x; a, b, c, d|q) = \frac{(z^2, z^{-2}; q)_\infty}{(az, a/z, bz, b/z, cz, c/z, dz, d/z; q)_\infty}.$$

The enumeration formula and the combinatorics of the moments of these polynomials were recently presented in [9, 10, 25, 26]. A recent catalog on known enumeration formulas for moments appeared in [28].

In this paper, we will consider the moments of orthogonal polynomials of the Askey scheme, and emphasize their relationship to combinatorial enumeration. In Section 2, we give a general interpretation of the orthogonal polynomials and their moments using paths [33]. In Section 3, we introduce some basic combinatorics related to the moments of Hermite, Charlier and Laguerre polynomials and their q-analogues. This was first started in the memoir of Viennot [33] and then in a series of papers by several authors. In Section 4, we present an odd-even trick which is often useful to compute moments and can also give non-trivial combinatorial results. In Section 5, we focus on the case of the Askey-Wilson moments and their enumeration formula. In Section 6, we present some modified moments coming from the Askey-Wilson basis. We end in Section 7 with linearization coefficients.

This paper is a survey and contains a collection of diverse results; so that the interested reader can learn a series of important tools built for the study of moments of orthogonal polynomials.

2. Path interpretation of the polynomials and the moments

For general orthogonal polynomials which satisfy Theorem 1.3, the moments μ_n are always polynomials in the three-term recurrence coefficients with non-negative integral coefficients. For example,

$$\mu_0 = 1, \quad \mu_1 = b_0, \quad \mu_2 = b_0^2 + \lambda_1, \quad \mu_3 = b_0^3 + 2b_0\lambda_1 + b_1\lambda_1.$$

The orthogonal polynomials $p_n(x)$ may also be given as polynomials in x and the three-term recurrence coefficients, but not necessarily positive. For example,

$$p_0(x) = 1, \quad p_1(x) = x - b_0, \quad p_2(x) = x^2 - (b_0 + b_1)x + (b_0b_1 - \lambda_1).$$

The purpose of this section is to give combinatorial interpretations for these two phenomena using paths.

Paths live in the quarter plane $\Pi = \mathbb{N} \times \mathbb{N}$. A *path* ω is a sequence $\omega = (s_0, \dots, s_n)$ where $s_i = (x_i, y_i) \in \Pi$. The s_i's are the *vertices* of the path. The starting vertex is s_0, the ending vertex is s_n and (s_i, s_{i+1}) is an *elementary step* of the path.

The step (s_i, s_{i+1}) is called

- *North* if $x_i = x_{i+1}$ and $y_i + 1 = y_{i+1}$;
- *North-North* if $x_i = x_{i+1}$ and $y_i + 2 = y_{i+1}$;
- *North-East* if $x_i + 1 = x_{i+1}$ and $y_i + 1 = y_{i+1}$;
- *South-East* if $x_i + 1 = x_{i+1}$ and $y_i - 1 = y_{i+1}$; and
- *East* if $x_i + 1 = x_{i+1}$ and $y_i = y_{i+1}$.

The paths are weighted. There is a partial application $\mathrm{wt} : \Pi \times \Pi \to \mathbb{K}$ which associates to each step (s_i, s_{i+1}) a *weight* $\mathrm{wt}(s_i, s_{i+1}) \in \mathbb{K}$. The *weight* of the path ω is the product of the weight of the steps :

$$\mathrm{wt}(\omega) = \prod_{i=0}^{n-1} \mathrm{wt}(s_i, s_{i+1}).$$

Definition 2.1. [33] A *Favard path* is a path $\eta = (s_0, \dots, s_n)$ starting at $s_0 = (0,0)$ with three types of elementary steps: North, North-North and North-East. The weight of the elementary step (s_i, s_{i+1}) is x if the step is North-East and $-b_k$ (resp. $-\lambda_{k+1}$) if the step is North (resp. North-North) and the y-coordinate of s_i is k. The *length* of the path is the y-coordinate of s_n.

Let Fav_n be the set of Favard paths of length n.

Lemma 2.2. [33] *Let $\{P_n(x)\}$ be a sequence of polynomials which satisfies the recurrence of Theorem 1.3 and with boundary conditions $P_0(x) = 1$ and $P_1(x) = x - b_0$. Then*

$$P_n(x) = \sum_{\omega \in \mathrm{Fav}_n} \mathrm{wt}(\omega).$$

A *Motzkin path* of length n is a path consisting of North-East steps, East steps, and South-East steps which lies in the first quadrant. Let $\mathrm{Mot}_{n,k,\ell}$ be the set of Motzkin paths from $(0, k)$ to (n, ℓ), and let $\mathrm{Mot}_n = \mathrm{Mot}_{n,0,0}$.

Suppose that two sequences $b = \{b_n\}_{n\ge 0}$ and $\lambda = \{\lambda_n\}_{n\ge 1}$ are given. For a Motzkin path P, we define $\mathrm{wt}_{b,\lambda}(P)$ to be the product of the weight of every step in P, where the weight of each step is defined as follows.

(1) A North-East step has weight 1.
(2) An East step has weight b_k, where k is the y-coordinate of starting point.
(3) A South-East step has weight λ_k, where k is the y-coordinate of starting point.

For the remainder of this section we assume that $P_n(x)$ are orthogonal polynomials given by $P_{-1}(x) = 0$, $P_0(x) = 1$, and for $n \ge 1$,

$$P_{n+1}(x) = (x - b_n)P_n(x) - \lambda_n P_{n-1}(x),$$

with the normalized linear functional $\mathcal{L}$ and the nth moment $\mu_n = \mathcal{L}(x^n)$.

Theorem 2.3. [33] *We have*

$$\mu_n = \sum_{\omega \in \mathrm{Mot}_n} \mathrm{wt}_{b,\lambda}(\omega).$$

We will sketch a proof of a generalization of Theorem 2.3 that also gives a combinatorial proof of the orthogonality.

Theorem 2.4. [33] *For all n, k, ℓ,*

$$\mathcal{L}(x^n P_k(x) P_\ell(x)) = \lambda_1 \cdots \lambda_\ell \sum_{\omega \in \mathrm{Mot}_{n,k,\ell}} \mathrm{wt}(\omega). \tag{2}$$

Proof. (Sketch [33].) The proof of this theorem uses the combinatorial interpretation of the polynomials as Favard paths. Let $E_{n,k,\ell}$ be the set of triplets made of

- A Favard path f of length k
- A Favard path g of length ℓ
- A Motzkin path ω of length $n + p(f) + p(g)$ where $p(f)$ is the number of North-East steps of f.

Let $F_{n,k,\ell}$ be the subset of $E_{n,k,\ell}$ where $p(f) = k$, $p(g) = \ell$ and ω starts with k North-East steps and ends with ℓ South-East steps. Proving the theorem is equivalent to defining a sign-reversing involution θ on $E_{n,k,\ell}$ whose fixed points are exactly the elements in the set $F_{n,k,\ell}$. Let j be the minimal index such that (s_j, s_{j+1}) is an East step or a South-East step in ω. If $j < k$ then let $h(\omega) = y_j$ otherwise $h(\omega) = \infty$. Let f_j be the first step of f which is a North or a North-North step. Let $h(f) = j$ if such a j exists otherwise $h(f) = \infty$. If $h(\omega)$ or $h(f)$ is finite, then if $h(f)-1$ is smaller than or equal to $h(\omega)$, if the $h(f)^{th}$ step of f is North-North, we add a North-East step and a South-East step to ω in position $h(f)-1$ and change the North-North step to two North-East steps; otherwise (it is a North step), we add an East step to ω in position $h(f)$. The step in f is changed to one North-East step. Otherwise ($h(f) - 1 > h(\omega)$), we take off the first East step or the first South-East step and its previous North-East step from ω and change two North-East steps in f by a North-North step or one North-East step in f by a North step. One can easily check that this changes the sign. If both h's are infinite, we apply a similar algorithm on g and the last ℓ steps of ω. The only triplets where the algorithm cannot be applied are exactly the ones in $F_{n,k,\ell}$. □

Note that this implies that

$$\mathcal{L}(P_k(x)P_\ell(x)) = \lambda_1 \ldots \lambda_\ell \delta_{k\ell}.$$

and this gives a proof of the orthogonality relation.

Let

$$C_i(z) = \cfrac{1}{1 - b_{0+i}z - \cfrac{\lambda_{1+i}z^2}{1 - b_{1+i}z - \cfrac{\lambda_{2+i}z^2}{\ddots}}}.$$

Using the theory of Flajolet [12] we have

Corollary 2.5. *We have*

$$\sum_{n\geq 0} \mu_n z^n = \cfrac{1}{1 - b_0 z - \cfrac{\lambda_1 z^2}{1 - b_1 z - \cfrac{\lambda_2 z^2}{\ddots}}} = C_0(z);$$

$$\sum_{n\geq 0} \mathcal{L}(x^n P_\ell(x)) z^n = \lambda_1 \cdots \lambda_\ell z^\ell \prod_{i=0}^{\ell} C_i(z);$$

$$\sum_{n\geq 0} \mathcal{L}(x^n P_\ell(x) P_k(x)) z^n = \sum_{j=0}^{\min(k,\ell)} \lambda_1 \cdots \lambda_j C_j(z) \prod_{i=j+1}^{\ell} \lambda_i z C_i(z) \prod_{i=j+1}^{k} \lambda_i z C_i(z).$$

The last equation in Corollary 2.5 can be proved by observing the fact that we can uniquely decompose a Motzkin path $\omega \in \mathrm{Mot}_{n,k,\ell}$ as

$$\omega = \omega_{k-j} D \cdots \omega_1 D \omega_0 U \omega'_1 \cdots U \omega'_{\ell-j},$$

where j is a nonnegative integer, each ω_i and ω'_i is a Motzkin path in which the starting and ending vertices have the smallest y-coordinates, D is a South-East step, and U is a North-East step.

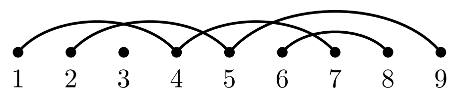

FIGURE 1. The diagram representing the set partition $\{\{1,4,7\},\{2,5,9\},\{3\},\{6,7\}\}$.

