FAKE GAUSSIAN SEQUENCES

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Abstract. Some positivity conjectures are made generalizing known results for Gaussian posets.

For a sequence of non-negative integers $\vec{a} = (a_1, a_2, \cdots, a_n)$, and a non-negative integer m, MacMahon [1, p. 137] considered the rational function of q,

$$F(\vec{a}, m, q) = \frac{\prod_{i=1}^{n}(1-q^{m+i})^{a_i}}{\prod_{i=1}^{n}(1-q^i)^{a_i}}. \tag{1.1}$$

He was interested in the values of $\vec{a}$ such that $F(\vec{a}, m, q)$ is a polynomial in q for all non-negative integers m. For example,

$$F((1,1,\cdots,1), m, q) = \begin{bmatrix} m+n \\ n \end{bmatrix}_q,$$

or if a_i is decreasing, $F(\vec{a}, m, q)$ is a q-multinomial coefficient.

We say $\vec{a}$ has the *polynomial property* if $F(\vec{a}, m, q)$ is a polynomial in q for all non-negative integers m. It is easy to restate this polynomiality condition as a condition on $a = (a_1, a_2, \cdots, a_n)$.

Proposition 1. *$\vec{a}$ has the polynomial property if, and only if,*

$$\sum_i a_{ip} \leq \sum_i a_{ip+r}$$

for all positive integers $1 \leq r < p \leq n$.

Proof. We just check the power of the cyclotomic polynomial $\phi_p(q)$. In the denominator it appears $\sum_i a_{ip}$ times, in the numerator $\sum_i a_{ip+r}$ times, where $m \equiv -r \mod p$. $\square$

Proposition 1 with $p = n$ clearly implies that $a_r \geq a_n$, and since we may assume that $a_n > 0$, this implies each entry of $\vec{a}$ is strictly positive. Also Proposition 1 says that the allowed set of $\vec{a}$ is the set of integral points in a convex polyhedral cone C_n [4, §4.6].

One may ask if $F(\vec{a}, m, q)$ has non-negative coefficients if $\vec{a}$ has the polynomial property. If $(a_1, a_2, \cdots, a_n)$ are the level numbers of a known Gaussian poset, this is known to be true [4, p. 270]. This we can think of any $\vec{a}$ with the polynomial property as a *fake Gaussian sequence*. In general it is not true, for example

$$F((1,3,1,1,1,1,1,1,2,1,1,1,1,1,1,1,1), 1, q) = 1 + q + \cdots - q^7 + \cdots,$$

and it can be shown that $\vec{a} = (1, 3, 1^{2s}, 2, 1^{2s+2})$ also contains $-q^7$ for any $s \geq 3$. I do not know of any such examples with fewer than 17 parts.

Nonetheless, there are some special families of $\vec{a}$, which are not the level numbers of a known Gaussian poset, for which positivity does hold.

Proposition 2. *If $\vec{a}$ has at most 6 parts and the polynomial property, then $F(\vec{a}, m, q)$ has non-negative coefficients.*

Proof. The extreme rays of the cone C_6 are

$$\{(1,0,0,0,0,0),(1,1,0,0,0,0),(1,2,1,0,0,0),(1,1,1,0,0,0),(1,1,1,1,0,0),\\ (1,2,2,1,0,0),(1,2,3,2,1,0),(1,3,3,2,1,0),(1,2,2,2,1,0),(1,3,2,1,1,0),\\ (1,1,2,1,1,0),(1,1,1,1,1,0),(1,2,1,1,1,0),(1,2,3,3,2,1),(1,2,2,2,2,1),\\ (1,1,1,1,1,1)\}.$$

All are known to be level numbers of Gaussian posets [4, p. 270], except

$$\{(1,3,3,2,1,0),(1,2,1,1,1,0)\}.$$

There is simplicial triangulation of C_6 using these extreme rays and

$$\{(1,2,2,1,1,0),(2,3,3,3,2,1)\}.$$

These facts have been verified using `Porta`. Thus any integer point in C_6 must be a positive integral combination of these 18 vectors. Since

$$F(\vec{a}+\vec{b}, m, q) = F(\vec{a}, m, q)F(\vec{b}, m, q),$$

we must verify non-negativity at these remaining four points. A computer verification for a stronger statement using (1.3) is indicated below. □

It is of interest to consider the symmetric case: $a_i = a_{n+1-i}$ for all i, this occurs for connected Gaussian posets. MacMahon [1, p. 141-144] listed the extreme rays for $n \leq 8$.

