[1] **SUMMABLE SUMS OF HYPERGEOMETRIC SERIES**

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Dedicated to Mizan Rahman.

Abstract. New expansions for certain ${}_2F_1$'s as a sum of r higher order hypergeometric series are given. When specialized to the binomial theorem, these r hypergeometric series sum. The results represent cubic and higher order transformations, and only Vandermonde's theorem is necessary for the elementary proof. Some q-analogues are also given.

1. Introduction. A hypergeometric series may always be written as a sum of two other hypergeometric series by splitting the series into its even and odd terms, for example

$$(1-x)^{-\alpha}=\sum_{k=0}^{\infty}\frac{(\alpha)_{2k}}{(2k)!}x^{2k}+\sum_{k=0}^{\infty}\frac{(\alpha)_{2k+1}}{(2k+1)!}x^{2k+1}.$$

One may also write a series as a sum of r series by reorganizing the terms modulo r.

In this paper a variation of this idea is considered, where the terms modulo r differ by a power of a linear function. We give in §2 four examples of this phenomena, Theorems 1-4, which may be considered as rth-degree transformations. Special cases of these expansions give new expansions for $(1-x)^{-\alpha}$: four cubic expansions are explicitly given as Corollaries 1-4 in §2. q-analogues may also be given, we state two such in §4: Theorems $1q$ and $2q$. Thus these results are in the same spirit as those of Mizan Rahman in [5], [6], and [7], and his work with George Gasper [1], [2], although these are at a much lower level. They are motivated by Corollary 6 for $r=2$, which was the key lemma in [4].

2. Main Results. In this section we state and prove the main results of this paper, Theorems 1-4.

One may expand a formal power series $F(x)$ in x as a formal power series in $y=x(1-x)^{-1/r}$

$$F(x)=\sum_{k=0}^{\infty}a_k\frac{x^k}{(1-x)^{k/r}}.$$

The coefficients a_k may be found from the Lagrange inversion formula. If r is a positive integer, then this series may be rewritten as

$$F(x)=\sum_{i=0}^{r-1}\sum_{t=0}^{\infty}b_t^{(i)}\frac{x^{rt+i}}{(1-x)^{t+i/r}}. \tag{2.1}$$

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We shall consider variations of (2.1) where the denominator exponent is either t or $t+1$ instead of $t+i/r$. We shall find expansions for the function

$$\sum_{k=0}^{\infty} \frac{(\alpha)_k}{(\beta)_k} x^k = {}_2F_1\left(\begin{matrix} \alpha, & 1 \\ & \beta \end{matrix} \middle| x \right).$$

First we consider what happens if all of the denominator exponents in (2.1) are t.

Proposition 1. *Let $F(x) = \sum_{s=0}^{\infty} F_s x^s$ be a formal power series in x, and let $r \geq 2$ be an integer. If*

$$F(x) = \sum_{i=0}^{r-1} \sum_{t=0}^{\infty} b_t^{(i)} \frac{x^{rt+i}}{(1-x)^t}$$

then

$$\sum_{s=0}^{t} \binom{t}{s} (-1)^s F_{rt+i-s} = \begin{cases} b_t^{(i)} & \text{if } 1 \leq i \leq r-1, \\ b_t^{(0)} - b_{t-1}^{(r-1)} & \text{if } i = 0. \end{cases}$$

Proof. Considering the coefficient of x^{rt+i}, $0 \leq i \leq r-1$, in

$$F(t)(1-x)^t = \sum_{j=0}^{r-1} \sum_{s=0}^{t} b_s^{(j)} x^{rs+j} (1-x)^{t-s} + \sum_{j=0}^{r-1} \sum_{s=t+1}^{\infty} b_s^{(j)} x^{rs+j} (1-x)^{t-s}$$

we find

$$b_t^{(i)} = \sum_{s=0}^{t} \binom{t}{s} (-1)^s F_{rt+i-s}, \quad 1 \leq i \leq r-1.$$