3. COMBINATORICS

In this section we show that the moments of Hermite, Charlier, and Laguerre polynomials are equal to the numbers of perfect matchings, set partitions, and permutations respectively.

A *set partition* of a set X is a collection of mutually disjoint nonempty subsets of X whose union is X. We denote by Π_n the set of set partitions of $[n] = \{1,2,\dots,n\}$. Let $\pi = \{B_1, B_2, \dots, B_k\} \in \Pi_n$. Each element B_i of π is called a *block*. The *size* of a block B_i is the cardinality $|B_i|$. An *edge* of π is a pair (i,j) of integers such that $i<j$, both i and j are contained in the same block, and the block containing i and j does not contain any integer k with $i<k<j$.

We will represent a set partition of $[n]$ as a diagram by placing integers $1,2,\dots,n$ in a row and connecting i and j for each edge (i,j) of the set partition, see Figure 1. In this pictorial representation, the following definitions are natural.

Let $\pi \in \Pi_n$. An integer $i \in [n]$ is called a *singleton*, an *opener*, a *closer*, or a *continuation* of π, if i is the unique element in a block of size 1, the minimum of a block of size at least 2, the maximum of a block of size at least 2, or none of these, respectively. For example, if π is the set partition in Figure 1, then 3 is a singleton, $1,2,6$ are openers, $7,8,9$ are closers, and $4,5$ are continuations.

A *crossing* of π is a set of two edges (a,b) and (c,d) such that $a<c<b<d$. Let crossing(π) denote the number of crossings of π. For example, if π is the set partition in Figure 1, then crossing$(\pi)=4$ since there are four crossings $\{(1,4),(2,5)\}$, $\{(2,5),(4,7)\}$, $\{(4,7),(5,9)\}$, $\{(4,7),(6,8)\}$.

A *matching* is a set partition in which every block has one or two elements. A matching is called *perfect* if every block has 2 elements. Let $\mathcal{M}_n$ denote the set of perfect matchings of $[n]$. One can easily see that $|\mathcal{M}_{2n+1}|=0$ and

$$|\mathcal{M}_{2n}| = (2n-1)!! = 1\cdot 3 \cdots (2n-1).$$

Since $\pi \in \mathcal{M}_{2n}$ is also a set partition, we can represent π using a diagram, and crossings of π are defined.

3.1. Hermite polynomials.

The *Hermite polynomials* $H_n(x)$ are defined by

$$H_{-1}(x)=0, \quad H_0(x)=1, \text{ and}$$

$$H_{n+1}(x) = xH_n(x) - nH_{n-1}(x), \text{ for } n\ge 0.$$

A *Dyck path* of length n is a Motzkin path in Mot_n without East steps. A *Hermite history of length* n is a Dyck path of length n in which every South-East step between the lines $y=i$ and $y=i-1$ is labeled by an integer in $\{0,1,\dots,i-1\}$.

By Theorem 2.3, the nth moment μ_n of the Hermite polynomials is

$$\mu_n = \sum_{P\in \mathrm{Mot}_n} \mathrm{wt}_{b,\lambda}(P),$$

where $b=\{b_n\}_{n\ge0}$ and $\lambda=\{\lambda_n\}_{n\ge1}$ are given by $b_n=0$ for all $n\ge0$ and $\lambda_n=n$ for $n\ge1$. Let $P\in\mathrm{Mot}_n$. Then $\mathrm{wt}_{b,\lambda}(P)=0$ if P has an East step. Otherwise, P is a Dyck path and

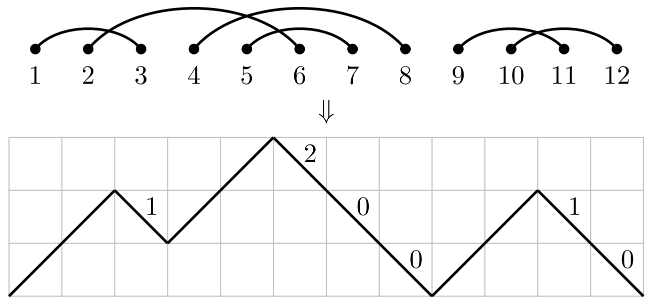

FIGURE 2. An example of the bijection ψ_H sending perfect matchings to Hermite histories.

$\mathrm{wt}_{b,\lambda}(P)$ is equal to the number of Hermite histories whose underlying Dyck path is P. Thus μ_n is equal to the number of Hermite histories of length n.

We now give a bijection $\psi_H : \mathcal{M}_n \to \mathcal{H}_H(n)$ from the set $\mathcal{M}_n$ of perfect matchings of $[n]$ to the set $\mathcal{H}_H(n)$ of Hermite histories of length n, which appears in [11, Section 2]. It suffices to consider the case when n is even since both sets are empty otherwise. Let $\pi \in \mathcal{M}_{2n}$ be a perfect matching of $[2n]$. Then the corresponding Hermite history $\psi_H(\pi)$ is defined as follows. For $i = 1, 2, \dots, 2n$, if i is an opener of π, then the ith step of $\psi_H(\pi)$ is a North-East step. If i is a closer of π, then the ith step of $\psi_H(\pi)$ is a South-East step labeled by the number of crossings $\{(a,b),(c,d)\}$ of π with $a < c < b < d$ and $b = i$. See Figure 2 for an example. One can check that ψ_H is a bijection. Thus we have the following theorem.

Theorem 3.1. *The nth moment of the Hermite polynomial is equal to the number of perfect matchings of $[n]$. In other words, $\mu_{2n+1} = 0$ and $\mu_{2n} = (2n-1)!!$.*

Classically, Theorem 3.1 may be derived using the representing measure for the Hermite linear functional

$$\mathcal{L}_{\mathrm{Hermite}}(p(x)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} p(x) e^{-x^2/2} dx.$$

The *q-Hermite polynomials* $H_n(x|q)$ are defined by $H_{-1}(x|q) = 0$, $H_0(x|q) = 1$, and for $n \geq 0$,

$$H_{n+1}(x|q) = 2xH_n(x|q) - (1-q^n)H_{n-1}(x|q).$$

For our purposes, it is more convenient to consider the *rescaled q-Hermite polynomials*

$$\widetilde{H}_n(x|q) = \frac{H_n(\frac{\sqrt{1-q}}{2}x|q)}{(1-q)^{n/2}}.$$

Then by Proposition 1.4, we have $\widetilde{H}_{-1}(x|q) = 0$, $\widetilde{H}_0(x|q) = 1$, and for $n \geq 1$,

$$\widetilde{H}_{n+1}(x|q) := x\widetilde{H}_n(x|q) - [n]_q \widetilde{H}_{n-1}(x|q).$$

where we have used

$$[n]_q = \frac{1-q^n}{1-q} = 1 + q + \cdots + q^{n-1}.$$

We see that the nth moment of $\widetilde{H}_n(x|q)$ is

$$\mu_n = \sum_{\eta \in \mathcal{H}_H(n)} q^{|\eta|},$$

where $|\eta|$ is the sum of labels of South-East steps. The bijection $\psi_H : \mathcal{M}_n \to \mathcal{H}_H(n)$ satisfies $\mathrm{crossing}(\pi) = |\psi_H(\eta)|$. Thus we obtain the following theorem.

Theorem 3.2. [19, Eq. (3.8)] *The nth moment of the rescaled q-Hermite polynomials $\widetilde{H}_n(x|q)$ is*

$$\mu_n = \sum_{\pi\in\mathcal{M}_n} q^{\mathrm{crossing}(\pi)}.$$

The rescaled *discrete q-Hermite polynomials* $\widetilde{h}_n(x;q)$ are defined by $\widetilde{h}_{-1}(x;q) = 0$, $\widetilde{h}_0(x;q) = 1$, and for $n \ge 0$,

$$\widetilde{h}_{n+1}(x;q) = x\widetilde{h}_n(x;q) - q^{n-1}[n]_q\widetilde{h}_{n-1}(x;q).$$

Again the odd moments are 0. The analogous version to Theorem 3.2 is the next result.

Theorem 3.3. [30, p. 310, (5.4)] *The $2n^{th}$ moment of the rescaled discrete q-Hermite polynomials $\widetilde{h}_n(x;q)$ is*

$$\mu_{2n} = [1]_q[3]_q\cdots[2n-1]_q = \sum_{\pi\in\mathcal{M}_{2n}} q^{\mathrm{crossing}(\pi)+2\,\mathrm{nesting}(\pi)},$$

where $\mathrm{nesting}(\pi)$ *is the number of pairs of two edges (a,b) and (c,d) of π such that $a < c < d < b$.*

3.2. **Charlier polynomials.** The *Charlier polynomials* $C_n(x)$ are defined by

$$C_{n+1}(x) = (x-n-1)C_n(x) - nC_{n-1}(x),$$

for $n \ge 1$ with initial conditions $C_{-1}(x) = 0$ and $C_0(x) = 1$.

Theorem 3.4. *The nth moment of the Charlier polynomials is equal to the number of set partitions of $[n]$.*

Theorem 3.4 may be derived using the representing measure for the Charlier linear functional

$$\mathcal{L}_{\mathrm{Charlier}}(p(x)) = \frac{1}{e}\sum_{n=0}^{\infty} p(n)\frac{1}{n!}.$$

We will instead prove a generalization of the above theorem using the *q-Charlier polynomials* $C_n(x,a;q)$ given by

$$C_{n+1}(x,a;q) = (x-a-[n]_q)C_n(x,a;q) - a[n]_qC_{n-1}(x,a;q).$$

A *Charlier history of length n* is a Motzkin path of length n in which every South-East step between the lines $y = i$ and $y = i-1$ is labeled by an integer in $\{0,1,\dots,i-1\}$ and every East on the line $y = i$ is either unlabeled or labeled by an integer in $\{0,1,\dots,i-1\}$.

