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Rim Hook Lattices

Abstract

A partial order is defined on partitions by the removal of rim hooks of a given length. This poset is isomorphic to a product of Young lattices, guaranteeing rim hook versions of Schensted correspondences. Analogous results are given for shifted shapes.

1. Main Results

A *shape* (Young diagram) is a finite order ideal of the lattice $\mathbb{P}^2 = \{(k,l) : \; k,l \geq 1\}$. Shapes form the so-called *Young lattice* $\mathbb{Y}$ (see, e.g., [St86]). An i'th *diagonal* of a shape λ is the set $\{(k,l) \; : l-k=i \;\}$. We use the so-called "English notation" for realizing shapes in the 4th quadrant. We denote by $\#\lambda$ the number of boxes in a shape λ.

A *rim hook* is a set of elements ("boxes") of $\mathbb{P}^2$ which forms a contiguous strip and has at most one box on each diagonal. Throughout the paper a positive integer r is fixed; all of the rim hooks contain exactly r boxes. (Exception: Definition 3.4(3).)

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1.1 Definition. Let λ and $\mu \subset \lambda$ be shapes such that $\lambda - \mu$ is a rim hook. Then we write $\lambda \gtrdot \mu$. We write $\lambda \succeq \mu$ if there exists a sequence of shapes $\lambda = \lambda_0 \gtrdot \lambda_1 \gtrdot \cdots \gtrdot \lambda_k = \mu$. In other words, $\lambda \succeq \mu$ means that μ can be obtained by deleting some rim hooks from λ. If $\lambda \succeq \phi$ then λ is said to be *r-decomposable*. Let RH_r denote the poset of all r-decomposable shapes ordered by $\succeq$. This poset is called the *rim hook lattice*. (We shall prove that it is really a lattice.)

The following result is essentially known; however we could not find it elsewhere stated in this explicit form.

1.2 Theorem. *The rim hook lattice RH_r is isomorphic to the cartesian product of r copies of the Young lattice:*

$$RH_r \cong \mathbb{Y}^r \ .$$

The proof of this theorem is given in Sec.2.

Figures 1 and 2 show the lattices RH_2 and RH_3, respectively; on the latter the underlying poset $3\mathbb{P}^2$ is highlighted.

1.3 Definition. Let P be a graded poset, K a field of zero characteristic, and KP a vector space with a basis P. Define the *up* and *down operators* $U, D \in End(KP)$ by

$$Ux = \sum_{y \text{ covers } x} y \ ,$$

$$Dy = \sum_{y \text{ covers } x} x \ .$$

1.4 Proposition. (see, e.g., [St88]) *The up and down operators in the Young lattice $\mathbb{Y}$ satisfy*

$$DU - UD = I$$

where I is the identity transformation.

Thus $\mathbb{Y}$ is a *self-dual graph*[Fo1] or a *differential poset*[St88].

Generally, two graded graphs G_1 and G_2 with a common set of vertices and a common rank function are called *r-dual*[Fo1,Fo3] if the up operator U_1 in G_1 and the down operator D_2 in G_2 satisfy

$$D_2U_1 - U_1D_2 = rI \ .$$

It is easy to see (cf. [Fo1, Lemma 2.2.3]) that if the graphs G_1 and G_2 are r-dual, and the graphs H_1 and H_2 are s-dual, then $G_1 \times H_1$ and $G_2 \times H_2$ are $(r+s)$-dual.

1.5 Corollary (see, e.g., [SS90, Sec.9, (9)])**.** *The up and down operators in the rim hook lattice RH_r satisfy*

$$DU - UD = rI \ . \quad \square$$

So RH_r is an r-self-dual graph (r-differential poset). Hence one can apply to RH_r each of the enumerative results concerning such graphs (see [St88, Fo1, Ro91, etc.]). Moreover, it allows us to construct an analogue of the Schensted algorithm for the rim hook lattice (see [Fo2, Fo3]). This algorithm establishes a bijection between pairs of paths in RH_r ("standard rim hook tableaux"; see [SW85]) and permutations colored in r colors. It is clear from Theorem 1.2 that this algorithm is essentially a "direct product" of r copies of independently running standard Schensted algorithms (see, e.g., [Sa90]). It coincides with the algorithm of [SW85] which was originally described in terms of "insertion" procedures.

