MOMENT DUALITY FOR ORTHOGONAL POLYNOMIALS

MOURAD E.H. ISMAIL AND DENNIS STANTON

Abstract. The Laguerre, Charlier, and Meixner polynomials are polynomials in two variables: in x they are classical orthogonal polynomials, and in a parameter b they are type R_I orthogonal polynomials. Thus they have two types of orthogonality relations. Remarkably, the type R_I moments are identified as the orthogonal polynomial moments for another set of classical polynomials. A general notion of moment duality is introduced for polynomials in two variables. This program is continued for two and three parameter Askey-Wilson polynomials. The equality of the moments is equivalent to the equality of two continued fractions.

Dedicated to George Andrews and Bruce Berndt for their 85th birthdays

1. Introduction

The classical orthogonal polynomials, for example Jacobi, Gegenbauer, Laguerre, Meixner, and Charlier, depend upon parameters besides the polynomial variable x (see [1]). They may be considered as polynomials in two variables: x and one of the parameters. In the variable x they satisfy classical orthogonality relations. As monic polynomials in a parameter they may satisfy another type of orthogonality relation: type R_I orthogonality.

We will consider orthogonal polynomials in x, $p_n(x,b)$, which depend on a parameter b. If the recurrence coefficients b_n and λ_n in (2.1) are linear polynomials in b, then the moments $L(x^n)$ are also polynomials in b. Additionally, $p_n(x,b)$ is a degree n type R_I polynomial in b which depends on a parameter x. The type R_I moments are polynomials in x.

We introduce moment duality for these polynomials in two variables.

Definition 1.1. *Suppose that $\{p_n(x,b)\}_{n\geq 0}$ and $\{P_n(b,x)\}_{n\geq 0}$ are both polynomials in x and b of degree n. Suppose that $p_n(x,b)$ is orthogonal in x and type R_I orthogonal in b, while $P_n(b,x)$ is orthogonal in b and type R_I orthogonal in x. We say $p_n(x,b)$ and $P_n(b,x)$ have moment duality if*

(1) *the type R_I moments in b of $p_n(x,b)$ equal the moments in b of $P_n(b,x)$,*
(2) *the type R_I moments in x of $P_n(b,x)$ equal the moments in x of $p_n(x,b)$.*

The R_1 and R_{II} biorthogonal rational functions were introduced by Ismail and Masson [8] through continued fractions which extended the T-fractions of Thron, [9]. Later Zhedanov [14] showed that the R_I and R_{II} biorthogonal polynomials arise through the

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generalized eigenvalue problem $AX = \lambda MX$, where M is a positive definite matrix. When M is the identity matrix, we have orthogonal polynomials. M. Derevyagin [4] formulated and explored the connection between spectral theory of operators and the multipoint diagonal Pade approximation through the theory of R-fractions. Cooper, Jones, and Thron studied the orthogonal Laurent polynomials associated with the log-normal distribution, see [3].

The purpose of this paper is to identify five examples of classical polynomials which have moment duality: Laguerre, Charlier, Meixner, Al-Salam-Chihara, and continuous dual q-Hahn, and find the explicit linear functional for their type R_I orthogonality.

2. Orthogonal polynomials and type R_I polynomials

Monic orthogonal polynomials $p_n(x)$ satisfy the three-term recurrence relation [2, Theorem 4.1 p. 18]

$$p_{n+1}(x) = (x - b_n)p_n(x) - \lambda_n p_{n-1}(x), \quad n \geq 0, \quad p_{-1}(x) = 0, \quad p_0(x) = 1. \tag{2.1}$$

Proposition 2.1. *There is a linear functional L,* [2, (1.9)]*, defined on the vector space of polynomials in x which satisfies*

$$L(p_n(x)p_m(x)) = 0, \quad n \neq m.$$

The moments $\mu_n = L(x^n)$ are non-negative polynomials in the three-term recurrence coefficients $\{b_k\}_{k\geq 0}$ and $\{\lambda_k\}_{k\geq 1}$. For example

$$\mu_2 = b_0^2 + \lambda_1.$$

For the classical orthogonal polynomials an explicit representing measure $d\mu(x)$ is known for L,

$$L(x^n) = \int_{-\infty}^{\infty} x^n d\mu(x).$$

For example for the Laguerre polynomials $L_n^a(x), a \geq -1$, we have $d\mu(x) = x^a e^{-x} dx$ on $[0, \infty)$ and $\mu_n = (a+1)\cdots(a+n) = (a+1)_n$. We will use the linear functional rather than explicit measures in this paper.

The Pochhammer notation for shifted factorials and the q-shifted factorials found in [6],[7] will be used.

The moment generating function has a continued fraction expansion, [5],[13].

Proposition 2.2. *The moment generating function for orthogonal polynomials, as a formal power series in t, satisfies*

$$\sum_{n=0}^{\infty} \mu_n t^n = \cfrac{1}{1 - b_0 t - \cfrac{\lambda_1 t^2}{1 - b_1 t - \cfrac{\lambda_2 t^2}{1 - b_2 t - \cfrac{\lambda_3 t^2}{\cdots}}}}$$

Type R_I polynomials $P_n(x)$ [8, (1.1)] allow a more general recurrence.