By Theorem 2.3, the nth moment μ_n of the Charlier polynomials is

$$\mu_n = \sum_{P\in\mathrm{Mot}_n} \mathrm{wt}_{b,\lambda}(P),$$

where $b = \{b_n\}_{n\ge0}$ and $\lambda = \{\lambda_n\}_{n\ge1}$ are given by $b_n = n + a$ for all $n \ge 0$ and $\lambda_n = an$ for $n \ge 1$. Let $P \in \mathrm{Mot}_n$. Then

$$\mathrm{wt}_{b,\lambda}(P) = \sum_{\eta} a^{\mathrm{unlabeled}(\eta)} q^{|\eta|},$$

where the sum is over all Charlier histories η with underlying Motzkin path P, $\mathrm{unlabeled}(\eta)$ is the number of unlabeled steps of η and $|\eta|$ is the sum of labels in η.

We now give a bijection $\psi_C : \Pi_n \to \mathcal{H}_C(n)$ from the set Π_n of set partitions of $[n]$ to the set $\mathcal{H}_C(n)$ of Charlier histories of length n. This map can be considered as a special case of

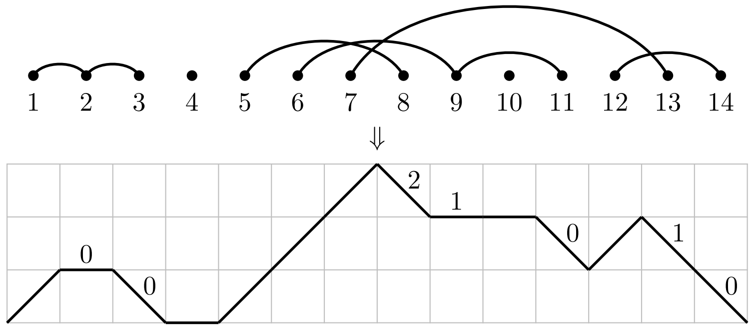

FIGURE 3. An example of the bijection ψ_C sending set partitions to Charlier histories.

Foata and Zeilberger's map [13], which will be described in the next subsection. Let $\pi \in \Pi_n$ be a set partition of $[n]$. Then the corresponding Charlier history $\psi_C(\pi) = \eta$ is defined as follows. For $i = 1, 2, \dots, n$, if i is an opener of π, then the ith step of η is a North-East step. If i is a closer of π, then the ith step of η is a South-East step labeled by the number of crossings $\{(a, b), (c, d)\}$ of π with $a < c < b < d$ and $b = i$. If i is a singleton of π, then the ith step of η is an unlabeled East step. If i is a continuation of π, then the ith step of η is an East step labeled by the number of crossings $\{(a, b), (c, d)\}$ of π with $a < c < b < d$ and $b = i$. See Figure 3 for an example.

It is easy to see that ψ_C is a bijection such that if $\psi_C(\pi) = \eta$ then $\mathrm{unlabeled}(\eta) = \mathrm{block}(\pi)$ and $|\eta| = \mathrm{crossing}(\pi)$. This implies the following theorem due to Kim, Stanton, and Zeng [24, Theorem 4].

Theorem 3.5. *The* q-Charlier polynomials $C_n(x, a; q)$ *given by*

$$C_{n+1}(x, a; q) = (x - a - [n]_q)C_n(x, a; q) - a[n]_q C_{n-1}(x, a; q)$$

has the nth moment

$$\mu_n = \sum_{\pi \in \Pi_n} a^{\mathrm{block}(\pi)} q^{\mathrm{crossing}(\pi)}.$$

3.3. **Laguerre polynomials.** The *Laguerre polynomials* $L_n(x)$ are defined by

$$L_{n+1}(x) = (x - 2n - 1)L_n(x) - n^2 L_{n-1}(x).$$

Theorem 3.6. *The nth moment of the Laguerre polynomials $L_n(x)$ is equal to the number of permutations of $[n]$, that is, $\mu_n = n!$.*

Theorem 3.6 may be derived using the representing measure for the Laguerre linear functional

$$\mathcal{L}_{\mathrm{Laguerre}}(p(x)) = \int_0^\infty p(x)e^{-x}dx.$$

We will consider the following generalization of the above theorem.

The *q-Laguerre polynomials* $L_n(x; q)$ are defined by

$$L_{n+1}(x; q) = (x - [n]_q - y[n+1]_q)L_n(x; a) - y[n]_q^2 L_{n-1}(x; q).$$

A *Laguerre history of length n* is a Motzkin path of length n in which every North-East or South-East step between the lines $y = i$ and $y = i-1$ is labeled by an integer in $\{0, 1, \dots, i-1\}$ and every East step on the line $y = i$ is either unlabeled or labeled by an integer in $\{-i, -i+1, \dots, -1, 0, 1, \dots, i-1\}$.

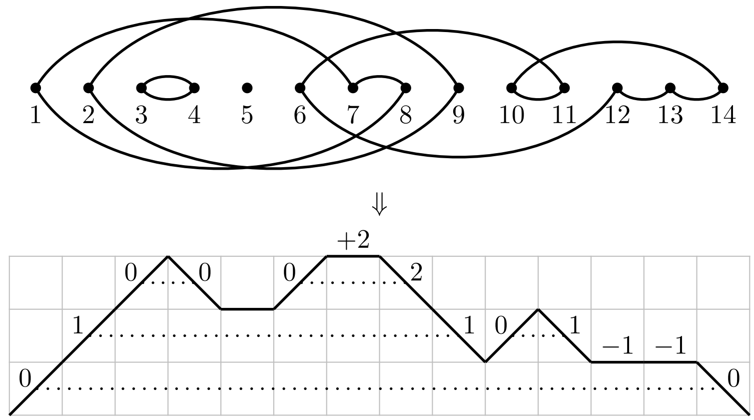

FIGURE 4. An example of the bijection ϕ_L sending permutations to Laguerre histories. The matching North-East steps and South-East steps are connected by dotted lines.

By Theorem 2.3, the nth moment μ_n of the Laguerre polynomials is

$$\mu_n = \sum_{P\in \mathrm{Mot}_n} \mathrm{wt}_{b,\lambda}(P),$$

where $b = \{b_n\}_{n\ge 0}$ and $\lambda = \{\lambda_n\}_{n\ge 1}$ are given by $b_n = y[n+1]_q + [n]_q$ for all $n \ge 0$ and $\lambda_n = y[n]_q^2$ for $n \ge 1$. Let $P \in \mathrm{Mot}_n$. Then it is easy to see that

$$\mathrm{wt}_{b,\lambda}(P) = \sum_{\eta} y^{w(\eta)} q^{|\eta|},$$

where the sum is over all Laguerre histories η with underlying Motzkin path P, $w(\eta)$ is the total number of North-East steps, unlabeled East steps, and East steps labeled by a nonnegative integer, and $|\eta|$ is the sum of labels in η.

We now give a bijection $\psi_L : S_n \to \mathcal{H}_L(n)$ from the set S_n of permutations of $[n]$ to the set $\mathcal{H}_L(n)$ of Laguerre histories of length n, which is due to Foata and Zeilberger [13]. We need the following definitions to describe $\psi_L(\pi)$.

Let $\pi \in S_n$ be a permutation of $[n]$. An *upper edge* of π is a pair (i, j) such that $i < \pi(i) = j$. A *lower edge* of π is a pair (i, j) such that $i = \pi(j) < j$. We represent π using the diagram obtained by putting n vertices labeled by $1, 2, \dots, n$ and placing an upper or lower arc connecting i and j for each upper or lower edge (i, j). For example, the upper diagram in Figure 4 represent the following permutation in the two-line notation:

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 \\ 7 & 9 & 4 & 3 & 5 & 11 & 8 & 1 & 2 & 14 & 10 & 6 & 12 & 13 \end{pmatrix},$$

where the entry in the second row and in the ith column means $\pi(i)$. We say that i is an *opener* of π if $i < \pi(i)$ and $i < \pi^{-1}(i)$, a *closer* of π if $\pi^{-1}(i) < i$ and $\pi(i) < i$, a *singleton* of π if $\pi(i) = i$, and a *transient* of π otherwise. An *upper crossing* of π is a set $\{(a, b), (c, d)\}$ of two upper edges of π with $a < c < b < d$ and $b = i$. A *lower crossing* of π is a set $\{(a, b), (c, d)\}$ of two lower edges of π with $a < c \le b < d$ and $b = i$.

Let $\pi \in S_n$ be a permutation of $[n]$. Then the corresponding Laguerre history $\psi_L(\pi) = \eta$ is defined as follows.

For $i = 1, 2, \ldots, n$, if i is an opener of π, then the ith step of η is a North-East step whose label will be determined later. If i is a closer of π, then the ith step of η is a South-East step. In this case suppose that the jth step and ith step are the matching North-East step and South-East step. Then the label of the jth step (resp. ith step) is the number of upper crossings (resp. lower crossings) $\{(a, b), (c, d)\}$ of π with $a < c < b < d$ and $b = i$. If i is a singleton, then the ith step of η is an unlabeled East step. Finally suppose that i is a continuation of π. Then the ith step of η is an East step whose labeled determined as follows. If the vertex i is incident to two upper edges, then the label is $+k$, where k is the number of upper crossings $\{(a, b), (c, d)\}$ of π with $a < c < b < d$ and $b = i$. If the vertex i is incident to two lower edges, then the label is $-k$, where k is the number of lower crossings $\{(a, b), (c, d)\}$ of π with $a < c \le b < d$ and $b = i$. See Figure 4 for an example.

One can check that ϕ_L is a bijection such that if $\phi_L(\pi) = \eta$ then $\mathrm{wex}(\pi) = w(\eta)$ and $\mathrm{crossing}(\pi) = |\eta|$. Here $\mathrm{wex}(\pi)$ is the number of *weak excedances*, which are integers i such that $i \le \pi(i)$. Thus we obtain the following theorem, which is shown in [23, Theorem 2]. An equivalent version of this theorem in terms of continued fractions has been proved earlier, see [30, 6].