We rewrite below two enumerative formulae concerning general r-differential posets, when applied to the rim hook lattice.

Let $e_r(\lambda)$ denote the number of saturated chains $\lambda = \lambda_0 \gtrdot \lambda_1 \gtrdot \cdots \gtrdot \lambda_n = \phi$ (i.e., the number of standard rim hook tableaux of shape λ). Similarly, we let $e_r(\lambda/\alpha)$ denote the number of "standard skew rim hook tableaux of shape λ/α, i.e., the number of paths $\lambda = \lambda_0 \gtrdot \lambda_1 \gtrdot \cdots \gtrdot \lambda_n = \alpha$.

1.6 Corollary. *For $\lambda \in RH_r$,*

$$\text{(i)} \sum_{\#\lambda = rn} e_r^2(\lambda) = r^n n! \ ;$$

$$\text{(ii)} \sum_{\#\lambda = rn} e_r(\lambda) = \#\{r\text{-colored involutions in the symmetric group } S_n\}$$

where "r-colored involution" means a symmetric $n \times n$-matrix containing exactly one nonzero entry in each row and column; this entry should be one of $1, 2, \ldots, r$.

Proof. See, e.g., [Fo1, (1.5.19)] and [Fo2, Corollary 3.9.4]. □

The following result is an analogue of the formulae of [SS90].

1.7 Corollary. *Let $\alpha, \beta \in RH_r$, $\#\alpha = rk$, $\#\beta = r(k+n-m)$ (n and m are fixed). Then*

$$\sum_{\#\lambda = r(k+n)} e_r(\lambda/\alpha)\ e_r(\lambda/\beta) \quad = \quad \sum_j r^j \binom{m}{j} \binom{n}{j} j! \sum_{\#\mu = r(k-m+j)} e_r(\alpha/\mu)\ e_r(\beta/\mu)\ .$$

Proof. See [Fo2, Corollary 3.8.3(iv)]. □

In Sec. 3 we give the analogues of the above results for the shifted shapes.

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2. Fairy Sequences

2.1 Definition. A map $f : \mathbb{Z} \longrightarrow \mathbb{Z}$ is called a *fairy sequence* if the following conditions hold:

$$\begin{array}{rl} \text{(i)} & f(i) \geq f(j) \quad \text{whenever} \quad 0 \leq i \leq j \ ; \\ \text{(ii)} & f(j) \leq f(i) \quad \text{whenever} \quad j \leq i \leq 0 \ ; \\ \text{(iii)} & |f(i) - f(i-1)| \leq 1 \quad \text{for all} \quad i \ ; \\ \text{(iv)} & f(i) = 0 \quad \text{if} \quad |i| \quad \text{is sufficiently large.} \end{array}$$

2.2 Lemma. *There is a bijective correspondence between shapes (Young diagrams) and fairy sequences. This bijection (denoted $\lambda \mapsto f_\lambda$) is given by*

$$f_\lambda(i) \ = \ \#\{ \textit{ boxes on the i'th diagonal of } \lambda \} \ . \quad \Box$$

2.3 Lemma. *A shape λ is r-decomposable if and only if the corresponding fairy sequence $f = f_\lambda$ satisfies the following condition:*

(1) $$\sum_{i \equiv a (\mod r)} f(i) \ = \sum_{i \equiv b (\mod r)} f(i) \quad \textit{for any} \quad a, b \in \mathbb{Z} \ . \quad \Box$$

2.4 Definition. Let $f^{(0)}, \ldots ,f^{(r-1)}$ be fairy sequences. Then $f = \langle f^{(0)}, \ldots, f^{(r-1)} \rangle$ will denote the sequence defined by

(2) $$f(i) \ = \ \sum_{k=0}^{r-1} f^{(k)} \left(\left[\frac{i+k}{r} \right] \right)$$

where $[\ldots]$ stands for the integer part.