Proposition 2.3. *Monic type R_I orthogonal polynomials $P_n(x)$ satisfy the three term recurrence*

$$P_{n+1}(x) = (x - B_n)P_n(x) - (A_n x + \Lambda_n)P_{n-1}(x), \quad n \geq 0, \quad P_{-1}(x) = 0, \quad P_0(x) = 1.$$

We assume $A_n \neq 0$, and let

$$D_n(x) = c_n \prod_{k=1}^{n} (A_k x + \Lambda_k)$$

for an appropriately chosen constant c_n. We also assume the non-degeneracy condition $P_n(-\Lambda_n/A_n) \neq 0$.

Let V be the vector space

$$V = span\{1, x, \cdots, x^n, \cdots, 1/D_1(x), 1/D_2(x), \cdots\}.$$

Note that V contains $r(x)/D_n(x)$ for any polynomial $r(x)$, see [10, Corollary 2.5].

The type R_I polynomials also have an orthogonality relation [8, Theorem 2.1].

Proposition 2.4. *There is a linear functional L on V such that*

$$L\left(P_k(x)\frac{P_n(x)}{D_n(x)}\right) = 0, \quad 0 \leq k \leq n-1.$$

The linear functional L acts on a larger vector space than polynomials, but the moments $L(x^n)$ still exist.

Proposition 2.5. *The moments $L(x^n)$ are polynomials in B_k, A_k, and Λ_k and may be found recursively* [10, Corollary 3.12, $m = 0$] *from*

$$L(P_n(x)) = A_1 A_2 \cdots A_n.$$

The value of the linear functional $L\left(\frac{1}{D_n(x)}\right)$ may be found recursively from

$$L\left(\frac{P_n(x)}{D_n(x)}\right) = 0, \quad n \geq 1.$$

For example,

$$L(x^2) = A_1^2 + A_1 A_2 + 2A_1 B_0 + B_0^2 + A_1 B_1 + \Lambda_1.$$

The moment generating function for type R_I polynomials also has a continued fraction expansion, [10, Proposition 5.5].

Proposition 2.6. *The moment generating function for type R_I orthogonal polynomials, as a formal power series in t, satisfies*

$$\sum_{n=0}^{\infty} \mu_n t^n = \cfrac{1}{1 - B_0 t - \cfrac{A_1 t + \Lambda_1 t^2}{1 - B_1 t - \cfrac{A_2 t + \Lambda_2 t^2}{1 - B_2 t - \cfrac{A_3 t + \Lambda_3 t^2}{\cdots}}}}$$

3. b-Laguerre polynomials

In this section we consider the Laguerre polynomials $L_n^b(x)$ as polynomials in x and b.

Definition 3.1. *The b-Laguerre polynomials are*

$$P_n(b) = \sum_{k=0}^{n} \frac{(-n)_k}{k!}(b+1+k)_{n-k}x^k, \quad n \geq 0.$$

The three-term recurrence for the Laguerre polynomials $L_n^b(x)$ becomes a type R_I recurrence for the b-Laguerre polynomials.

Proposition 3.2. *For $n \geq 0$,*

$$P_{n+1}(b) = (b - x + 2n + 1)P_n(b) - n(n+b)P_{n-1}(b).$$

Proposition 3.3. *The b-Laguerre polynomials satisfy the type R_I recurrence relation in Proposition 2.3*

$$P_{n+1}(b) = (b - B_n)P_n(b) - (A_n b + \Lambda_n)P_{n-1}(b)$$
$$B_n = x - 2n - 1, \quad A_n = n, \quad \Lambda_n = n^2.$$

The non-degeneracy condition is $x \neq 0$ because

$$P_n(-n) = (-x)^n.$$

So there is a linear functional L on

$$V = span\{1, b, \cdots, b^n, \cdots, 1/(b+1), 1/(b+1)(b+2), \cdots\}$$

which satisfies

$$L\left(b^k \frac{P_n(b)}{(b+1)(b+2)\cdots(b+n)}\right) = 0, \quad 0 \leq k \leq n-1.$$

The moments $L(b^n)$ of L are uniquely determined by

$$L(1) = 1, \quad L(P_n(b)) = A_1 A_2 \cdots A_n = n!. \tag{3.1}$$

Theorem 3.4. *The type R_I moments of L for the b-Laguerre polynomials are the same as the Charlier moments,*

$$L(b^n) = \sum_{k=1}^{n} S(n,k)x^k.$$

Proof. We first verify the connection relation to Charlier polynomials

$$P_n(b) = \sum_{k=0}^{n} \binom{n}{k}^2 (n-k)! q_k(b) \tag{3.2}$$

where

$$q_k(b) = (-x)^k \sum_{s=0}^{k} \frac{(-k)_s}{s!}(-b)_s(-1/x)^s.$$

Assuming (3.2) is true, applying the Charlier linear functional C gives

$$C(P_n(b)) = n!,$$

so $C = L$ on polynomials.