Theorem 3.7. *The q-Laguerre polynomials $L_n(x; q)$ has the nth moment*

$$\mu_n = \sum_{\pi \in S_n} y^{\mathrm{wex}(\pi)} q^{\mathrm{crossing}(\pi)}.$$

Corteel et al. [8] showed that the moment has several interpretations in terms of permutations, permutation tableaux, matrix anszatz, etc. Simion and Stanton [30] consider octabasic Laguerre polynomials (whose recurrence relation has eight independent q's) and express their moments using various statistics of permutations.

4. The odd-even trick

In this section we give a relation between moments for two different sets of orthogonal polynomials. We call this occurrence *the odd-even trick*. It occurs for the three cases considered in Section 3, and offers alternative combinatorial interpretations for the moments. This idea appears in [5, p. 40].

Definition 4.1. For an orthogonal polynomial satisfying the Theorem 2.4, let

$$\mu_n = \mu_n(\{b_k\}_{k\ge0}, \{\lambda_k\}_{k\ge1})$$

denote the n^{th} moment.

From Theorem 2.4, μ_n is a positive polynomial function of the three-term recurrence coefficients. The odd-even trick is the observation that if $b_k = 0$ for all k, the polynomials $p_n(x)$ are alternately even and odd. Moreover if

$$p_{2n}(x) = e_n(x), \quad p_{2n+1}(x) = x o_n(x^2),$$

the even and polynomials $e_n(x)$ and $o_n(x)$ are themselves orthogonal polynomials. The moment sequences for $e_n(x)$ and $o_n(x)$ are related to the moment sequence of $p_n(x)$.

Proposition 4.2. *Given a sequence λ_k with $\lambda_0 = 0$ we have*

$$\begin{aligned}\mu_{2n}(0, \{\lambda_k\}_{k\ge1}) &= \mu_n(\{\lambda_{2k} + \lambda_{2k+1}\}_{k\ge0}, \{\lambda_{2k-1}\lambda_{2k}\}_{k\ge1}),\\ \mu_{2n+2}(0, \{\lambda_k\}_{k\ge1}) &= \lambda_1 \mu_n(\{\lambda_{2k+1} + \lambda_{2k+2}\}_{k\ge0}, \{\lambda_{2k}\lambda_{2k+1}\}_{k\ge1}).\end{aligned}$$

Proof. There are several ways to prove this. We give a sketch of the combinatorial proof due to Viennot [34]. The first statement is equivalent to a bijection between

- Dyck paths of length $2n$ where South-East steps starting at y-coordinate k are weighted by λ_k,
- Motzkin paths of length n where the East steps (resp. South-East steps) starting at y-coordinate k are weighted by $\lambda_{2k} + \lambda_{2k+1}$ (resp. $\lambda_{2k}\lambda_{2k-1}$).

Starting with a Dyck path of length $2n$, $(s_0, \dots, s_{2n})$. For i from 0 to $n-1$, suppose that $s_{2i} = (2i, 2k)$

- If (s_{2i}, s_{2i+1}) and (s_{2i+1}, s_{2i+2}) are North-East steps, change them to one North-East step;
- If (s_{2i}, s_{2i+1}) and (s_{2i+1}, s_{2i+2}) are South-East steps, change them to one South-East step weighted by $\lambda_{2k}\lambda_{2k-1}$;
- If (s_{2i}, s_{2i+1}) is a South-East step and (s_{2i+1}, s_{2i+2}) is a North-East step, change them to one East step weighted by λ_{2k};
- If (s_{2i}, s_{2i+1}) is a North-East step and (s_{2i+1}, s_{2i+2}) is a South-East step, change them to one East step weighted by λ_{2k+1}.

This gives a Motzkin path of length n with the correct weights. The second statement is equivalent to a bijection between

- Dyck paths of length $2n+2$ where South-East steps starting at y-coordinate k are weighted by λ_k
- Motzkin paths of length n where the East steps (resp. South-East step) starting at y-coordinate k are weighted by $\lambda_{2k+1} + \lambda_{2k+2}$ (resp. $\lambda_{2k+1}\lambda_{2k}$).

One needs to delete the first and last step of the Dyck path and apply the same bijection and adjust the weights accordingly. □

Example 4.3. The Hermite polynomials have $b_k = 0$, $\lambda_k = k$, so

$$(2n-1)!! = \mu_{2n}(0, \{k\}_{k\ge1}) = \mu_n(\{4k+1\}_{k\ge0}, \{2k(2k-1)\}_{k\ge1}).$$

Example 4.4. The Laguerre polynomials have $b_k = 2k+1$, $\lambda_k = k^2$, so

$$n! = \mu_n(\{2k+1\}_{k\ge0}, \{k^2\}_{k\ge1}) = \mu_{2n}(0, \{\lfloor (k+1)/2 \rfloor\}_{k\ge1}).$$

Example 4.5. The q-Charlier polynomials have $b_k = a + [k]_q$, $\lambda_k = a[k]_q$, so in this case

$$\mu_n(\{b\}_{k\ge0}, \{\lambda_k\}_{k\ge1}) = \mu_{2n}(0, \{\Lambda_k\}_{k\ge1}),$$

where

$$\Lambda_k = \begin{cases} [k/2]_q, & \text{if } k \text{ is even,} \\ a, & \text{if } k \text{ is odd.} \end{cases}$$

Example 4.6. The renormalized Askey-Wilson polynomials $Q_n(y)$ (see Proposition 5.1)

$$y = a + a^{-1} - 2x, \quad Q_n(y) = (-1)^n P_n(x)$$

satisfy Theorem 1.3 with

$$b_k = A_{k-1} + C_k, \quad \lambda_k = A_k C_k.$$

In this case

$$\mu_n(\{b_k\}_{k\ge0}, \{\lambda_k\}_{k\ge1}) = \mu_{2n}(0, \{\Lambda_k\}_{k\ge1}),$$

where

$$\Lambda_k = \begin{cases} C_{k/2}, & \text{if } k \text{ is even,} \\ A_{(k-1)/2}, & \text{if } k \text{ is odd.} \end{cases}$$

The next Theorem follows from Example 4.6.

Theorem 4.7. *Let*

$$\theta_{2n} = \mu_{2n}(0, \{\Lambda_k\}_{k\geq 1}).$$

The Askey-Wilson moments satisfy

$$2^n \mu_n(a,b,c,d|q) = \sum_{s=0}^{n} \binom{n}{s} (a+1/a)^s (-1)^{n-s} \theta_{2n-2s}.$$

Using the Proposition 4.2 we see

Corollary 4.8. *Given a sequence a_k for $k \geq 0$ with $a_0 = 0$ we have*

$$\mu_n(\{a_{2k+1}+a_{2k}\}_{k\geq 0}, \{a_{2k}a_{2k-1}\}_{k\geq 1}) = a_1 \mu_{n-1}(\{a_{2k+2}+a_{2k+1}\}_{k\geq 0}, \{a_{2k+1}a_{2k}\}_{k\geq 1}).$$

Example 4.9. The Laguerre polynomials have $b_k = 2k+1$, $\lambda_k = k^2$, so $a_k = \lceil k/2 \rceil$

$$(n+1)! = \mu_{n+1}(\{2k+1\}_{k\geq 0}, \{k^2\}_{k\geq 1}) = \mu_n(\{2k+2\}_{k\geq 0}, \{k(k+1)\}_{k\geq 1}).$$

5. Askey-Wilson moments

In this section we consider the Askey-Wilson polynomials and explicit forms for their moments. Recall that these polynomials in x have five parameters: a, b, c, d, and q. In Subsection 5.1 we give explicit formulas for the moments. In Subsection 5.2 we use a non-linear change of variables on the parameters to obtain four new parameters α, β, γ, and δ. Using these parameters we give a combinatorial interpretation for the moments as weights of special tableaux.

Recall from (1.5) that the Askey-Wilson polynomial $p_n(x;a,b,c,d|q)$ is not monic and has the leading term $(abcd;q)_n(2x)^n$. It is true, but not obvious, that the Askey-Wilson polynomial $p_n(x;a,b,c,d|q)$ is symmetric in all four parameters a, b, c, and d, not just b, c, and d.

Let

$$P_n(x) = \frac{1}{(abcd;q)_n} p_n(x;a,b,c,d|q)$$

be the normalized Askey-Wilson (AW) polynomial whose leading term is $(2x)^n$. The moment sequences for $P_n(x)$ and $p_n(x;a,b,c,d|q)$ are the same. For historical reasons, and to agree with the literature, we keep the $2x$.

Proposition 5.1. *The normalized Askey-Wilson polynomials $P_n(x)$ satisfy the three-term recurrence relation :*

$$P_{n+1}(x) = (2x - b_n)P_n(x) - \lambda_n P_{n-1}(x)$$

for $n \geq 0$ with $P_{-1}(x) = 0$ and $P_0(x) = 1$,

$$b_n = (a + a^{-1} - A_n - C_n), \quad \lambda_n = A_{n-1}C_n,$$

where

$$A_n = \frac{(1-abq^n)(1-acq^n)(1-adq^n)(1-abcdq^{n-1})}{a(1-abcdq^{2n-1})(1-abcdq^{2n})},$$

$$C_n = \frac{a(1-q^n)(1-bcq^{n-1})(1-bdq^{n-1})(1-cdq^{n-1})}{(1-abcdq^{2n-2})(1-abcdq^{2n-1})}.$$

In this section, we will start by giving enumeration formulas for the moments of these polynomials. We will then present the combinatorics coming from staircase tableaux. We will end the section with some special cases.

5.1. **Enumeration formulas.** The moments of the AW polynomials, denoted $\mu_n(a,b,c,d|q)$, are not polynomials in a, b, c, d and q. But it was recently proven [25, Proposition 2.1] that $2^n(abcd;q)_n\mu_n(a,b,c,d|q)$ are polynomials in a,b,c,d,q with integer coefficients. Several explicit formulas for $\mu_n(a,b,c,d|q)$ are known. The following is the simplest known expression as a double sum.