For $i = 0$ there are two terms ($j = 0$, $s = t$, $(b_t^{(0)})$ and $j = r-1$, $s = t-1$, $(b_{t-1}^{(r-1)})$) which contribute to the coefficient of x^{rt}. $\square$

Theorem 1. *For any integer $r \geq 2$ we have*

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{(\alpha)_k}{(\beta)_k} x^k &= \frac{\beta-1}{\beta-\alpha-1} - \\ &\frac{\alpha}{\beta-\alpha-1}\, {}_{r+1}F_r\left(\begin{matrix} \frac{\alpha+1}{r-1}, & \ldots, & \frac{\alpha+r-1}{r-1}, & \beta-\alpha-1, & 1 \\ & \frac{\beta}{r}, & \ldots, & \frac{\beta+r-1}{r} & \end{matrix} \middle| \frac{(r-1)^{r-1} x^r}{r^r (x-1)} \right) + \\ &\sum_{i=1}^{r-1} \frac{(\alpha)_i x^i}{(\beta)_i}\, {}_{r+1}F_r\left(\begin{matrix} \frac{\alpha+i}{r-1}, & \ldots, & \frac{\alpha+i+r-2}{r-1}, & \beta-\alpha, & 1 \\ & \frac{\beta+i}{r}, & \ldots, & \frac{\beta+i+r-1}{r} & \end{matrix} \middle| \frac{(r-1)^{r-1} x^r}{r^r (x-1)} \right). \end{aligned}$$

Proof. In Proposition 1 take

$$F_s = \frac{(\alpha)_s}{(\beta)_s}$$

and use the Vandermonde evaluation

$$\sum_{s=0}^{t} \binom{t}{s} (-1)^s F_{A-s} = \frac{(\alpha)_A (\beta-\alpha)_t}{(\beta)_A (1-A-\alpha)_t}. \quad \square \tag{2.2}$$

When $\beta = 1$ in Theorem 1, the left side sums to $(1-x)^{-\alpha}$ by the binomial theorem. We explicitly state the $r = 3$ case as a corollary.

Corollary 1.

$$\begin{aligned}(1-x)^{-\alpha} = & \quad {}_3F_2\left(\left.\begin{matrix}-\alpha, & (\alpha+1)/2, & (\alpha+2)/2\\ & 1/3, & 2/3\end{matrix}\,\right|\, \frac{-4x^3}{27(1-x)}\right)+\\ & \alpha x\; {}_3F_2\left(\left.\begin{matrix}1-\alpha, & (\alpha+1)/2, & (\alpha+2)/2\\ & 2/3, & 4/3\end{matrix}\,\right|\, \frac{-4x^3}{27(1-x)}\right)+\\ & \frac{\alpha(\alpha+1)x^2}{2}\,{}_3F_2\left(\left.\begin{matrix}1-\alpha, & (\alpha+2)/2, & (\alpha+3)/2\\ & 4/3, & 5/3\end{matrix}\,\right|\, \frac{-4x^3}{27(1-x)}\right).\end{aligned}$$

Next we consider a variation of Theorem 1 in which the denominator exponents for $b_t^{(i)}$ are $t+1$ for $1 \le i \le r-1$ and t for $b_t^{(0)}$.

Proposition 2. *Let $F(x) = \sum_{s=0}^{\infty} F_s x^s$ be a formal power series in x, and let r be a positive integer. If*

$$F(x) = \sum_{t=0}^{\infty} b_t^{(0)} \frac{x^{rt}}{(1-x)^t} + \sum_{i=1}^{r-1}\sum_{t=0}^{\infty} b_t^{(i)} \frac{x^{rt+i}}{(1-x)^{t+1}}$$

then

$$b_t^{(0)} = \sum_{s=0}^{t} \binom{t}{s} (-1)^s F_{rt-s}, \tag{1}$$

$$b_t^{(1)} = \sum_{s=0}^{t} \binom{t}{s} (-1)^s F_{rt+1-s}, \tag{2}$$

$$b_t^{(i)} = \sum_{s=0}^{t} \binom{t}{s} (-1)^s (F_{rt+i-s} - F_{rt+i-1-s}), \quad 2 \le i \le r-1. \tag{3}$$