So it remains to prove (3.2). Using

$$\binom{b+j}{p} = \sum_{s=0}^{j} \binom{j}{p-s}\binom{b}{s}$$

with $j = n$, $p = n-k$, we have

$$P_n(b) = \sum_{k=0}^{n} \frac{(-n)_k}{k!} x^k (n-k)! \sum_{s=0}^{n-k} \binom{n}{k+s}\binom{b}{s}.$$

So the coefficient of $\binom{b}{s}$ in $P_n(b)$ is

$$\sum_{k=0}^{n-s} \frac{(-n)_k}{k!} x^k (n-k)! \binom{n}{k+s}$$

while the same coefficient on the right side of (3.2) is

$$\sum_{k=s}^{n} \binom{n}{k}^2 (n-k)!(-k)_s(-x)^{k-s}(-1)^s. \tag{3.3}$$

These are equal, after replacing k by $k+s$ in (3.3). □

The values of the b-Laguerre functional on the rational functions may be found.

Theorem 3.5. *The values of the b-Laguerre linear functional on the rational functions are given by*

$$L\left(\frac{1}{(b+1)(b+2)\cdots(b+n)}\right) = \frac{1}{x^n}.$$

Proof. We prove this by induction on n using

$$L\left(\frac{P_n(b)}{(b+1)(b+2)\cdots(b+n)}\right) = 0$$

for $n \geq 1$.

The defining relation becomes

$$\frac{P_n(b)}{(b+1)(b+2)\cdots(b+n)} = \sum_{k=0}^{n} \frac{(-n)_k}{k!} \frac{1}{(b+1)(b+2)\cdots(b+k)} x^k.$$

Applying L and induction we have

$$0 = \sum_{k=0}^{n-1} \frac{(-n)_k}{k!} + \frac{(-n)_n}{n!} L\left(\frac{1}{(b+1)(b+2)\cdots(b+n)}\right) x^n$$

which implies

$$L\left(\frac{1}{(b+1)(b+2)\cdots(b+n)}\right) = \frac{1}{x^n}.$$

□

The moment generating function for $L(b^n)$ has two continued fraction expansions. One for the classical Charlier polynomials (see Section 4), and the other for the type R_I b-Laguerre polynomials.

Theorem 3.6.

$$\sum_{n=0}^{\infty}\left(\sum_{k=1}^{n} S(n,k)x^k\right)t^n = \cfrac{1}{1-b_0t-\cfrac{\lambda_1t^2}{1-b_1t-\cfrac{\lambda_2t^2}{1-b_2t-\cfrac{\lambda_3t^2}{\cdots}}}}$$
$$= \cfrac{1}{1-B_0t-\cfrac{A_1t+\Lambda_1t^2}{1-B_1t-\cfrac{A_2t+\Lambda_2t^2}{1-B_2t-\cfrac{A_3t+\Lambda_3t^2}{\cdots}}}}$$

where

$$b_n = n+x, \quad \lambda_n = nx$$
$$B_n = x-2n-1, \quad A_n = n, \quad \Lambda_n = n^2.$$

and $S(n,k)$ are the Stirling numbers of the second kind.

4. b-Charlier polynomials

The b-Charlier polynomials as a monic polynomial of b of degree n are

$$P_n(b) = b^n \sum_{k=0}^{n} \frac{(-n)_k(-x)_k}{k!}(-1/b)^k.$$

In this section we consider these as type R_I polynomials in b.

The Charlier recurrence becomes a type R_I recurrence relation (2.3)

$$P_{n+1}(b) = (b-B_n)P_n(b) - (A_nb+\Lambda_n)P_{n-1}(b)$$
$$B_n = x-n, \quad A_n = n, \quad \Lambda_n = 0.$$

Thus the vector space V for the b-Charlier linear functional L is

$$V = span\{1, b, b^2, \cdots, 1/b, 1/b^2, \cdots\}.$$

The non-degeneracy condition is $x \neq 0, 1, 2, \cdots$ because

$$P_n(0) = (-x)_n.$$

Theorem 4.1. *The moments of the b-Charlier linear functional L are the same as the Laguerre moments,*

$$L(b^n) = (x+1)(x+2)\cdots(x+n).$$

Proof. Again we use induction and

$$L(P_n(b)) = A_1A_2\cdots A_n = n!. \tag{4.1}$$

This time (4.1) becomes

$$\sum_{k=0}^{n} \frac{(-n)_k}{k!}(-x)_k(-1)^k(x+1)_{n-k} = n!$$

which is a special case of the Chu-Vandermonde theorem. □

Theorem 4.2. *The negative moments of the b-Charlier linear functional L are*

$$L(b^{-n}) = \frac{1}{x(x-1)\cdots(x-n+1)}.$$

Proof. Again we use induction and

$$L(P_n(b)/b^n) = 0, \quad n \geq 1. \tag{4.2}$$

This time (4.2) becomes

$$\sum_{k=0}^{n} \frac{(-n)_k}{k!}\frac{(-x)_k}{x(x-1)\cdots(x-k+1)}(-1)^k = 0, \quad n \geq 1,$$

which is a special case of the binomial theorem. □

Corollary 4.3. *The Laguerre polynomials $p_n(x,b) = L_n^b(x)$ and the Charlier polynomials $P_n(b,x) = C_n(b;x)$ have moment duality.*

Again we have two expressions for the moment generating function: Laguerre and b-Charlier.