Theorem 5.2. [9, Theorem 1.13] *We have*

$$\mu_n(a,b,c,d|q) = \frac{1}{2^n}\sum_{m=0}^{n}\frac{(ab,ac,ad;q)_m}{(abcd;q)_m}q^m\sum_{j=0}^{m}\frac{q^{-j^2}a^{-2j}(aq^j+q^{-j}/a)^n}{(q,q^{1-2j}/a^2;q)_j(q,q^{2j+1}a^2;q)_{m-j}}.$$

Theorem 5.2 was proved using techniques built in [17, 18]. The formula has two clear defects. It does not demonstrate the polynomiality of $2^n(abcd;q)_n\mu_n(a,b,c,d|q)$ in a, and it not obviously symmetric in the four parameters a, b, c, and d.

A fivefold sum formula was found where the polynomiality is clear.

Corollary 5.3. [25, Theorem 5.6] *We have*

$$2^n\mu_n(a,b,c,d|q) = \sum_{k=0}^{n}\left(\binom{n}{\frac{n-k}{2}}-\binom{n}{\frac{n-k}{2}-1}\right)\sum_{u+v+w+x+2t=k}a^ub^vc^wd^x$$
$$\times\frac{(ac;q)_v(bd;q)_w}{(abcd;q)_{v+w}}(-1)^tq^{\binom{t+1}{2}}\begin{bmatrix}u+v+w+t\\u\end{bmatrix}_q\begin{bmatrix}v+w+x+t\\v,w,x+t\end{bmatrix}_q\begin{bmatrix}u+x+t\\x\end{bmatrix}_q,$$

where the second sum is over all integers $u,v,w,x\ge 0$ *and* $-k\le t\le k/2$ *satisfying* $u+v+w+x+2t=k$.

A combinatorial approach to the proof of Corollary 5.3 is given in Section 7. When $d=0$, the moments $2^n\mu_n(a,b,c,0|q)$ are polynomials and there is an explicit polynomial formula which also establishes symmetry.

Corollary 5.4. [25, Theorem 2.3] *The Askey-Wilson moments for* $d=0$ *are*

$$2^n\mu_n(a,b,c,0|q) = \sum_{k=0}^{n}\left(\binom{n}{\frac{n-k}{2}}-\binom{n}{\frac{n-k}{2}-1}\right)$$
$$\times\sum_{u+v+w+2t=k}a^ub^vc^w(-1)^tq^{\binom{t+1}{2}}\begin{bmatrix}u+v+t\\v\end{bmatrix}_q\begin{bmatrix}v+w+t\\w\end{bmatrix}_q\begin{bmatrix}w+u+t\\u\end{bmatrix}_q,$$

where the second sum is over all integers $u,v,w\ge 0$ *and* $-k\le t\le k/2$ *satisfying* $u+v+w+2t=k$.

There is a result showing symmetry using very-well poised basic hypergeometric series.

Theorem 5.5. [25, Theorem 2.10] *For an arbitrary* A,

$$2^n\mu_n(a,b,c,d|q) = \sum_{m=0}^{n}\frac{(aA,bA,cA,dA;q)_m}{(A^2,abcd;q)_m}(-q)^m{}_8W_7(m)\sum_{s=0}^{n+1}\left(\binom{n}{s}-\binom{n}{s-1}\right)$$
$$\times\sum_{p=0}^{n-2s-m}A^{-n+2s+2p}\begin{bmatrix}m+p\\m\end{bmatrix}_q\begin{bmatrix}n-2s-p\\m\end{bmatrix}_q q^{m(-n+2s+p)+\binom{m}{2}},$$

where

$${}_8W_7(m) = {}_8W_7(A^2/q;A/a,A/b,A/c,A/d,q^{-m};q,abcdq^m),$$

and [14, Chap. 2.1]

$$_8W_7(a_0;a_1,\dots,a_5;q,z)=\sum_{k=0}^{\infty}\frac{(1-a_0q^{2k})}{(1-a_0)}\frac{(a_0;q)_k}{(q;q)_k}\prod_{i=1}^{5}\frac{(a_i;q)_k}{(qa_0/a_i;q)_k}z^k.$$

The expression for $2^n(abcd;q)_n\mu_n(a,b,c,d|q)$ in Theorem 5.5 is clearly symmetric in a, b, c, and d. From Theorem 5.5 it may be shown that it is a polynomial in these four parameters. We do not give the details here. Instead we offer another representation which shows polynomiality but breaks symmetry

$$\text{(3)}\qquad \frac{(aA,bA,cA,dA;q)_m}{(A^2;q)_m}\,{}_8W_7(m)=\sum_{j=0}^{m}\begin{bmatrix}m\\ j\end{bmatrix}_q(cd)^j(A/c,A/d;q)_j(ab;q)_j \times(Aaq^j,Abq^j,cd;q)_{m-j}.$$

The details appear in [25].

The results in Corollaries 5.3 and 5.4, and Theorem 5.5, all involve a product of differences of binomial coefficients and q-binomial coefficients. The reason for this unusual behavior is the form of Theorem 5.2. The power in the numerator leads to binomial coefficients, while the denominator terms lead to differences of q-binomial coefficients. This difference can be switched to the binomial coefficients, giving the differences that are displayed.

The proofs of these results are analytic and use properties of the Askey-Wilson functional $\mathcal{L}_{\mathrm{AW}}$. There is a combinatorial proof of the $c=0$ case of Corollary 5.4 using weighted Motzkin paths in [25]. Josuat-Vergès also did this case (which is the Al-Salam-Chihara polynomials), see [20, Theorem 6.1.1]. The differences of binomial coefficients occur naturally in these combinatorial approaches, see (14). A combinatorial proof of a general result remains open.

5.2. **Combinatorics of the moments.** We define here a combinatorial object that was defined to study the stationary distribution of the asymmetric exclusion process with open boundaries [32].

Definition 5.6. [10] A *staircase tableau* of size n is a Young diagram of "staircase" shape $(n,n-1,\dots,2,1)$ such that boxes are either empty or labeled with α,β,γ, or δ, subject to the following conditions:

- no box along the diagonal is empty;
- all boxes in the same row and to the left of a β or a δ are empty;
- all boxes in the same column and above an α or a γ are empty.

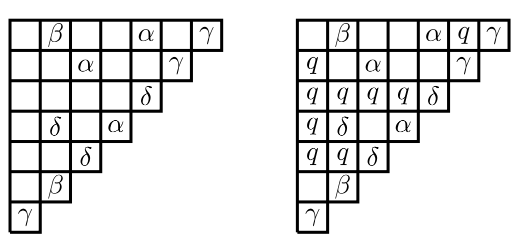

FIGURE 5. A staircase tableau of size 7

See the left entry of Figure 5 for an example of a staircase tableau.

Definition 5.7. [10] The *weight* $\mathrm{wt}(\mathcal{T})$ of a staircase tableau $\mathcal{T}$ is a monomial in $\alpha, \beta, \gamma, \delta$, and q, which we obtain as follows. Some blank boxes of $\mathcal{T}$ are assigned a q, based on the label of the closest labeled box to its right in the same row and the label of the closest labeled box below it in the same column, such that:

- every blank box which sees a δ to its right gets assigned a q;
- every blank box which sees an α or γ to its right, and a β or γ below it, gets assigned a q.

After this assignment, the *weight* of $\mathcal{T}$, $\mathrm{wt}(\mathcal{T})$ is then defined as the product of all labels in all boxes.

The right entry of Figure 5 shows that this staircase tableau has weight $\alpha^3\beta^2\gamma^3\delta^3q^9$. The *type* of the tableau $\mathcal{T}$ is the number of α's plus the number of δ's on the diagonal. We denote it by $t(\mathcal{T})$.

Let

$$Z_n(y, \alpha, \beta, \gamma, \delta, q) = \sum_{\mathcal{T}} \mathrm{wt}(\mathcal{T}) y^{t(\mathcal{T})}, \tag{4}$$

where the sum is taken on the staircase tableaux of size n.

We now link this generating polynomial to the moments of the Askey-Wilson polynomials. Let

$$\begin{aligned}
a &= \frac{1-q-\alpha+\gamma+\sqrt{(1-q-\alpha+\gamma)^2+4\alpha\gamma}}{2\alpha},\\
c &= \frac{1-q-\alpha+\gamma-\sqrt{(1-q-\alpha+\gamma)^2+4\alpha\gamma}}{2\alpha},\\
b &= \frac{1-q-\beta+\delta+\sqrt{(1-q-\beta+\delta)^2+4\beta\delta}}{2\beta},\\
d &= \frac{1-q-\beta+\delta-\sqrt{(1-q-\beta+\delta)^2+4\beta\delta}}{2\beta}.
\end{aligned}$$

Proposition 5.8 (see [9]; Theorem 1.11)**.** *The generating polynomial $Z_n(y; \alpha, \beta, \gamma, \delta; q)$ is*

$$(abcd; q)_n \sqrt{y}^n (\alpha\beta)^n \times \mu_n$$

where μ_n are the moments of the orthogonal polynomials defined by

$$G_{n+1}(x) = (x - B_n)G_n(x) - \Lambda_n G_{n-1}(x),$$

where

$$B_n = \frac{1/\sqrt{y} + \sqrt{y} + b_n}{1-q}, \quad \Lambda_n = \frac{A_{n-1}C_n}{(1-q)^2}$$

are given by the Askey-Wilson recurrence with the substitutions $a \to a/\sqrt{y}$, $b \to b\sqrt{y}$, $c \to c/\sqrt{y}$, $d \to d\sqrt{y}$.

Remark 5.9. This proposition uses a result of [10] that relates the partition function of the ASEP and the generating polynomial of the staircase tableaux. The proof is quite complicated. It is still an open problem to find a simple combinatorial proof.

Remark 5.10. The Askey-Wilson moments $2^n(abcd; q)_n\mu_n(a, b, c, d|q)$ do not have positive coefficients as polynomials in a, b, c, d, and q. The change of variables in the parameters miraculously does make these moments positive.