Proof. Let's again find the coefficient of x^{rt+i}, $0 \le i \le r-1$, in

$$\begin{aligned}F(t)(1-x)^t = & \sum_{s=0}^{t} a_s x^{rs}(1-x)^{t-s} + \sum_{s=t+1}^{\infty} a_s x^{rs}(1-x)^{t-s}+\\ & \sum_{j=1}^{r-1}\sum_{s=0}^{t-1} b_s^{(j)} x^{rs+j}(1-x)^{t-s-1} + \sum_{j=1}^{r-1}\sum_{s=t}^{\infty} b_s^{(j)} x^{rs+j}(1-x)^{t-s-1}.\end{aligned}$$

If $i=0$ or 1, only the $s=t$ term contributes, and we obtain (1) and (2). For $2 \le i \le r-1$, the term $s=t$ contributes for $1 \le j \le i$ and we see that

$$\sum_{j=1}^{i} b_t^{(j)} = \sum_{s=0}^{t} \binom{t}{s} (-1)^s F_{rs+i-s},$$

which implies (3). $\square$

Theorem 2. *For any integer $r \geq 2$ we have*

$$\sum_{k=0}^{\infty}\frac{(\alpha)_k}{(\beta)_k}x^k = {}_{r+1}F_r\left(\begin{matrix}\frac{\alpha}{r-1}, & \ldots, & \frac{\alpha+r-2}{r-1}, & \beta-\alpha, & 1\\ \frac{\beta}{r}, & \ldots, & \frac{\beta+r-1}{r} & & \end{matrix}\,\middle|\,\frac{(r-1)^{r-1}x^r}{r^r(x-1)}\right)+$$
$$\frac{\alpha x}{\beta(1-x)}\;{}_{r+1}F_r\left(\begin{matrix}\frac{\alpha+1}{r-1}, & \ldots, & \frac{\alpha+r-1}{r-1}, & \beta-\alpha, & 1\\ \frac{\beta+1}{r}, & \ldots, & \frac{\beta+r}{r} & & \end{matrix}\,\middle|\,\frac{(r-1)^{r-1}x^r}{r^r(x-1)}\right)+$$
$$\sum_{i=2}^{r-1}\frac{(\alpha)_{i-1}(\alpha-\beta)x^i}{(\beta)_i(1-x)}$$
$$\times\,{}_{r+1}F_r\left(\begin{matrix}\frac{\alpha+i-1}{r-1}, & \ldots, & \frac{\alpha+i+r-3}{r-1}, & \beta-\alpha+1, & 1\\ \frac{\beta+i}{r}, & \ldots, & \frac{\beta+i+r-1}{r} & & \end{matrix}\,\middle|\,\frac{(r-1)^{r-1}x^r}{r^r(x-1)}\right).$$

Proof. In Proposition 2 take

$$F_s = \frac{(\alpha)_s}{(\beta)_s}$$

and use (2.2). $\square$

The $\beta = 1$ case is the next corollary.