Theorem 4.4.

$$\begin{aligned}\sum_{n=0}^{\infty}(x+1)_n t^n &= \cfrac{1}{1-b_0t-\cfrac{\lambda_1t^2}{1-b_1t-\cfrac{\lambda_2t^2}{1-b_2t-\cfrac{\lambda_3t^2}{\cdots}}}}\\ &= \cfrac{1}{1-B_0t-\cfrac{A_1t+\Lambda_1t^2}{1-B_1t-\cfrac{A_2t+\Lambda_2t^2}{1-B_2t-\cfrac{A_3t+\Lambda_3t^2}{\cdots}}}}\end{aligned}$$

where

$$\begin{aligned} b_n &= 2n+x+1, \quad \lambda_n = n(n+x)\\ B_n &= \ x-n, \quad A_n = n, \quad \Lambda_n = 0.\end{aligned}$$

5. β-Meixner polynomials

The monic Meixner polynomials satisfy (2.1) with

$$b_n = \frac{n + (\beta + n)c}{1 - c}, \quad \lambda_n = \frac{n(n + \beta - 1)c}{(1 - c)^2}. \tag{5.1}$$

Thus as polynomials in β they are type R_I polynomials.

Definition 5.1. *The monic β-Meixner polynomials are defined by*

$$P_n(\beta) = (\beta)_n \sum_{k=0}^{n} \frac{(-n)_k(-x)_k}{k!(\beta)_k}(1 - 1/c)^k.$$

Proposition 5.2. *The monic β-Meixner polynomials satisfy the type R_I recurrence relation in* (2.3)

$$\begin{aligned} P_{n+1}(\beta) &= (\beta - B_n)P_n(\beta) - (A_n\beta + \Lambda_n)P_{n-1}(\beta) \\ B_n &= ((1 - c)x - n(1 + c))/c, \quad A_n = n/c, \quad \Lambda_n = n(n - 1)/c. \end{aligned}$$

The non-degeneracy condition is $x \neq 0, 1, 2, \cdots$ and $c \neq 1$ because

$$P_n(1 - n) = (-x)_n(1 - 1/c)^n.$$

The vector space in this case is

$$V = span\{1, \beta, \beta^2, \cdots, 1/\beta, 1\beta(\beta + 1), \cdots, \}.$$

The b-Laguerre and the b-Charlier polynomials had two nice features. Their moments were equal to moments of classical orthogonal polynomials, and there were simple explicit formulas for the rational part of the linear functional. This phenomenon persists for the Meixner polynomials.

First we relate the β-Meixner moments back to classical Meixner moments. Let $L_{Meix,\beta,c}(x^n) = \mu_n(\beta, c)$ be the moments for the Meixner polynomials given by (5.1).

Proposition 5.3. *The type R_I moments of the β-Meixner polynomials are the moments of the orthogonal polynomials with*

$$b_n = (2n + 1 - nc + (1 - c)x)/c, \quad \lambda_n = n(1 - c)(n + x)/c^2.$$

This may be rewritten as

$$L(\beta^n) = \sum_{k=0}^{n} \binom{n}{s} \mu_s(x + 1, 1 - c) = L_{Meix,x+1,1-c}((\beta + 1)^n).$$

Proof. Let $M_n(x; \beta, c)$ be the monic Meixner polynomial in x of degree n

$$M_n(x; \beta, c) = (\beta)_n(1 - 1/c)^{-n} \sum_{k=0}^{n} \frac{(-n)_k(-x)_k}{k!(\beta)_k}(1 - 1/c)^k.$$

We will show the connection coefficient relation

$$P_n(\beta + 1) = \sum_{k=0}^{n} \binom{n}{k}^2 \frac{(n - k)!}{c^{n-k}} M_k(\beta; x + 1, 1 - c) \tag{5.2}$$

Applying the Meixner functional $L_{Meix,x+1,1-c}$, only the constant term $k=0$ survives on the right side

$$L_{Meix,x+1,1-c}(P_n(\beta+1)) = n!/c^n = A_1A_2\cdots A_n.$$

So the composition of $L_{Meix,x+1,1-c}$ with the translation $\beta \to \beta+1$ must agree with the type R_I linear functional L,

$$L(\beta^n) = L_{Meix,x+1,1-c}((\beta+1)^n) = \sum_{k=0}^{n} \binom{n}{s} \mu_s(x+1, 1-c).$$

The proof of (5.2) is similar to the proof of (3.2). □

Remark 5.4. *Proposition 5.3 is equivalent to*

$$L((\beta-1)(\beta-2)\cdots(\beta-n)) = \frac{(1-c)^n}{c^n}(x+1)(x+2)\cdots(x+n).$$

We give the value of the β-Meixner linear functional on the rational part.