We also know an enumeration formula :

Theorem 5.11. [9, Theorem 1.14]

$$Z_n(y;\alpha,\beta,\gamma,\delta;q) = (abcd;q)_n \left(\frac{\alpha\beta}{1-q}\right)^n \sum_{k=0}^{n} \frac{(ab,ac/y,ad;q)_k}{(abcd;q)_k} q^k$$
$$\times \sum_{j=0}^{k} q^{-(k-j)^2} (a^2/y)^{j-k} \frac{(1+y+q^{k-j}a+q^{j-k}y/a)^n}{(q,q^{2j-2k+1}y/a^2;q)_{k-j}(q,a^2q^{1-2j+2k}/y;q)_j}.$$

This formula is a polynomial in $y,\alpha,\beta,\gamma,\delta,q$ with positive coefficients whose sum of coefficients is $4^n n!$. This is not easy to see from Theorem 5.11. The special case $y=q=1$ does make this clear.

Proposition 5.12. *The generating polynomial of the staircase tableau of size n satisfies*

$$Z_n(1,\alpha,\beta,\gamma,\delta,1) = \prod_{i=0}^{n-1}(\alpha+\beta+\gamma+\delta+i(\alpha+\gamma)(\beta+\delta)). \tag{5}$$

We give here a simple bijective proof based on [7]. A *colored inversion table* of size n is a table T such that for $i \in \{1,...,n\}$, $T[i]=(i-1)_x$ with $x \in \{\alpha,\beta,\gamma,\delta\}$ or $T[i]=j_{x,y}$ with $0 \le j < i-1$, $x \in \{\alpha,\gamma\}$ and $y \in \{\beta,\delta\}$. The weight of a colored inversion table is the product of its colors in $\{\alpha,\beta,\gamma,\delta\}$. Computing the generating polynomial of the colored inversion table is trivial and it is equal to :

$$\prod_{i=0}^{n-1}(\alpha+\beta+\gamma+\delta+i(\alpha+\gamma)(\beta+\delta)).$$

Proposition 5.13. *There exists a weight preserving bijection between staircase tableaux (with $q=1$) of size n and colored inversion tables of size n.*

Proof. Start with a staircase tableau of size n and number the columns from 1 to n from left to right and the rows from 1 to n from top to bottom. The bijection is obtained doing the following steps. For each column i, look at the topmost Greek letter in column i and count the number of cells j directly to its left that does not contain any Greek letter.

- If this letter, say x, is topmost and leftmost, record $T[i]=i_x$.
- Otherwise let y be the first Greek letter to the left of x and let z be the first Greek letter under y. Then $T[i]=j_{x,z}$.

We present the inverse of the algorithm. Start from a colored inversion table T of size n. Mark all the cells of the tableau as free. For i from n down to 1,

- if $T_i=(i-1)_x$, put an x in in the ith column, as high as possible. Mark all the cells to its left as reserved.
- Otherwise, T_i is equal to some $j_{x,y}$. Put an x in the ith column, as high as possible. Mark j boxes to its immediate left as reserved. Suppose that x is inserted in row k.
- Insert a y in column $i-j-1$ as high as possible but in a row with a label larger than k (such a cell always exists as one can check that the diagonal box of column $i-j-1$ is still free). Mark all the cells to its left as reserved.

□

For example, we obtain the inversion table $T=(0_\gamma,1_\beta,2_\alpha,1_{\alpha,\beta},2_{\alpha,\delta},2_{\gamma,\delta},1_{\gamma,\delta})$ from the tableau on the left of Figure 5, . Note that the proof of Proposition 5.13 gives a proof of (5).

5.3. **Special cases.** We give alternative proofs of special cases for $Z_n(y;\alpha,\beta,\gamma,\delta;q)$ proven in [9, Table 1]. These were proven using the staircase tableaux and the matrix ansatz. We use here the moments and the explicit three-term recurrence relation. The first result is new and does not have (yet) a combinatorial proof.

Proposition 5.14. *If $\delta = -\beta/y$, then*

$$Z_n(y;\alpha,\beta,\gamma,\delta;q) = \prod_{j=0}^{n-1}(\alpha y + \gamma q^j).$$

Proof. First let's check that the choice $\delta = -\beta/y$ forces $\Lambda_1 = 0$, so that in Proposition 5.8 $\mu_n = B_0^n$.

In general we have $bd = -\delta/\beta$. So Proposition 5.1 and Proposition 5.8 imply that Λ_1 has the numerator factor $(1 - bdy) = (1 + y\delta/\beta) = 0$.

Since $abcd = \gamma\delta/\alpha\beta$, we have

$$Z_n(y;\alpha,\beta,\gamma,\delta;q) = (\gamma\delta/\alpha\beta;q)_n \sqrt{y}^n (\alpha\beta)^n \times B_0^n.$$

Take the version of Proposition 5.1 which uses b instead of a. In this case $A(0) = C(0) = 0$, and

$$B_0 = \frac{1/\sqrt{y} + \sqrt{y} + b\sqrt{y} + 1/b\sqrt{y}}{1-q} = \frac{\sqrt{y}}{\beta}.$$

□

Proposition 5.12 may be shown using the recurrence relation instead of the bijection of Proposition 5.13. We need the fact, which follows from Proposition 1.4, that the rescaled Laguerre polynomials which have

$$b_n = A(2n+\theta+1), \quad \lambda_n = A^2 n(n+\theta),$$

have moments of

$$\mu_n = A^n \prod_{i=1}^{n}(\theta + i).$$

We do not give these details.

The next result can be proven bijectively by adapting the bijection between staircase tableaux and inversion tables given in the preceding subsection, see [7].

Proposition 5.15. *If $\alpha = \delta = 0$, then*

$$Z_n(y;\alpha,\beta,\gamma,\delta;q) = \prod_{j=0}^{n-1}(\beta + \beta\gamma[j]_q + \gamma q^j).$$

Proof. This analogously comes from the q-Laguerre polynomials studied by de Médicis and Viennot [11] which have

$$b_n = Aq^n\left([n]_q + B + q + \cdots + q^n\right), \quad \lambda_n = A^2 q^{2n-1}[n]_q\left(B + q + \cdots + q^{n-1}\right),$$
$$\mu_n = A^n \prod_{j=1}^{n}(B + q + \cdots + q^{j-1}).$$

□

6. Modified moments

The Askey-Wilson polynomials have the polynomial representation in Definition 1.5 using the Askey-Wilson basis of polynomials,

$$\phi_n(x;a) = \prod_{k=0}^{n-1}(1-2axq^k+a^2q^{2k}) = (az, a/z;q)_n, \quad z=e^{i\theta}, \quad x=\cos\theta.$$

However it is not clear how Theorem 1.2 and Theorem 5.2 lead to the explicit formula in Definition 1.5. One may ask if an analogue of Theorem 1.2 exists for the Askey-Wilson basis $\phi_n(x;a)$ instead of x^n. We shall see that for the Askey-Wilson polynomials it does exist. It was given by Wilson in his 1978 thesis and is implicit in [35].

Suppose that polynomial sequences $\{\psi_k(x)\}_{k=0}^{\infty}$ and $\{\phi_k(x)\}_{k=0}^{\infty}$ are given with

$$\deg(\psi_k(x)) = \deg(\phi_k(x)) = k.$$

For the linear functional $\mathcal{L}$ define a matrix M by

$$M_{i,j} = \mathcal{L}(\psi_i\phi_j).$$

Proposition 6.1. *The orthogonal polynomial $p_n(x)$ for $\mathcal{L}$ is*

$$p_n(x) = \frac{1}{B_n}\begin{vmatrix} M_{0,0} & M_{0,1} & \cdots & M_{0,n} \\ M_{1,0} & M_{1,1} & \cdots & M_{1,n} \\ \vdots & \vdots & \ddots & \vdots \\ M_{n-1,0} & M_{n-1,1} & \cdots & M_{n-1,n} \\ \phi_0(x) & \phi_1(x) & \cdots & \phi_n(x) \end{vmatrix},$$

where

$$B_n = \begin{vmatrix} M_{0,0} & M_{0,1} & \cdots & M_{0,n-1} \\ M_{1,0} & M_{1,1} & \cdots & M_{1,n-1} \\ \vdots & \vdots & \ddots & \vdots \\ M_{n-1,0} & M_{n-1,1} & \cdots & M_{n-1,n-1} \end{vmatrix}.$$

The determinant B_n in Proposition 6.1 is non-zero and $p_n(x)$ is a polynomial of exact degree n with leading term $\phi_n(x)$. To prove this, let D_n be the matrix of the determinant Δ_n in Theorem 1.2. Let Y and Z be the $n \times n$ non-singular lower triangular matrices defined by $\psi_i(x) = \sum_{s=0}^{i} Y_{is}x^s$, $\phi_j(x) = \sum_{t=0}^{j} Z_{jt}x^t$. Then $M_{i,j}$ is the ij-entry of the matrix YD_nZ^T, which is invertible.

Theorem 1.2 is the special case $\psi_k(x) = \phi_k(x) = x^k$ of Proposition 6.1.

Whenever we have two bases of polynomials for which $M_{i,j}$ may be explicitly computed, we have a determinantal formula for $p_n(x)$. The explicit representation for the polynomials is reduced to evaluating the determinant

$$\det(M_{i,j})_{\substack{0\le i\le n-1\\ 0\le j\le n, j\ne k}},$$

$$p_n(x) = \frac{1}{B_n}\sum_{k=0}^{n}(-1)^k\phi_k(x)\det(M_{ij})_{\substack{0\le i\le n-1\\ 0\le j\le n, j\ne k}} \tag{6}$$

The Askey-Wilson bases provide such an example for the Askey-Wilson polynomials. We use a normalized representing measure for $0<q<1$, $0\le |a|,|b|,|c|,|d|<1$, which is [3]

$$\mathcal{L}_{\mathrm{AW}}(p(x)) = \frac{1}{2\pi}\frac{(ab,ac,ad,bc,bd,cd;q)_\infty}{(abcd;q)_\infty} \times \int_0^\pi p(\cos(\theta))\frac{(e^{2i\theta},e^{-2i\theta};q)_\infty \quad d\theta}{(ae^{i\theta},ae^{-i\theta},be^{i\theta},be^{-i\theta},ce^{i\theta},ce^{-i\theta},de^{i\theta},de^{-i\theta};q)_\infty}.$$

In this form we have the Askey-Wilson integral

$$\mathcal{L}_{\mathrm{AW}}(1)=1.$$

Proposition 6.2. *If $\psi_i(x)=\phi_i(x;d)$, $\phi_j(x)=\phi_j(x;a)$, then*

$$\mathcal{L}_{\mathrm{AW}}(\psi_i\phi_j) = M_{i,j} = \frac{(ab,ac;q)_j(bd,cd;q)_i(ad;q)_{i+j}}{(abcd;q)_{i+j}}.$$

Proof. Finding $\mathcal{L}_{\mathrm{AW}}(\psi_i\phi_j)$ amounts to shifting a to aq^j and d to dq^i in the Askey-Wilson integral. □

To find the coefficient of $\phi_k(x;a)$ for the Askey-Wilson polynomials in Definition 1.5, the following determinant evaluation of Wilson ([35, p. 1155]) is used in (6).