Corollary 2.

$$\begin{aligned}(1-x)^{-\alpha} = &\;{}_3F_2\left(\begin{matrix}1-\alpha, & \alpha/2, & (\alpha+1)/2\\ & 1/3, & 2/3\end{matrix}\,\middle|\,\frac{-4x^3}{27(1-x)}\right)\\ &+\frac{\alpha x}{1-x}{}_3F_2\left(\begin{matrix}1-\alpha, & (\alpha+1)/2, & (\alpha+2)/2\\ & 2/3, & 4/3\end{matrix}\,\middle|\,\frac{-4x^3}{27(1-x)}\right)\\ &+\frac{\alpha(\alpha-1)x^2}{2(1-x)}{}_3F_2\left(\begin{matrix}2-\alpha, & (\alpha+1)/2, & (\alpha+2)/2\\ & 4/3, & 5/3\end{matrix}\,\middle|\,\frac{-4x^3}{27(1-x)}\right).\end{aligned}$$

Proposition 2 may be generalized by taking the denominator exponents for $b_t^{(i)}$ to be $t+1$ for $v+1 \leq i \leq r-1$, and the exponents for $b_t^{(i)}$ to be t, $0 \leq i \leq v$, for some $0 \leq v \leq r-2$.

Proposition 3. *Let $F(x) = \sum_{s=0}^{\infty} F_s x^s$ be a formal power series in x, and let $0 \leq v \leq r-2$ be non-negative integers. If*

$$F(x) = \sum_{i=0}^{v}\sum_{t=0}^{\infty} b_t^{(i)}\frac{x^{rt+i}}{(1-x)^t} + \sum_{i=v+1}^{r-1}\sum_{t=0}^{\infty} b_t^{(i)}\frac{x^{rt+i}}{(1-x)^{t+1}}$$

then

$$b_t^{(i)} = \sum_{s=0}^{t}\binom{t}{s}(-1)^s F_{rt+i-s}, \quad 0 \leq i \leq v+1$$
$$b_t^{(i)} = \sum_{s=0}^{t}\binom{t}{s}(-1)^s (F_{rt+i-s} - F_{rt+i-1-s}), \quad v+2 \leq i \leq r-1.$$

Theorem 2 is the $v = 0$ case of Theorem 3.

Theorem 3. *For integers $0 \le v \le r-2$ we have*

$$\sum_{k=0}^{\infty}\frac{(\alpha)_k}{(\beta)_k}x^k = {}_{r+1}F_r\left(\begin{matrix}\frac{\alpha}{r-1}, & \dots, & \frac{\alpha+r-2}{r-1}, & \beta-\alpha, & 1\\ & \frac{\beta}{r}, & \dots, & \frac{\beta+r-1}{r} & \end{matrix}\,\middle|\,\frac{(r-1)^{r-1}x^r}{r^r(x-1)}\right)+$$

$$\sum_{i=1}^{v}\frac{(\alpha)_i x^i}{(\beta)_i}\,{}_{r+1}F_r\left(\begin{matrix}\frac{\alpha+i}{r-1}, & \dots, & \frac{\alpha+i+r-2}{r-1}, & \beta-\alpha, & 1\\ & \frac{\beta+i}{r}, & \dots, & \frac{\beta+i+r-1}{r} & \end{matrix}\,\middle|\,\frac{(r-1)^{r-1}x^r}{r^r(x-1)}\right)+$$

$$\frac{(\alpha)_{v+1}x^{v+1}}{(\beta)_{v+1}(1-x)}$$

$$\times\,{}_{r+1}F_r\left(\begin{matrix}\frac{\alpha+v+1}{r-1}, & \dots, & \frac{\alpha+v+r-1}{r-1}, & \beta-\alpha, & 1\\ & \frac{\beta+v+1}{r}, & \dots, & \frac{\beta+v+r}{r} & \end{matrix}\,\middle|\,\frac{(r-1)^{r-1}x^r}{r^r(x-1)}\right)+$$

$$\sum_{i=v+2}^{r-1}\frac{(\alpha)_{i-1}(\alpha-\beta)x^i}{(\beta)_i(1-x)}$$

$$\times\,{}_{r+1}F_r\left(\begin{matrix}\frac{\alpha+i-1}{r-1}, & \dots, & \frac{\alpha+i+r-3}{r-1}, & \beta-\alpha+1, & 1\\ & \frac{\beta+i}{r}, & \dots, & \frac{\beta+i+r-1}{r} & \end{matrix}\,\middle|\,\frac{(r-1)^{r-1}x^r}{r^r(x-1)}\right).$$