Proposition 5.5. *The type R_I linear functional L of the β-Meixner polynomials satisfies*

$$L\left(\frac{1}{\beta(\beta+1)\cdots(\beta+n-1)}\right) = \frac{c^n}{(1-c)^n}\frac{1}{x(x-1)\cdots(x-n+1).}$$

Proof. We have by induction that this choice works. If $n \geq 1$,

$$\begin{aligned} L\left(\frac{P_n(\beta)}{(\beta)_n}\right) &= \sum_{k=0}^{n} \frac{(-n)_k(-x)_k}{k!}(1-1/c)^k L\left(\frac{1}{(\beta)_k}\right) \\ &= \sum_{k=0}^{n} \frac{(-n)_k(-x)_k}{k!}(1-1/c)^k \frac{(-c)^k}{(1-c)^k(-x)_k} \\ &= 0 \end{aligned}$$

again by the binomial theorem. □

The Meixner duality results can be summarized by the next Proposition.

Proposition 5.6. *The polynomials*

$$p_n(x,b) = M_n(x;b,c) \text{ and } P_n(b,x) = M_n(b-1;x+1,1-c)$$

have moment duality.

Proof. This follows by iterating the map $M_n(x;\beta,c) \to M_n(\beta-1;x+1,1-c)$. □

Finally we have the two forms of the moment generating function.

Theorem 5.7. *The moment generating function for the β-Meixner is*

$$\sum_{n=0}^{\infty} L(\beta^n)t^n = \cfrac{1}{1-b_0t-\cfrac{\lambda_1t^2}{1-b_1t-\cfrac{\lambda_2t^2}{1-b_2t-\cfrac{\lambda_3t^2}{\cdots}}}}$$

$$= \cfrac{1}{1-B_0t-\cfrac{A_1t+\Lambda_1t^2}{1-B_1t-\cfrac{A_2t+\Lambda_2t^2}{1-B_2t-\cfrac{A_3t+\Lambda_3t^2}{\cdots}}}}$$

where

$$b_n = (2n+1-nc+(1-c)x)/c, \quad \lambda_n = n(1-c)(n+x)/c^2$$
$$B_n = ((1-c)x-n(1+c))/c, \quad A_n = n/c, \quad \Lambda_n = n(n-1)/c.$$

6. b-Al-Salam-Chihara polynomials

The monic Al-Salam-Chihara polynomials [7, §15.1] are

$$Q_n(x/2;a,b|q) = \frac{1}{a^n}\sum_{k=0}^{n}\frac{(q^{-n};q)_k}{(q;q)_k}(abq^k;q)_{n-k}q^k\prod_{j=0}^{k-1}(1-axq^j+a^2q^{2j}).$$

which satisfy (2.1) with

$$b_n = (a+b)q^n, \quad \lambda_n = (1-q^n)(1-abq^{n-1}).$$

This is a polynomial in b with a type R_I recurrence.

Definition 6.1. *The monic b-Al-Salam-Chihara polynomials are defined by*

$$P_n(b) = \frac{1}{(-a)^nq^{\binom{n}{2}}}\sum_{k=0}^{n}\frac{(q^{-n};q)_k}{(q;q)_k}(abq^k;q)_{n-k}q^k\prod_{j=0}^{k-1}(1-axq^j+a^2q^{2j}).$$

Proposition 6.2. *The monic b-Al-Salam-Chihara polynomials satisfy the type R_I recurrence relation* (2.3)

$$P_{n+1}(b) = (b-B_n)P_n(b)-(A_nb+\Lambda_n)P_{n-1}(b),$$
$$B_n = xq^{-n}-a, \quad A_n = -aq^{-n}(1-q^n), \quad \Lambda_n = q^{1-2n}(1-q^n).$$

The non-degeneracy condition is $x \neq aq^j+q^{-j}/a, j=0,1,2,\cdots$, because

$$P_n(q^{1-n}/a) = q^{-\binom{n}{2}}(-1)^n(az,a/z;q)_n/a^n.$$

The vector space in this case is

$$V = span\{1,b,b^2,\cdots,1/(1-ab),1/(ab;q)_2,\cdots,\}.$$

Theorem 6.3. *The type R_I moments of the b-Al-Salam-Chihara polynomials are the moments of the orthogonal polynomials with* (2.1)

$$b_n = (-a - aq + aq^{n+1} + xq^{n+1})/q^{2n+1}, \quad \lambda_n = -q^{1-n}(1-q^{-n})(1-q^{-n}a/z)(1-q^{-n}az).$$

where $x = z + z^{-1}$.