Proposition 6.3. *If $0\le k\le n$, we have*

$$\det\left(\frac{(Aq^i;q)_j}{(Dq^i;q)_j}\right)_{0\le i\le n-1, 0\le j\le n, j\ne k} = \Delta_{nn}\pi_k/\pi_n,$$

where

$$\Delta_{nn} = \prod_{r=0}^{n-1}\frac{(q,D/A;q)_r}{(Dq^r;q)_{n-1}}A^r q^{r(r-1)}$$

$$\pi_k = \frac{(q^{-n},Dq^{n-1};q)_k}{(q,A;q)_k}(-q)^k$$

Proposition 6.3 with $A=ad$ and $D=abcd$ and (6) give the explicit formula in Definition 1.5 for the Askey-Wilson polynomials.

Another example occurs with the Hahn polynomials $Q_n(x;\alpha,\beta,N)$, $0\le n\le N$, whose measure is purely discrete (the hypergeometric distribution), and is located at the integers $x=0,1,\cdots,N$,

$$w(x;\alpha,\beta,N) = \frac{\binom{\alpha+x}{x}\binom{\beta+N-x}{N-x}}{\binom{\alpha+\beta+1+N}{N}}.$$

In this case the choices of

$$\psi_i(x) = (-x)_i = (-x)(-x+1)\cdots(-x+i-1),$$
$$\phi_j(x) = (-N+x)_j = (-N+x)(-N+x+1)\cdots(-N+x+j-1)$$

yield

$$M_{i,j} = \frac{(\alpha+1)_i(\beta+1)_j}{(\alpha+\beta+2)_{i+j}}\frac{N!}{(N-i-j)!}(-1)^{i+j}, \quad \text{for } i+j\le N.$$

Here another determinant evaluation leads to the explicit hypergeometric representation [27] for $Q_n(x;\alpha,\beta,N)$.

7. Linearization coefficients

Suppose that $\{P_n(x)\}_{n\geq 0}$ is an orthogonal polynomial sequence with respect to a linear functional $\mathcal{L} : \mathbb{K}[x] \to \mathbb{K}$. Since $\{P_n(x)\}_{n\geq 0}$ is a basis of the ring $\mathbb{K}[x]$, we can express the product $P_n(x)P_m(x)$ of two polynomials in the basis as follows:

$$P_n(x)P_m(x) = \sum_{k=0}^{n+m} c_{n,m}^k P_k(x). \tag{7}$$

If we multiply both sides of (7) by $P_\ell(x)$ and apply the linear functional $\mathcal{L}$, then by the orthogonality we obtain

$$c_{n,m}^\ell = \frac{\mathcal{L}(P_n(x)P_m(x)P_\ell(x))}{\mathcal{L}(P_\ell(x)^2)}.$$

By Theorem 2.4, $\mathcal{L}(P_\ell(x)^2)$ is obtained immediately once we know the three-term recurrence relation. Thus computing the coefficients $c_{n,m}^\ell$ is equivalent to computing the quantity $\mathcal{L}(P_n(x)P_m(x)P_\ell(x))$. Although it is more common to call $c_{n,m}^\ell$ the linearization coefficient, for brevity, we will instead call the quantity $\mathcal{L}(P_{n_1}(x)P_{n_2}(x)\cdots P_{n_k}(x))$ the *linearization coefficient* of the orthogonal polynomials $P_n(x)$.

Using the three-term recurrence relation, Lemma 2.2 allows us to consider orthogonal polynomials $P_n(x)$ as generating functions of Favard paths. By Theorem 2.3, there is a combinatorial meaning to the moments $\mu_n = \mathcal{L}(x^n)$ of $P_n(x)$. Therefore it is possible to understand the linearization coefficients of $P_n(x)$ combinatorially. When $P_n(x)$ are q-Hermite, q-Charlier, or q-Laguerre polynomials, there is a nice combinatorial expression for the linearization coefficients.

In this section we will consider the linearization coefficients of the Hermite polynomials. Then we will see a connection between the linearization coefficients of the q-Hermite polynomials and the moments of Askey-Wilson polynomials.

Recall that the Hermite polynomials $H_n(x)$ are defined by $H_{-1}(x) = 0$, $H_0(x) = 1$, and for $n \geq 0$,

$$H_{n+1}(x) = xH_n(x) - nH_{n-1}(x), \tag{8}$$

and the moment $\mu_n = \mathcal{L}(x^n)$ is equal to the number of perfect matchings of $[n]$. Using the three-term recurrence relation (8), one can easily see that

$$H_n(x) = \sum_{\pi\in \mathrm{Matching}(n)} (-1)^{\mathrm{edge}(\pi)} x^{\mathrm{fix}(\pi)}, \tag{9}$$

where $\mathrm{fix}(\pi)$ is the number of singletons in π, which are also called *fixed points*.

Recall that $\mathcal{M}_n$ is the set of perfect matchings of $[n]$. Suppose that $n = n_1 + n_2 + \cdots + n_k$ and for $1 \leq i \leq k$, let

$$S_i = \{a_{i-1} + 1, a_{i-1} + 2, \dots, a_{i-1} + n_i\}, \tag{10}$$

where $a_{i-1} = n_1 + n_2 + \cdots + n_{i-1}$ for $i \geq 2$ and $a_0 = 0$. Then $[n]$ is a disjoint union of S_i's. An edge (i, j) of $\pi \in \mathcal{M}_n$ is called *homogeneous* if $i, j \in S_r$ for some $r \in [k]$, and *inhomogeneous* otherwise. If every edge is inhomogeneous in $\pi \in \mathcal{M}_n$, then we call π an *inhomogeneous perfect matching*. We denote by $\mathcal{M}(n_1, n_2, \dots, n_k)$ the set of inhomogeneous perfect matchings of $[n]$.

The following theorem is due to Azor, Gillis, and Victor [4] and Godsil [15].

Theorem 7.1. *The linearization coefficient of the Hermite polynomials is equal to the number of inhomogeneous perfect matchings:*

$$\mathcal{L}(H_{n_1}(x)\cdots H_{n_k}(x)) = |\mathcal{M}(n_1, n_2, \dots, n_k)|.$$

Proof. By (9), $H_{n_1}(x)\cdots H_{n_k}(x)$ is equal to

$$\sum_{\pi_1\in\mathrm{Matching}(n_1)}\cdots\sum_{\pi_k\in\mathrm{Matching}(n_k)}(-1)^{\mathrm{edge}(\pi_1)+\cdots+\mathrm{edge}(\pi_k)}x^{\mathrm{fix}(\pi_1)+\cdots+\mathrm{fix}(\pi_k)}.$$

Since $\mathcal{L}(x^m)$ is the number of perfect matchings of $[m]$, we have

$$\mathcal{L}(H_{n_1}(x)\cdots H_{n_k}(x))=\sum_{(\pi_1,\dots,\pi_k,\pi_0)\in X}(-1)^{\mathrm{edge}(\pi_1)+\cdots+\mathrm{edge}(\pi_k)},$$

where X is the set of all $(k+1)$-tuples $(\pi_1,\dots,\pi_k,\pi_0)$ such that

$$\pi_1\in\mathrm{Matching}(n_1),\dots,\pi_k\in\mathrm{Matching}(n_k),$$

and π_0 is a perfect matching of the union of the sets of fixed points in $\pi_1,\pi_2,\dots,\pi_k$. As before we say that an edge (a,b) is *homogeneous* if $(a,b)\in S_i$ for some $1\le i\le k$, and *inhomogeneous* otherwise, where S_i is given in (10). Note that all edges in $\pi_1,\dots,\pi_k$ are homogeneous and π_0 may have both homogeneous and inhomogeneous edges.

We will construct a sign-reversing involution ρ on X. For $(\pi_1,\dots,\pi_k,\pi_0)\in X$, we define $\rho((\pi_1,\dots,\pi_k,\pi_0))=(\pi_1',\dots,\pi_k',\pi_0')$ as follow. If $\pi_1,\dots,\pi_k$ are all empty and π_0 has only inhomogeneous edges, then $\pi_i'=\pi_i$ for all $0\le i\le k$. Otherwise, there is a homogeneous edge (a,b) in one of $\pi_1,\pi_2,\dots,\pi_k$ or π_0. Take the homogeneous edge (a,b) such that b is minimal. If $(a,b)\in\pi_i$ for some $1\le i\le k$, then let $\pi_i'=\pi_i\setminus\{(a,b)\}$ and $\pi_0'=\pi_0\cup\{(a,b)\}$, and $\pi_j'=\pi_j$ for $j\ne 0,i$. If $(a,b)\in\pi_0$, then let i be the index for which $a,b\in S_i$ and let $\pi_i'=\pi_i\cup\{(a,b)\}$ and $\pi_0'=\pi_0\setminus\{(a,b)\}$, and $\pi_j'=\pi_j$ for $j\ne 0,i$.

It is not hard to check that ρ is a sign-reversing involution on X whose fixed points are the $(k+1)$-tuples $(\pi_1,\dots,\pi_k,\pi_0)\in X$ such that $\pi_1,\dots,\pi_k$ are all empty and π_0 is an inhomogeneous perfect matchings of $S_1\cup\cdots\cup S_k$. This completes the proof. □

From now on we put $x=\cos\theta$.