2.5 Lemma. *We have*

(3) $$f(i) - f(i-1) \ = \ f^{(k)} \left(\frac{i+k}{r} \right) - f^{(k)} \left(\frac{i+k}{r} - 1 \right)$$

where $k = (-i) \mod r$. $\Box$

2.6 Lemma. *For any fairy sequences $f^{(0)}, \ldots ,f^{(r-1)}$, the sequence $f = \langle f^{(0)}, \ldots, f^{(r-1)} \rangle$ is fairy and r-decomposable.*

Proof. The first property follows from Lemma 2.5, the second one — from

$$\sum_{i \equiv a (\mod r)} f(i) \ = \ \sum_k \sum_j f^{(k)}(j) \ . \quad \Box$$

2.7 Lemma. *Any fairy and r-decomposable sequence f can be uniquely represented as $f = \langle f^{(0)}, \dots, f^{(r-1)} \rangle$ where $f^{(0)}$, ... ,$f^{(r-1)}$ are some fairy sequences.*

Proof. Given such a sequence f, one can inductively use (2) to find the values $f^{(k)}(j)$, starting with $f^{(k)}(j) = 0$ for $j \ll 0$. The equality (3) guarantees that the resulting sequences satisfy Definition 2.1,(i)-(iii). The condition (1) for $b = a - 1$ gives, together with (3), the equality

$$\lim_{j\to\infty} f^{(k)}(j) - \lim_{j\to-\infty} f^{(k)}(j) = 0 \ ,$$

and the proof follows. □

The following statement contains Theorem 1.2.

Proof of Theorem 1.2. We will prove that the bijection

$$f \ \longleftrightarrow \ (f^{(0)}, \dots, f^{(r-1)}) \tag{4}$$

induces an isomorphism between RH_r and $\mathbb{Y}^r$. According to (2), the following are equivalent:

(i) adding 1 to some $f^{(k)}(j)$;

(ii) adding 1 to each of $f(i), f(i+1), \dots, f(i+r-1)$ for some i.

In view of Lemmas 2.6-2.7, the operations (i) and (ii) either both preserve the "fairyness" or both don't. To complete the proof, note that (i) corresponds to adding a box to the respective shape, and (ii) to adding a rim hook. □

Theorem 1.2 can be also proved by means of the approach of [JK81].

3. Shifted shapes

In this section we define the shifted rim hooks (Definition 3.4) and find the analogous isomorphism theorem (Theorem 3.7) for shifted shapes.

3.1 Definition. Let *SemiPascal* be the set

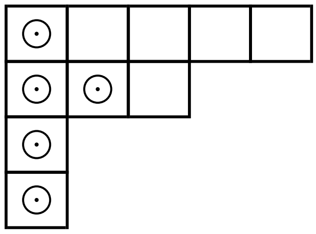
$$\{(k,l) \in \mathbb{Z}^2 \ : \ l > k \geq 1\}$$

ordered by inclusion. The finite order ideals in *SemiPascal* are called *shifted shapes*; the corresponding distributive lattice is denoted $\mathbb{SY}$. For any $\sigma \in \mathbb{SY}$, let $\lambda(\sigma)$ denote the "symmetrized" shape being a union of σ and a flipped shifted shape σ'; formally,

$$\sigma' \ = \ \{(l, k-1) \ : \ (k,l) \in \sigma\} \ , \quad \lambda(\sigma) = \sigma \cup \sigma' \ .$$

Informally, this means that we fold the shifted shape σ about a diagonal line just to the left of σ's main diagonal, and add it to σ.