Corollary 3.

$$\begin{aligned}(1-x)^{-\alpha} = {}& {}_3F_2\left(\begin{matrix}1-\alpha, & \alpha/2, & (\alpha+1)/2\\ & 1/3, & 2/3\end{matrix}\,\middle|\,\frac{-4x^3}{27(1-x)}\right)+\\ & \alpha x\,{}_3F_2\left(\begin{matrix}1-\alpha, & (\alpha+1)/2, & (\alpha+2)/2\\ & 2/3, & 4/3\end{matrix}\,\middle|\,\frac{-4x^3}{27(1-x)}\right)+\\ & \frac{\alpha(\alpha+1)x^2}{2(1-x)}{}_3F_2\left(\begin{matrix}1-\alpha, & (\alpha+2)/2, & (\alpha+3)/2\\ & 4/3, & 5/3\end{matrix}\,\middle|\,\frac{-4x^3}{27(1-x)}\right).\end{aligned}$$

A version of Proposition 3 exists for denominator exponents $t+1$ for $1 \le i \le v$, and t for $i=0$ or $v+1 \le i \le r-1$, for some $1 \le v \le r-2$.

Proposition 4. *Let $F(x) = \sum_{s=0}^{\infty} F_s x^s$ be a formal power series in x, and let $1 \le v \le r-2$ be non-negative integers. If*

$$F(x) = \sum_{t=0}^{\infty} b_t^{(0)}\frac{x^{rt}}{(1-x)^t} + \sum_{i=1}^{v}\sum_{t=0}^{\infty} b_t^{(i)}\frac{x^{rt+i}}{(1-x)^{t+1}} + \sum_{i=v+1}^{r-1}\sum_{t=0}^{\infty} b_t^{(i)}\frac{x^{rt+i}}{(1-x)^t}$$

then

$$\begin{aligned}b_t^{(0)} - b_{t-1}^{(r-1)} &= \sum_{s=0}^{t}\binom{t}{s}(-1)^s F_{rt-s},\\ b_t^{(1)} &= \sum_{s=0}^{t}\binom{t}{s}(-1)^s F_{rt+1-s},\\ b_t^{(i)} &= \sum_{s=0}^{t}\binom{t}{s}(-1)^s (F_{rt+i-s} - F_{rt+i-1-s}), \quad 2 \le i \le v\\ b_t^{(i)} &= \sum_{s=0}^{t}\binom{t}{s}(-1)^s (F_{rt+i-s} - F_{rt+v-s}), \quad v+1 \le i \le r-1.\end{aligned}$$

Although the equations in Proposition 4 may be solved using (2.2), we give here only the $r=3$, $v=1$ version.

Theorem 4.

$$\sum_{k=0}^{\infty}\frac{(\alpha)_k}{(\beta)_k}x^k = \frac{1-\beta}{\alpha-1}+$$
$$\frac{\alpha+\beta-2}{\alpha-1}\,{}_5F_4\left(\begin{matrix}\frac{\alpha-1}{2}, & \frac{\alpha}{2}, & \frac{\alpha+\beta+3}{5}, & \beta-\alpha, & 1\\ & \frac{\beta}{3}, & \frac{\beta+1}{3}, & \frac{\beta+1}{3} & \frac{\alpha+\beta-2}{5}\end{matrix}\;\middle|\;\frac{-4x^3}{27(1-x)}\right)+$$
$$\frac{\alpha x}{\beta(1-x)}\,{}_4F_3\left(\begin{matrix}\frac{\alpha+1}{2}, & \frac{\alpha+2}{2}, & \beta-\alpha, & 1\\ & \frac{\beta+1}{3}, & \frac{\beta+2}{3} & \frac{\beta+3}{3}\end{matrix}\;\middle|\;\frac{-4x^3}{27(1-x)}\right)+$$
$$\frac{\alpha(\alpha-\beta)x^2}{\beta(\beta+1)}\,{}_4F_3\left(\begin{matrix}\frac{\alpha+1}{2}, & \frac{\alpha+2}{2}, & \beta-\alpha+1, & 1\\ & \frac{\beta+2}{3}, & \frac{\beta+3}{3}, & \frac{\beta+4}{3}\end{matrix}\;\middle|\;\frac{-4x^3}{27(1-x)}\right).$$