Conjecture 1. *If $\vec{a}$ is symmetric and has the polynomial property, then $F(\vec{a}, m, q)$ has non-negative coefficients.*

Proposition 3. *Conjecture 1 holds for all $\vec{a}$ with at most 10 parts.*

Proof. We consider the cone $\tilde{C}_n$, which is C_n with the additional equalities

$$a_i = a_{n+1-i}, \quad 1 \leq i \leq n.$$

The extreme rays of $\tilde{C}_n$ for $n \leq 10$ are

$n=1$ $\{(1)\}$,
$n=2$ $\{(1,1)\}$,
$n=3$ $\{(1,1,1),(1,2,1)\}$,
$n=4$ $\{(1,1,1,1),(1,2,2,1),(1,1,1,1,1)\}$
$n=5$ $\{(1,1,2,1,1),(1,2,2,2,1),(1,2,3,2,1)\}$,
$n=6$ $\{(1,1,1,1,1,1),(1,2,2,2,2,1),(1,2,3,3,2,1)\}$,
$n=7$ $\{(1,1,1,1,1,1,1),(1,1,1,2,1,1,1),(1,2,2,2,2,2,1),(1,2,3,3,3,2,1),$
$(1,2,3,4,3,2,1)\}$,
$n=8$ $\{(1,1,1,1,1,1,1,1),(1,1,2,2,2,2,1,1),(1,2,2,2,2,2,2,1),$
$(1,2,2,3,3,2,2,1),(1,2,3,3,3,3,2,1),(1,2,3,4,4,3,2,1),$
$n=9$ $\{(1,2,3,4,5,4,3,2,1),(1,2,3,3,4,3,3,2,1),(1,1,2,2,3,2,2,1,1),$
$(1,2,2,2,3,2,2,2,1),(2,2,3,4,5,4,3,2,2),(1,1,1,1,2,1,1,1,1),$
$(1,2,3,4,4,4,3,2,1),(1,2,3,3,3,3,3,2,1),(1,2,2,2,2,2,2,2,1),$
$(1,1,1,1,1,1,1,1,1)\}$,
$n=10$ $\{(1,2,3,4,5,5,4,3,2,1),(2,3,5,5,6,6,5,5,3,2),(2,4,4,5,6,6,5,4,4,2),$
$(2,3,3,3,4,4,3,3,3,2),(1,2,3,4,4,4,4,3,2,1),(2,3,4,6,6,6,6,4,3,2),$
$(1,2,2,3,3,3,3,2,2,1),(2,2,3,4,4,4,4,3,2,2),(1,1,2,2,2,2,2,2,1,1),$
$(1,2,3,3,3,3,3,3,2,1),(1,2,2,2,2,2,2,2,2,1),(1,1,1,1,1,1,1,1,1,1)\}$,

while the possibly non-Gaussian extreme rays in these sets are

$\{(1,1,2,2,2,2,1,1),(1,2,2,3,3,2,2,1),(1,2,3,3,4,3,3,2,1),(1,2,2,2,3,2,2,2,1),$
$(2,2,3,4,5,4,3,2,2),(2,3,5,5,6,6,5,5,3,2),(2,4,4,5,6,6,5,4,4,2),$
$(2,3,3,3,4,4,3,3,3,2),(2,3,4,6,6,6,6,4,3,2),(1,2,2,3,3,3,3,2,2,1),$
$(2,2,3,4,4,4,4,3,2,2)\}$.

For any $n \leq 10$ there is a simplicial triangulation of $\tilde{C}_n$ using the appropriate extreme rays, thus we check only the possibly non-Gaussian extreme rays, using the technique below. □

To prove a particular special case by computer we verify a stronger positivity result. It is clear that $F(\vec{a},m,q)$ is a polynomial in q^m of degree $a_1+a_2+\cdots+a_n$, and thus may be expanded

$$(1.2) \qquad F(\vec{a},m,q)=\sum_{s=0}^{a_1+a_2+\cdots+a_n-1}\begin{bmatrix} m+a_1+a_2+\cdots+a_n-s \\ a_1+a_2+\cdots+a_n \end{bmatrix}_q W_s(\vec{a},q),$$

for some rational function $W_s(\vec{a},q)$ independent of m. Clearly, setting $m=0$ in (1.2) implies

$$W_0(\vec{a},q)=1$$

and recursively

$$W_i(\vec{a},q) = F(\vec{a},i,q) - \sum_{s=0}^{i-1} W_s(\vec{a},q) \begin{bmatrix} i+a_1+a_2+\cdots+a_n-s \\ a_1+a_2+\cdots+a_n \end{bmatrix}_q . \tag{1.3}$$