Definition 6.4. *The monic big q^{-1}-Laguerre polynomials* [12, §3.11] $R_n(b;\alpha,\beta;q^{-1})$ *in b are defined by* (2.1) *with*

$$b_n = \alpha q^{-n-1} + \beta q^{-n-1} + \alpha\beta q^{-n-1} - \alpha\beta q^{-1-2n} - \alpha\beta q^{-2-2n}$$
$$\lambda_n = -\alpha\beta q^{-1-n}(1-q^{-n})(1-\alpha q^{-n})(1-\beta q^{-n})$$

Theorem 6.5. *The orthogonal polynomials in Theorem 6.3 are big q^{-1}-Laguerre polynomials $(q/a)^n R_n(ab/q; a/z, az; q^{-1})$, where $x = z + z^{-1}$.*

Proof. Rescaling a monic orthogonal polynomial $p_n(x)$ to $s_n(x) = C^n p_n(x/C)$ multiplies (b_n, λ_n) for $p_n(x)$ in (2.1) by (C, C^2) for $s_n(x)$. The choice of $C = q/a$ shows the recurrence in (6.4) becomes that given in Conjecture 6.3. □

Proof. Again we use a connection coefficient relation. The rescaled big q^{-1}-Laguerre polynomials are []

$$t_n(b) = (q/a)^n (a/zq;q^{-1})_n (az/q;q^{-1})_n \sum_{k=0}^{n} \frac{(q^n;q^{-1})_k}{(q^{-1};q^{-1})_k} \frac{(baq^{-1};q^{-1})_k}{(a/zq;q^{-1})_k (az/q;q^{-1})_k} q^{-k}.$$

The connection relation between the b-Al-Salam-Chihara polynomials and $t_n(b)$ is

$$P_n(b) = \sum_{k=0}^{n} \frac{(q^n, q^{-1})_k^2}{(q;q)_k} (-a)^k q^{-2nk+3\binom{k+1}{2}-2k} t_{n-k}(b). \tag{6.1}$$

Applying the linear functional T for $t_n(b)$ we have

$$T(P_n(b)) = (-a)^n (q;q)_n q^{-\binom{n+1}{2}} = A_1 A_2 \cdots A_n$$

so T equals the b-Al-Salam-Chihara linear functional L on polynomials in b.

To prove (6.1), we use

$$(A/C;q)_m = (C;q^{-1})_m (-C)^{-m} q^{\binom{m}{2}} \sum_{s=0}^{m} \frac{(q^m;q^{-1})_s}{(q^{-1};q^{-1})_s} \frac{(A;q^{-1})_s}{(C;q^{-1})_s} q^{-s}$$

with

$$A = ab/q, \quad C = q^{-1-k}, \quad m = n-k$$

to rewrite $P_n(b)$ as a linear combination of $(ab/q;q^{-1})_s$. Equating the resulting coefficient of $(ab/q;q^{-1})_s$ on both sides is equivalent to

$${}_3\phi_2\left(\begin{matrix} q^{s-n}, & az, & a/z \\ & q^{s+1}, & 0 \end{matrix} \,\middle|\, q;q\right) = (a^2 q^{-1-s})^{n-s} \, {}_3\phi_2\left(\begin{matrix} q^{s-n}, & zq^{s+1}/a, & q^{s+1}/az \\ & q^{s+1}, & 0 \end{matrix} \,\middle|\, q;q\right).$$

This is a special case of a ${}_3\phi_2$ transformation, [6, (III.11), $e=0$] □

Proposition 6.6. *The type R_I linear functional L of the b-Al-Salam-Chihara polynomials satisfies*

$$L\left(\frac{1}{(ab;q)_k}\right) = \frac{1}{\prod_{j=0}^{k-1}(1-axq^j+a^2q^{2j})}.$$

Proof. Again we have by induction that this choice works. If $n \geq 1$,

$$\begin{aligned} L\left(\frac{P_n(b)}{(ab;q)_n}\right) &= \sum_{k=0}^{n} \frac{(q^{-n};q)_k}{(q;q)_k} q^k \prod_{j=0}^{k-1}(1-axq^j+a^2q^{2j}) L\left(\frac{1}{(ab;q)_k}\right) \\ &= \sum_{k=0}^{n-1} \frac{(q^{-n};q)_k}{(q;q)_k} q^k + \frac{(q^{-n};q)_n}{(q;q)_n} q^n \prod_{j=0}^{n-1}(1-axq^j+a^2q^{2j}) L\left(\frac{1}{(ab;q)_n}\right) \\ &= \sum_{k=0}^{n} \frac{(q^{-n};q)_k}{(q;q)_k} q^k = 0 \end{aligned}$$

by the q-binomial theorem. □

Theorem 6.7. *The moments for the Al-Salam-Chihara polynomials which have*

$$b_n = (a+b)q^n, \quad \lambda_n = (1-q^n)(1-abq^{n-1}).$$

are equal to the type R_I moments of the rescaled big q^{-1}-Laguerre polynomials which have

$$B_n = bq^n + (a+aq-aq^{n+1})/q^{n+1}, \quad A_n = a(1-q^{-n}), \quad \Lambda_n = (1-q^n)(1+a^2q^{-2n}).$$