Corollary 4.

$$(1-x)^{-\alpha} = {}_4F_3\left(\begin{matrix}1-\alpha, & \alpha/2, & (\alpha-1)/2, & (\alpha+4)/5\\ & 1/3, & 2/3, & (\alpha-1)/5\end{matrix}\;\middle|\;\frac{-4x^3}{27(1-x)}\right)+$$
$$\frac{\alpha x}{1-x}\,{}_3F_2\left(\begin{matrix}1-\alpha, & (\alpha+1)/2, & (\alpha+2)/2\\ & 2/3, & 4/3\end{matrix}\;\middle|\;\frac{-4x^3}{27(1-x)}\right)+$$
$$\frac{\alpha(\alpha-1)x^2}{2}\,{}_3F_2\left(\begin{matrix}2-\alpha, & (\alpha+1)/2, & (\alpha+2)/2\\ & 4/3, & 5/3\end{matrix}\;\middle|\;\frac{-4x^3}{27(1-x)}\right).$$

3. Previous results. In [3, (5.12)] another expansion is given for the left side in Theorems 1-4,

$$\sum_{k=0}^{\infty}\frac{(sb-a)_k}{(b+1)_k}x^k = (1-x)^a\sum_{k=0}^{\infty}\frac{b(-a-sk)_k}{k!(b+k)}(-x(1-x)^{s-1})^k. \tag{3.1}$$

If $s=r$, a positive integer, then (3.1) becomes

$$\sum_{k=0}^{\infty}\frac{(rb-a)_k}{(b+1)_k}x^k = (1-x)^a$$
$$\times\,{}_{r+1}F_r\left(\begin{matrix}\frac{a+1}{r}, & \cdots, & \frac{a+r}{r}, & b\\ \frac{a+1}{r-1}, & \cdots, & \frac{a+r-1}{r-1}, & b+1\end{matrix}\;\middle|\;\frac{r^r x(1-x)^{r-1}}{(r-1)^{r-1}}\right) \tag{3.2}$$

while for $s=-r$ a negative integer we have

$$\sum_{k=0}^{\infty}\frac{(-rb-a)_k}{(b+1)_k}x^k = (1-x)^a$$
$$\times\,{}_{r+2}F_{r+1}\left(\begin{matrix}\frac{-a}{r+1}, & \cdots, & \frac{-a+r}{r+1}, & b\\ \frac{-a}{r}, & \cdots, & \frac{-a+r-1}{r}, & b+1\end{matrix}\;\middle|\;\frac{-(r+1)^{r+1}x}{r^r(1-x)^{r+1}}\right). \tag{3.3}$$

One may consider Theorems 1-3, (3.2), and (3.3) as iterated higher order transformations.

4. q-analogues. It is a routine computation to find q-analogues of Propositions 1-4. The denominator terms of $(1-x)^t$ and $(1-x)^{t+1}$ are replaced by $(x;q)_t$ and $(x;q)_{t+1}$. What replaces (2.2) is the q-Vandermonde evaluation [1]

$$F_s = \frac{(a;q)_s}{(b;q)_s},$$

$$\sum_{s=0}^{t} \begin{bmatrix} t \\ s \end{bmatrix}_q (-1)^s q^{\binom{s}{2}} F_{A-s} = \frac{(a;q)_A (b/a;q)_t}{(b)_A (q^{1-A}/a;q)_t}. \tag{4.1}$$

We give the q-analogues of Theorems 1 and 2.