Clearly (1.3) implies that $W_s(\vec{a},q)$ is a polynomial in q, for example

$$W_1(\vec{a},q) = \prod_{j=1}^{n} \frac{(1-q^{j+1})^{a_j}}{(1-q^j)^{a_j}} - \begin{bmatrix} a_1+a_2+\cdots+a_n+1 \\ a_1+a_2+\cdots+a_n \end{bmatrix}_q .$$

Also it is easy to see that $W_s(\vec{a},q)=0$ for $s > a_1+a_2+\cdots+a_n-n$, by setting $m=-1,-2,\cdots,-(n-1)$ in (1.2).

Conjecture 2. *If $\vec{a}$ is symmetric and has the polynomial property, then the polynomials $W_s(\vec{a},q)$ have non-negative coefficients.*

It is clear that Conjecture 2 implies Conjecture 1. Conjecture 2 holds in the Gaussian poset case, when $W_s(\vec{a},q)$ is the generating function of linear extensions having s descents according to the major index. A special case of Conjecture 2 can be verified if $m=1$.

Proposition 4. *Let $c_i = a_i - a_{i-1}$, where $a_0 = 0$. If $c_1 \geq c_2 \geq \cdots \geq c_{\lfloor (n+1)/2 \rfloor}$, then Conjecture 2 holds for $m=1$.*

Proof. Put $p = \lfloor (n+1)/2 \rfloor$. We have

$$F(\vec{a},1,q) = \prod_{i=1}^{p} \left(\frac{1-q^{n+2-i}}{1-q^i}\right)^{c_i} = \begin{bmatrix} n+1 \\ p \end{bmatrix}_q^{c_p} \prod_{i=1}^{p-1} \begin{bmatrix} n+1 \\ p-i \end{bmatrix}_q^{c_{p-i}-c_{p-i+1}} .$$

If c_i is decreasing, then each term is a q-binomial coefficient raised to a non-negative power. □

Proposition 5. *Conjecture 2 holds if $\vec{a}$ has at most 10 parts.*

Proof. The proof of [3, Prop. 12.6(ii)] implies

$$W_s(\vec{a}+\vec{b},q) = \sum_{i,j} W_i(\vec{a},q) W_j(\vec{b},q) \begin{bmatrix} a_1+a_2+\cdots+a_n+j-i \\ s-i \end{bmatrix}_q \begin{bmatrix} b_1+b_2+\cdots+b_t+i-j \\ s-j \end{bmatrix}_q q^{(s-i)(s-j)},$$

so that the positivity of the coefficients of $W_s(\vec{a})$ is preserved under sum. Again we check $W_s(\vec{a})$ for the vectors $\vec{a}$ given in Proposition 3. This was done explicitly on a computer using (1.3). □

Proposition 2 was also verified, by checking the non-negativity of $W_s(\vec{a})$ for the vectors $\vec{a}$ listed there.

It is not hard to verify that

$$W_s(\vec{a},1/q) q^{s(a_1+a_2+\cdots+a_n)} = W_s(\vec{a},q).$$

If the coefficients of $W_s(\vec{a})$ are "centered' and unimodal, then (1.2) implies that the coefficients of $F(\vec{a}, m, q)$ are also unimodal. However the coefficients of $W_s(\vec{a})$ and $F(\vec{a}, m, q)$ are not always unimodal, e.g. $\vec{a} = (1,2,2,3,3,3,3,2,2,1)$, $m = 1$, and $s = 1$.

If $\vec{a}$ does arise from a Gaussian poset, then the coefficient of q^j in $F(\vec{a}, m, q)$ is an increasing function of $m \geq 0$. This property appears also to hold for $\vec{a}$ with the polynomial property.

Conjecture 3. *If $\vec{a}$ has the polynomial property, then the coefficient of q^j in $F(\vec{a}, m, q)$ is an increasing function of $m \geq 0$.*

If $W_s(\vec{a})$ has non-negative coefficients, then (1.2) verifies Conjecture 3. So the validity Conjecture 2 implies the validity Conjecture 3.