Proof. The inverse relation to (6.1) is

$$t_n(b) = \sum_{k=0}^{n} \frac{(q^n,q^{-1})_k^2}{(q;q)_k} a^k q^{-k(2n-k)} P_{n-k}(b). \tag{6.2}$$

This also holds as polynomials in x, and we need the degree n monic in x versions: $(-1)^n q^{\binom{n}{2}} t_n(b)$ and $q^{\binom{n}{2}} P_n(b)$,

$$(-1)^n q^{\binom{n}{2}} t_n(b) = (-1)^n \sum_{k=0}^{n} \frac{(q^n,q^{-1})_k^2}{(q;q)_k} a^k q^{-k(2n-k)+\binom{n}{2}-\binom{n-k}{2}} q^{\binom{n-k}{2}} P_{n-k}(b).$$

Applying the linear functional L for $q^{\binom{n}{2}} P_n(b)$ as a function of x we have

$$L((-1)^n q^{\binom{n}{2}} t_n(b)) = (q;q)_n (-a)^n q^{-\binom{n+1}{2}} = A_1 A_2 \cdots A_n$$

where in Theorem 6.7 $A_n = a(1-q^{-n})$. □

Corollary 6.8. *The continued fractions in Proposition 2.2 and Proposition 2.6 are equal if*

$$b_n = (a+b)q^n, \quad \lambda_n = (1-q^n)(1-abq^{n-1}).$$
$$B_n = bq^n + (a+aq-aq^{n+1})/q^{n+1}, \quad A_n = a(1-q^{-n}), \quad \Lambda_n = (1-q^n)(1+a^2q^{-2n}).$$

Corollary 6.9. *The Al-Salam-Chihara polynomials $p_n(x,b) = Q_n(x;a,b|q)$ and the rescaled big q^{-1}-Laguerre polynomials $P_n(b,x) = (q/a)^n R_n(ab/q;az,a/z;.q^{-1})$. have moment duality, $z+1/z=x$.*

7. CONTINUOUS DUAL q-HAHN POLYNOMIALS

The monic continuous dual q-Hahn polynomials in x are

$$p_n(x/2;a,b,c|q)=\frac{1}{a^n}\sum_{k=0}^{n}\frac{(q^{-n};q)_k}{(q;q)_k}(abq^k;q)_{n-k}(acq^k;q)_{n-k}q^k\prod_{j=0}^{k-1}(1-axq^j+a^2q^{2j}).$$

which satisfy (2.1) with

$$\begin{aligned}b_n=&(a+b+c)q^n+abcq^{n-1}-abcq^{2n}-abcq^{2n-1},\\ \lambda_n=&(1-q^n)(1-abq^{n-1})(1-acq^{n-1})(1-bcq^{n-1}).\end{aligned}$$

Note that these polynomials in x are symmetric in the parameters a,b,c. If $c=0$ these are the Al-Salam-Chihara polynomials.

As a function of b, λ_n is quadratic, not linear. So we must renormalize the parameters to have linear functions b in the recurrence, and thus a type R_I polynomial recurrence in b.

7.1. y-continuous dual q-Hahn polynomials.

Let

$$a=A/B,\quad b=Bz,\quad c=B/z,\ \text{where } z+1/z=y.$$

Now λ_n and b_n are linear polynomials in y,

$$\begin{aligned}b_n&=(A/B+By)q^n+ABq^{n-1}-ABq^{2n}-ABq^{2n-1}\\ \lambda_n&=(1-q^n)(1-Ayq^{n-1}+A^2q^{2n-2})(1-B^2q^{n-1}).\end{aligned}$$

We need the leading term in b_n to be y, so we replace y by y/B, and multiply by $(-1)^nq^{-\binom{n}{2}}$.

Definition 7.1. *The monic y-continuous dual q-Hahn polynomials are defined by*

$$\begin{aligned}P_n(y)=(-B/A)^nq^{-\binom{n}{2}}\sum_{k=0}^{n}\frac{(q^{-n};q)_k}{(q;q)_k}q^k\prod_{j=0}^{k-1}(1-Axq^j/B+A^2q^{2j}/B^2)\\ \times\prod_{j=k}^{n-1}(1-Ayq^j/B+A^2q^{2j})\end{aligned}$$

Proposition 7.2. *The monic y-continuous dual q-Hahn polynomials satisfy the type R_I recurrence relation* (2.3)

$$\begin{aligned}P_{n+1}(y)=&\ (y-B_n)P_n(y)-(A_ny+\Lambda_n)P_{n-1}(y),\\ B_n&=(-ABq^{n-1}-Aq^n/B+ABq^{2n}+ABq^{2n-1}+x)/(q^n)\\ A_n&=(1-q^n)(1-B^2q^{n-1})\frac{(-Aq^{-n})}{B},\\ \Lambda_n&=(1-q^n)(1-B^2q^{n-1})(1+A^2q^{2n-2})/q^{2n-1}.\end{aligned}$$