Theorem 1q. *For any integer $r \geq 2$,*

$$\begin{aligned}\sum_{k=0}^{\infty} \frac{(a;q)_k}{(b;q)_k} x^k =& 1 + \sum_{t=0}^{\infty} \frac{(a;q)_{(r-1)t+1}(b/a;q)_{t-1}}{(b;q)_{rt}} (-a)^{t-1} q^{rt(t-1)-\binom{t}{2}} \frac{x^{rt}}{(x;q)_t} + \\ & \sum_{i=1}^{r-1}\sum_{t=0}^{\infty} \frac{(a;q)_{(r-1)t+i}(b/a;q)_t}{(b;q)_{rt+i}} (-a)^t q^{t(rt+i-1)-\binom{t}{2}} \frac{x^{rt+i}}{(x;q)_t}.\end{aligned}$$

Theorem 2q. *For any integer $r \geq 2$,*

$$\begin{aligned}\sum_{k=0}^{\infty} \frac{(a;q)_k}{(b;q)_k} x^k =& \sum_{t=0}^{\infty} \frac{(a;q)_{(r-1)t}(b/a;q)_t}{(b;q)_{rt}} (-a)^t q^{t(rt-1)-\binom{t}{2}} \frac{x^{rt}}{(x;q)_t} + \\ & \sum_{t=0}^{\infty} \frac{(a;q)_{(r-1)t+1}(b/a;q)_t}{(b;q)_{rt+1}} (-a)^t q^{rt^2-\binom{t}{2}} \frac{x^{rt+1}}{(x;q)_{t+1}} + \\ & \sum_{i=2}^{r-1}\sum_{t=0}^{\infty} \frac{(a;q)_{(r-1)t+i-1}(b/a;q)_{t+1}}{(b;q)_{rt+i}} (-a)^{t+1} \\ & \times q^{(rt+i-1)(t+1)-\binom{t+1}{2}} \frac{x^{rt+i}}{(x;q)_{t+1}}.\end{aligned}$$

The $a = q^N$, $b = q$ case of Theorem $2q$ was the key Lemma used in [4] with $r = 2$. As corollaries we state the general $a = q^N$, $b = q$ case of Theorems $1q$ and $2q$.

Corollary 5. *For any integer $r \geq 2$,*

$$\begin{aligned}\frac{1}{(x;q)_N} =& 1 + \sum_{t=1}^{\infty} \begin{bmatrix} N+(r-1)t \\ rt \end{bmatrix}_q q^{rt(t-1)} \frac{x^{rt}}{(x;q)_t} + \\ & \sum_{i=1}^{r-1}\sum_{t=0}^{\infty} \begin{bmatrix} N+(r-1)t+i-1 \\ rt+i \end{bmatrix}_q q^{t(rt+i)} \frac{x^{rt+i}}{(x;q)_t}.\end{aligned}$$

Corollary 6. *For any integer $r \geq 2$,*

$$\frac{1}{(x;q)_N} = \sum_{t=0}^{\infty} \begin{bmatrix} N+(r-1)t-1 \\ rt \end{bmatrix}_q q^{rt^2} \frac{x^{rt}}{(x;q)_t} +$$
$$\sum_{t=0}^{\infty} \begin{bmatrix} N+(r-1)t \\ rt+1 \end{bmatrix}_q q^{t(rt+1)} \frac{x^{rt+1}}{(x;q)_{t+1}} +$$
$$\sum_{i=2}^{r-1} \sum_{t=0}^{\infty} \begin{bmatrix} N+(r-1)t+i-2 \\ rt+i \end{bmatrix}_q q^{(rt+i)(t+1)} \frac{x^{rt+i}}{(x;q)_{t+1}}.$$

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