The $m \to \infty$ limit of (1.2) can be rewritten

$$g(\vec{a}, q) = \frac{\prod_{i=1}^{a_1+a_2+\cdots+a_n}(1-q^i)}{\prod_{i=1}^{n}(1-q^i)^{a_i}} = \sum_{s=0}^{a_1+a_2+\cdots+a_n-n} W_s(\vec{a}, q). \tag{1.5}$$

Conjecture 4. *If $a_i > 0$ for all i and $g(\vec{a}, q)$ is a polynomial, then the coefficients of $g(\vec{a}, q)$ are non-negative.*

If $F(\vec{a}, m, q)$ is a polynomial for all non-negative m, then $g(\vec{a}, q)$ is also a polynomial. So Conjecture 2 implies Conjecture 4 in these cases. However, $\vec{a}$ may not have the polynomial property, yet $g(\vec{a}, q)$ is a polynomial (e.g. $\vec{a} = (1,3,2)$).

If $\vec{a}$ is decreasing, then $g(\vec{a}, q)$ is a q-multinomial coefficient. Conjecture 4 asks for an interpretation of $g(\vec{a}, q)$ as the generating function for a statistic on some set of multiset permutations. Similarly, Conjecture 2 asks for an interpretation of $W_s(\vec{a})$ as the generating function of some set of permutations with s descents.

Remarks

(1) It is easy to see that if $F(\vec{a}, m, q)$ is polynomial for all non-negative m, then it is also a Laurent polynomial for all non-positive m.

(2) $W_s(\vec{a})$ can have negative coefficients even though $F(\vec{a}, m, q)$ does not: $\vec{a} = (1,3,1,1,1,1,2,1,1,1,1,1,1)$, $m = 1$, and $s = 1$.

(3) The number of new extreme rays for C_n reported by `Porta` is given below.

n	1	2	3	4	5	6	7	8	9	10
# of new extreme rays for n	1	1	2	2	7	3	19	18	40	99

(4) The number of extreme rays for n in the symmetric version of Conjecture 1 is given below.

n	1	2	3	4	5	6	7	8	9	10
# of extreme symmetric rays for n	1	1	2	2	4	3	5	6	10	12

(5) For the 93 extreme rays $\vec{a}$ in C_9, $W_s(\vec{a})$ has non-negative coefficients.

(6) There is no connected Gaussian poset with level numbers $(1,1,2,2,2,2,1,1)$ or $(1,2,2,3,3,2,2,1)$, these are the first candidates for a new Gaussian poset.

(7) An explicit version of (1.2) is

$$F((1,1,2,2,2,2,1,1),q,m) = \begin{bmatrix} m+12 \\ 12 \end{bmatrix}_q +$$
$$(q^3+q^4+q^5+2q^6+q^7+q^8+q^9)\begin{bmatrix} m+11 \\ 12 \end{bmatrix}_q +$$
$$(q^8+q^9+2q^{10}+2q^{11}+3q^{12}+2q^{13}+2q^{14}+q^{15}+q^{16})\begin{bmatrix} m+10 \\ 12 \end{bmatrix}_q +$$
$$(q^{15}+q^{16}+q^{17}+2q^{18}+q^{19}+q^{20}+q^{21})\begin{bmatrix} m+9 \\ 12 \end{bmatrix}_q + \begin{bmatrix} m+8 \\ 12 \end{bmatrix}_q .$$

Is this the character of a natural sl_2 representation, as occurs for the known Gaussian posets [2]?

(8) Conjecture 1 has been verified for $\vec{a} = (Reverse(\lambda), \lambda)$, of length $n \le 20$, and partitions λ, $|\lambda| \le 20$.

(9) Conjecture 2 has been verified for $a_1 + a_2 + \cdots + a_n \le 15$

(10) Conjecture 4 has been verified for $a_1 + a_2 + \cdots + a_n \le 18$.

References

1. P. MacMahon, *Combinatory Analysis Vol. 2*, Cambridge Univ. Press, Cambridge, 1918.
2. R. Proctor, *Bruhat lattices, plane partition generating functions, and minuscule representations*, Europ. J. Combin. **5** (1984), 331-350.
3. R. Stanley, *Ordered structures and partitions*, Mem. AMS 119 (1972), AMS.
4. ———, *Enumerative Combinatorics*, Wadsworth, Monterey, 1986.

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