The non-degeneracy condition is $x \neq Aq^j/B + Bq^{-j}/A, j = 0, 1, 2, \cdots$ because

$$P_n(Bq^{1-n}/A + BAq^{n-1}) = (B/A)^n q^{-n^2+n} \prod_{j=0}^{n-1} (1 - Axq^j/B + A^2q^{2j}/B^2).$$

The vector space in this case, with $D_n(y) = \prod_{j=0}^{n-1}(1 - Ayq^j + A^2q^{2j})$, is

$$V = span\{1, y, y^2, \cdots, 1/D_1(y), 1/D_2(y), \cdots, \}.$$

Theorem 7.3. *The R_I moments of the y-continuous dual q-Hahn polynomials are the moments of the orthogonal polynomials* (2.1)

$$\begin{aligned} b_n =&(-A - Aq + AB^2q^n + Aq^{n+1} + Bxq^{n+1})/Bq^{2n+1}, \\ \lambda_n =&(1 - q^{-n})(1 - q^{1-n}/B^2)(1 - Awq^{-n}/B)(1 - Aq^{-n}/Bw)B^2. \end{aligned}$$

and $x = w + w^{-1}$. These monic orthogonal polynomials are the continuous dual q-Hahn polynomials

$$B^n p_n(y/2B; A/q, w/B, 1/Bz|q^{-1}).$$

Corollary 7.4. *The continued fractions in Proposition 2.2 and Proposition 2.6 are equal if*

$$\begin{aligned} b_n =&(-A - Aq + AB^2q^n + Aq^{n+1} + Bxq^{n+1})/Bq^{2n+1}, \\ \lambda_n =&(1 - q^{-n})(1 - q^{1-n}/B^2)(1 - Azq^{-n}/B)(1 - Aq^{-n}/Bz)B^2, \\ B_n =& (-ABq^{n-1} - Aq^n/B + ABq^{2n} + ABq^{2n-1} + x)/(q^n), \\ A_n =& (1 - q^n)(1 - B^2q^{n-1})\frac{(-Aq^{-n})}{B}, \\ \Lambda_n =& (1 - q^n)(1 - B^2q^{n-1})(1 + A^2q^{2n-2})/q^{2n-1}. \end{aligned}$$

Corollary 7.5. *The continuous dual q-Hahn polynomials*

$$r_n(x, y) = p_n(x/2; A/B, Bz, B/z|q) \text{ with } y = z + 1/z$$

and

$$R_n(y, x) = B^n p_n(y/2B; A/q, z/B, 1/Bz|q^{-1}) \text{ with } x = z + 1/z$$

have moment duality.

Proof. Again we use connection coefficients between the two sets of monic polynomials in y,. $P_n(y)$ and $R_n(y, x)$. It is

$$\begin{aligned} P_n(y) &= \sum_{k=0}^{n} c_{n,k} R_k(y, x), \text{ where} \\ c_{n,k} &= \frac{(q^n; q^{-1})_k^2}{(q; q)_k} (B^2q^{n-1}; q^{-1})_k \frac{(-A)^k}{B^k} q^{-2kn+3\binom{k+1}{2}-2k}. \end{aligned} \tag{7.1}$$

Applying the linear functional R for the y-polynomials $R_n(y, x)$ gives

$$R(P_n(y)) = c_{n,n} = (q; q)_n (B^2q^{n-1}; q^{-1})_n \frac{(-A)^n}{B^n} q^{-n(n+1)/2} = A_1A_2 \cdots A_n.$$

so that R coincides with the type R_I linear functional of Corollary 7.2 on polynomials in y.

Equation (7.1) may be proven inductively using the three-term recurrences in y for $P_n(y)$ and $R_k(y,x)$.

Moment duality follows from iterating the map from $r_n(x,y)$ to $R_n(y,x)$. $\square$

8. Concluding Remarks

Question 8.1. *Is there a general moment duality result which includes Corollary 4.3, Corollary 6.9, and Corollary 7.5?*

Question 8.2. *Type R_I versions of the Askey-Wilson and q-Racah polynomials are given in* [10, Theorem 8.33, Theorem 8.37]. *The corresponding moments are equal to the Askey-Wilson and q-Racah moments. Is there a moment duality result for these polynomials?*

Question 8.3. *Is there an Askey scheme for type R_I polynomials with corresponding moment dualities?*

Remark 8.4. *Another equality of moments and R_I moments is given in* [11, Cor. 5.13].

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Department of Mathematics, University of Central Florida, Orlando, Florida, 32816

Email address: mourad.eh.ismail@gmail.com

School of Mathematics, University of Minnesota, Minneapolis, Minnesota 55455

Email address: stanton@math.umn.edu