The generating function for the q-Hermite polynomials $H_n(x|q)$ is given by

$$H(\cos\theta,z):=\sum_{n=0}^{\infty}\frac{H_n(\cos\theta|q)}{(q;q)_n}z^n=\frac{1}{(ze^{i\theta};q)_\infty(ze^{-i\theta};q)_\infty}.$$

The orthogonality for the q-Hermite polynomials is

$$\int_0^\pi H_n(\cos\theta|q)H_m(\cos\theta|q)v(\cos\theta|q)d\theta=0,\qquad n\ne m,$$

where

$$v(\cos\theta|q)=\frac{(q;q)_\infty}{2\pi}(e^{2i\theta};q)_\infty(e^{-2i\theta};q)_\infty.$$

We now look at the Askey-Wilson polynomials defined in (1.5). The total mass

$$I_0(a,b,c,d)=\frac{(q;q)_\infty}{2\pi}\int_0^\pi w(\cos\theta,a,b,c,d;q)d\theta=\frac{(abcd;q)_\infty}{(ab,ac,ad,bc,bd,cd;q)_\infty}\tag{11}$$

of the measure is called the *Askey-Wilson integral.* Observe that the Askey-Wilson integral is the generating function for linearization coefficients of the q-Hermite polynomials

$$\begin{aligned} I_0(a,b,c,d) &= \int_0^\pi H(\cos\theta,a)H(\cos\theta,b)H(\cos\theta,c)H(\cos\theta,d)v(\cos\theta|q)d\theta \\ &= \sum_{n_1,n_2,n_3,n_4=0}^{\infty} \mathcal{L}(H_{n_1}(x|q)H_{n_2}(x|q)H_{n_3}(x|q)H_{n_4}(x|q)) \\ &\qquad \times \frac{a^{n_1}b^{n_2}c^{n_3}d^{n_4}}{(q;q)_{n_1}(q;q)_{n_2}(q;q)_{n_3}(q;q)_{n_4}}, \end{aligned}$$

where $\mathcal{L}$ is the linear function for $H_n(x|q)$ defined by

$$\mathcal{L}(p(x)) = \int_0^\pi p(\cos\theta)v(\cos\theta|q)d\theta,$$

Using this observation Ismail, Stanton, and Viennot [19] evaluated the Askey-Wilson integral combinatorially. More precisely, they considered the rescaled q-Hermite polynomials, which are more suitable to work with combinatorially, and showed the following generalization of Theorem 7.1.

Theorem 7.2. [19, Theorem 3.2] *Let $\widetilde{\mathcal{L}}$ be the normalized linear functional for $\widetilde{H}_n(x|q)$. Then*

$$\widetilde{\mathcal{L}}(\widetilde{H}_{n_1}(x|q)\cdots\widetilde{H}_{n_k}(x|q)) = \sum_{\pi\in\mathcal{M}(n_1,\dots,n_k)} q^{\mathrm{crossing}(\pi)},$$

where $\mathcal{M}(n_1,\dots,n_k)$ is the set of inhomogeneous perfect matchings of $[n_1]\uplus\cdots\uplus[n_k]$.

This idea can be extended to compute the moments of the Askey-Wilson polynomials as follows. Let

$$I_n(a,b,c,d) = \frac{(q;q)_\infty}{2\pi}\int_0^\pi (\cos\theta)^n w(\cos\theta,a,b,c,d;q)d\theta. \tag{12}$$

Then the normalized moment $\mu_n(a,b,c,d;q)$ of the Askey-Wilson polynomials is

$$\mu_n(a,b,c,d;q) = I_n(a,b,c,d)/I_0(a,b,c,d). \tag{13}$$

By the same observation as above we have

$$\begin{aligned} I_n(a,b,c,d) &= \int_0^\pi (\cos\theta)^n H(\cos\theta,a)H(\cos\theta,b)H(\cos\theta,c)H(\cos\theta,d)v(\cos\theta|q)d\theta \\ &= \sum_{n_1,n_2,n_3,n_4=0}^{\infty} \mathcal{L}(x^n H_{n_1}(x|q)H_{n_2}(x|q)H_{n_3}(x|q)H_{n_4}(x|q)) \\ &\qquad \times \frac{a^{n_1}b^{n_2}c^{n_3}d^{n_4}}{(q;q)_{n_1}(q;q)_{n_2}(q;q)_{n_3}(q;q)_{n_4}}. \end{aligned}$$

Using the above formula Kim and Stanton [25] found the formula for $\mu_n(a,b,c,d;q)$ in Corollary 5.3. In what follows we briefly explain the idea of their proof.

First, in [25] they consider the rescaled q-Hermite polynomials and show the following theorem.

Theorem 7.3. [25, Theorem 5.1] *Let $\widetilde{\mathcal{L}}$ be the normalized linear functional for $\widetilde{H}_n(x|q)$. Then*

$$\widetilde{\mathcal{L}}(x^n\widetilde{H}_{n_1}(x|q)\cdots\widetilde{H}_{n_k}(x|q)) = \sum_{\pi\in\mathcal{M}(n;n_1,\dots,n_k)} q^{\mathrm{crossing}(\pi)},$$

where $\mathcal{M}(n; n_1, \dots, n_k)$ *is the set of perfect matchings* π *of* $S_1 \cup \cdots \cup S_k \cup S_{k+1}$ *such that if* π *contains a homogeneous edge* $(a, b) \in S_i$*, then* $i = k+1$*. Here,* S_i *is given in* (10) *for* $1 \le i \le k$*, and*

$$S_{k+1} = \{n_1 + \cdots + n_k + 1, n_1 + \cdots + n_k + 2, \dots, n_1 + \cdots + n_k + n\}.$$

In order to evaluate the sum in the left hand side of the above theorem, they [25] decompose $\pi \in \mathcal{M}(n; n_1, n_2, n_3, n_4)$ as a pair (π_0, π_1) of a matching π_0 of $[n]$ with m fixed points and an inhomogeneous perfect matching $\pi_1 \in \mathcal{M}(m, n_1, n_2, n_3, n_4)$ for some integer m. In this decomposition we have $\mathrm{crossing}(\pi) = \mathrm{crossing}^*(\pi_0) + \mathrm{crossing}(\pi_1)$, where $\mathrm{crossing}^*(\pi_0)$ is the number of pairs (e_1, e_2) such that

- $e_1 = \{a, b\}$ and $e_2 = \{c, d\}$ are edges of π_0 with $a < c < b < d$, or
- $e_1 = \{a, b\}$ is an edge of π_0 and $e_2 = \{c\}$ is a singleton of π_0 with $a < c < b$.

Then they use the following formula due to Josuat-Vergès [22, Proposition 5.1]:

$$(1-q)^{(n-m)/2} \sum_{\pi \in \mathcal{M}^*(n,m)} q^{\mathrm{crossing}^*(\pi)} = \sum_{k \ge 0} \left(\binom{n}{\frac{n-k}{2}} - \binom{n}{\frac{n-k}{2} - 1} \right) (-1)^{(k-m)/2} q^{\binom{(k-m)/2+1}{2}} \begin{bmatrix} \frac{k+m}{2} \\ \frac{k-m}{2} \end{bmatrix}_q, \tag{14}$$

where $\mathcal{M}^*(n, m)$ is the set of matchings of $[n]$ with m fixed points.

The appearance of the difference of binomial coefficients in (14) can be explained as follows. A *Dyck prefix* is a path obtained from a Dyck path by taking the first m steps for some m. By a similar argument using Hermite histories, the left hand side of (14) is the generating function for Dyck prefixes whose South-East steps are labeled by 1 or $-q^i$, where i is the y-coordinate of the starting point. Then we can use Penaud's idea [29] which decomposes such a labeled Dyck prefix into a Dyck prefix and a certain labeled Dyck path. The number of Dyck prefixes is given by a difference of binomial coefficients.

There are analogous combinatorial interpretations for the linearization coefficients of q-Charlier polynomials and q-Laguerre polynomials.

Theorem 7.4. [1, p. 127] *Let* $\mathcal{L}_C$ *be the normalized linear functional for the* q*-Charlier polynomials* $C_n^a(x; q)$*. Then*

$$\mathcal{L}_C(C_{n_1}^a(x; q) \cdots C_{n_k}^a(x; q)) = \sum_{\pi \in \Pi(n_1, \dots, n_k)} a^{\mathrm{block}(\pi)} q^{\mathrm{crossing}(\pi)},$$

where $\Pi(n_1, \dots, n_k)$ *is the set of partitions of* $S_1 \cup \cdots \cup S_k$ *which do not have homogeneous edges. Here,* S_i *is given in* (10) *for* $1 \le i \le k$*.*

Kim, Stanton and Zeng [24] found a combinatorial proof of Theorem 7.4. Kasraoui, Stanton, and Zeng [23] showed the following theorem using recurrence relations.

Theorem 7.5. [23, Theorem 5] *Let* $\mathcal{L}_L$ *be the normalized linear functional for the* q*-Laguerre polynomials* $L_n(x; q)$*. Then*

$$\mathcal{L}_L(L_{n_1}(x; q) \cdots L_{n_k}(x; q)) = \sum_{\pi \in D(n_1, \dots, n_k)} y^{\mathrm{wex}(\pi)} q^{\mathrm{crossing}(\pi)},$$

where $D(n_1, \dots, n_k)$ *is the set of permutations* π *of* $S_1 \cup \cdots \cup S_k$ *such that if* $\pi(a) = b$ *then* a *and* b *are in different* S_i*'s. Here,* S_i *is given in* (10) *for* $1 \le i \le k$*.*

Note that in Theorem 7.5, if $n_1 = \cdots = n_k = 1$, then $D(n_1, \ldots, n_k)$ becomes the set of *derangements* of $[k]$, i.e., permutations with no fixed points. Permutations in $D(n_1, \ldots, n_k)$ are called multi-derangements. There is a nice generating function expression for the number of multi-derangements, see [16, p. 563] and references therein.

Ismail, Kasraoui, and Zeng [16] found a general approach to find the linearization coefficients using recurrence relations.

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