For example, if $\sigma = 41$, then $\lambda(\sigma) = 5311$:



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Similarly, two fairy sequences f and g are said to be *folded* to each other if $f(k) = g(1-k)$ for all $k \in \mathbb{Z}$. This means that their respective shapes are related by folding across the diagonal in the definition of $\lambda(\sigma)$. We call f *self-folded* if $f(k) = f(1-k)$ for all $k \in \mathbb{Z}$. The next lemma follows immediately from the definition of $\lambda(\sigma)$.

3.2 Lemma. *For any shifted shape σ, the fairy sequence $f = f_{\lambda(\sigma)}$ is self-folded; namely,*

$$f(k) = f(1-k) \quad \textit{for} \quad k \in \mathbb{Z} \quad . \tag{5}$$

Conversely, any fairy sequence satisfying (5) *has a unique representation of the form $f = f_{\lambda(\sigma)}$ for an appropriate shifted shape σ.* $\square$

It is clear that self-conjugate (left-justified) shapes have fairy sequences f which are symmetric, $f(k) = f(-k)$, for all $k \in \mathbb{Z}$. We need these fairy sequences for the next lemma, which applies (4) to self-folded shapes $\lambda(\sigma)$.

3.3 Lemma (cf. [Ol87,MY86,GKS90])**.** *The bijection* (4)*, when restricted to the shifted shape case, reduces to a bijection between*

(i) *r-decomposable self-folded fairy sequences and*

(ii) *r-tuples $(f^{(0)}, \ldots, f^{(r-1)})$ of fairy sequences where $f^{(0)}$ is self-folded, $f^{(i)}$ is folded to $f^{(r-i)}$ for $1 \le i \le \frac{r-1}{2}$, and if r is even then, in addition, $f^{(r/2)}$ is symmetric.* $\square$

We next define shifted rim hooks. This definition will allow us to define the shifted rim hook lattice, SRH_r, on all r-decomposable shifted shapes.

3.4 Definition. A *shifted rim hook* is a *convex* subset of *SemiPascal* which satisfies one of the three following conditions:

(1) for some $i \ge r$, h has exactly one box on each of the diagonals $i-r+1, \ldots, i$;

(2) for some i, $r/2 < i < r$, h has two boxes on each of the diagonals $1, \ldots, r-i$ and one box on the diagonals $r-i+1, \ldots, i$;

(3) for $i = r/2$, h has one box on each of the diagonals $1, \ldots, r/2$.

(In the cases (1)-(2) a shifted rim hook contains r boxes, in the case (3) r is even and the shifted rim hook has $r/2$ boxes.)

The following picture describes some shifted rim hooks which one can add to the shifted shape 41:

$\odot$					E	D	C	B	A
$\odot$	$\odot$		J	G	F				
$\odot$	M	L	K	H					
$\odot$				I					

For $r = 4$: $BCDE$, $EFGJ$, JK; for $r = 5$: $ABCDE$, $DEFGJ$, $EFGJK$, $GHIJK$.

Shifted rim hooks define the covering relation in SRH_r. The posets (lattices) $\mathbb{SY}$ and SRH_2 are given in Figures 3 and 4 respectively; the join-irreducible elements are highlighted. The rank function $rank_r$ on SRH_r is standard for r odd: $rank_r(\sigma) = \#\sigma/r$, but this does not hold for r even. For example, for $r = 2$, $rank_2(\sigma)$ is the number of boxes of σ lying on odd diagonals; $rank_2(542) = 6$.

3.5 Lemma. *The bijection $\sigma \leftrightarrow f_{\lambda(\sigma)}$ is a poset isomorphism between $\mathbb{SY}$ and the coordinate-wise partial order on self-folded fairy sequences.* □

3.6 Lemma. *The coordinate-wise partial order on the set of symmetric fairy sequences is isomorphic to $\mathbb{SY}$.* □

Thus we obtain the following result.

3.7 Theorem. *If r is odd then*

$$SRH_r \cong \mathbb{SY} \times \mathbb{Y}^{\frac{r-1}{2}} \ ;$$

if r is even then

$$SRH_r \cong \mathbb{SY} \times \mathbb{SY} \times \mathbb{Y}^{\frac{r-2}{2}} \ . \quad \square$$

3.8 Lemma [Fo2]. *Let $\mathbb{SY}^*$ be the graph obtained from $\mathbb{SY}$ by doubling the edges which correspond to adding boxes lying outside the first diagonal. Then $\mathbb{SY}$ and $\mathbb{SY}^*$ are dual.*

We now define the graphs SRH_r^*. To get SRH_r^*, take the shifted rim hook lattice SRH_r and double its edges which correspond to adding rim hooks of type (1) (see Definition 3.4) with $i = rj$, $j \geq 2$, and, in case r is even, with $i = r(j + \frac{1}{2}))$, $j \geq 1$ as well.

3.9 Corollary. *The graphs SRH_r and SRH_r^* are $\left[\frac{r+2}{2}\right]$-dual. In addition,*
if r is odd then $SRH_r^* \cong \mathbb{SY}^* \times \mathbb{Y}^{\frac{r-1}{2}}$,
if r is even then $SRH_r^* \cong \mathbb{SY}^* \times \mathbb{SY}^* \times \mathbb{Y}^{\frac{r-2}{2}}$. □

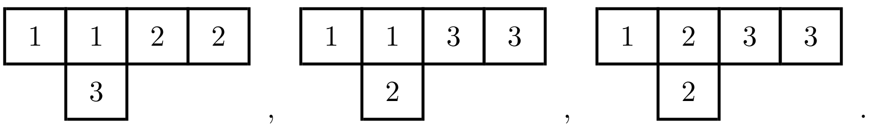
Once a dual graph is constructed, the corresponding Schensted algorithm arises (see [Fo2, Fo3]). In this case it falls into $\left[\frac{r-1}{2}\right]$ independently running ordinary Schensteds and 1 or 2 (for r odd and even, respectively) independently running "shifted Schensteds" (see [Wo84,Sa87,Ha89,Fo2]).

Now we can apply to the shifted rim hook lattices all of the results concerning arbitrary dual graphs (see [St88, St90, Fo1, Fo2]). For example, we get the following analogue of the Young-Frobenius identity. To state it, define the numbers $d_r(\sigma)$ as follows. Take any decomposition of a shifted shape σ into shifted rim hooks. Let $d_r(\sigma)$ be the number of shifted rim hooks whose value of i in Definition 3.4 is $r(j+1)$ or $r(j+\frac{1}{2})$ for $j \geq 1$.

3.10 Corollary. *Let $e_r(\sigma)$ denote the number of shifted rim hook tableaux of shape σ. Then*

$$\sum_{\substack{\sigma \in SRH_r \\ rank(\sigma)=n}} e_r^2(\sigma)\, 2^{d_r(\sigma)} = \left[\frac{r+2}{2}\right]^n n!\ . \qquad \square$$

As an example of Corollary 3.10 we take $r=2$ and $n=3$. Note that for $r=2$, $d_2(\sigma)$ is the number of shifted rim hooks whose value of i is not 1 or 2. The elements of rank 3 in SRH_2 are 6, 42, 41, 32, 31, and 5, which respectively have 1,1,3,3,1, and 1 shifted rim hook tableaux for $r=2$. For example, for 41, the shifted rim hook tableaux are



The shapes 6, 42, 41, 32, 31, and 5 have respective weights $2^{d_r(\sigma)}$ of 4, 2, 2, 2, 2, and 4. We see that Corollary 3.10 becomes $4+2+18+18+2+4=48=2^3\,3!$.

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