THE ODLYZKO CONJECTURE AND O'HARA'S UNIMODALITY PROOF

Dennis Stanton* and Doron Zeilberger**

Abstract. We observe that Andrew Odlyzko's conjecture that the Maclaurin coefficients of $1/[(1+q)(1+q+q^2)\cdots(1+q+\cdots+q^{k-1})]$ have alternating signs is an almost immediate consequence of an identity that is implied by Kathy O'Hara's recent magnificent combinatorial proof of the unimodality of the Gaussian coefficients.

To a true combinatorialist, a combinatorial result is not properly proved until it receives a *direct combinatorial proof.* This is why Kathy O'Hara's long-sought-for constructive proof of the unimodality of the Gaussian polynomials ([4], [5], see also [6]) generated so much excitement in combinatorial circles. However to non-combinatorialists, a direct combinatorial proof is "just another proof". O'Hara's proof is longer than most of the dozen previous proofs, and probably would not add any insight to anyone who is not a genuine combinatorialist. Moreover, it does not seem to be generalizable at first sight. Yet it turned out to imply a deep result (KOH) to which hitherto there was no known proof of any kind.

In this note we shall prove and generalize a conjecture of Odlyzko, using O'Hara's result. Odlyzko's results imply that for k sufficiently large, the first k coefficients in

$$\frac{1}{(1+q)(1+q+q^2)\cdots(1+q+\cdots+q^{k-1})}=\frac{(1-q)^k}{(1-q)(1-q^2)\cdots(1-q^k)}$$

alternate in sign. He conjectured that in fact for every $k \geq 0$, all of the coefficients of the above series alternate in sign. We prove the sharper result

Theorem 1. *For any integer k,*

$$\frac{(1-q)^{[(k+1)/2]}}{(1-q)(1-q^2)\cdots(1-q^k)}$$

has coefficients which alternate in sign.

Note that the exponent of $(1-q)$ is best possible, since if $[(k+1)/2]$ is replaced by $[(k-1)/2]$ then the pole $q=1$ has the highest order among all the poles, all

*School of Mathematics, University of Minnesota, Minneapolis, MN 55455. This work was partially supported by NSF grant DMS:8700995.
**Department of Mathematics, Drexel University, Philadelphia, PA, 19104. This work was partially supported by NSF grant DMS:8800663.

Typeset by $\mathcal{AMS}$-TEX

of which are roots of unity, so a partial fraction expansion would yield that the coefficients are asymptotically of the same sign.

Odlyzko has informed the authors that Theorem 1 can be used to shorten the proof in [3] by at least one third.

We will prove a more general result. Recall that the Gaussian polynomials are defined for nonnegative integers k and n by

$$\text{(GP)} \qquad G(n,k) = \begin{bmatrix} n+k \\ k \end{bmatrix}_q = \frac{(1-q^{n+1})(1-q^{n+2})\cdots(1-q^{n+k})}{(1-q)(1-q^2)\cdots(1-q^k)}.$$

If n is negative, we put $G(n,k)=0$. We will prove:

Theorem 2. *For nonnegative integers n and k, with nk even, $G(n,k)(1-q)^m$ has coefficients which alternate in sign, where $m = min\{[(k+1)/2],[(n+1)/2]\}$.*

Theorem 1 follows from Theorem 2 upon taking n even and letting $n \to \infty$.

Theorem 2 will follow from the following amazing q-binomial identity that was derived in [7], by "algebrizing" O'Hara's main theorem ([4], [5], [6]).

$$\text{(KOH)} \qquad G(n,k) = \sum_{\lambda \vdash k} q^{2n(\lambda)} \prod_{i=0}^{k-1} G((k-i)n - 2i + \sum_{j=0}^{i-1} 2(i-j)d_{k-j}, d_{k-i}),$$

where

$$n(\lambda) = \sum_i (i-1)\lambda_i.$$

The sum in (KOH) is over all partitions $\lambda = (\lambda_1, \lambda_2, \dots)$ of k. The integer d_i is the multiplicity of i in λ, thus in frequency notation $\lambda = 1^{d_1}2^{d_2}\cdots i^{d_i}\cdots$. In this notation,

$$2n(\lambda) = \sum_{i=1}^{k}(D_i^2 - D_i)$$

where

$$D_r = \sum_{i=r}^{k} d_i.$$

Proof of Theorem 2. By symmetry in n and k, we may assume that n is even. We proceed by induction on n and k. Theorem 2 clearly holds for $n=0$ and $k=1$.

Let

$$F(n,k) := (1-q)^{[(k+1)/2]}G(n,k).$$

Then (KOH) can be rewritten as

$$\text{(KOH')} \quad F(2n,k) = \sum_{\lambda \vdash k}(1-q)^{\alpha(\lambda)}q^{2n(\lambda)} \prod_{i=0}^{k-1} F(2(k-i)n - 2i + \sum_{j=0}^{i-1} 2(i-j)d_{k-j}, d_{k-i}).$$

where

$$\alpha(\lambda) := m - \sum_{i=1}^{k}[(d_i+1)/2]$$

Suppose we show that $\alpha(\lambda)\ge 0$. If $\lambda\ne 1^k$, then each F on the right side of (KOH') has a second argument less than k. If $\lambda=1^k$, the first argument of F is less than $2n$. Thus by induction each F is alternating. Since $(1-q)^{\alpha(\lambda)}$ is alternating, and the power of q is even, the left side must be alternating. So it remains to verify that $\alpha(\lambda)\ge 0$.

First suppose that $n\ge[(k+1)/2]$, so $m=[(k+1)/2]$. Then we will show that for any partition λ of k, we have the inequality

$$(*)\qquad [(k+1)/2]-\sum_{i=1}^{k}[(d_i+1)/2]\ge 0.$$

It is easy to see that (*) is

$$[(k+1)/2]-(\text{number of parts of }\lambda\text{+number of }i\text{ with }d_i\text{ odd})/2.$$

This is nonnegative, since any part $i>1$ of λ can contribute at most one i which has d_i odd.

Next suppose that $n<[(k+1)/2]$, so $m=n$. First we show

$$(**)\qquad n+1-\sum_{i=1}^{k}d_i\ge 0$$

for all partitions λ of k which occur in (KOH'). The key observation is that F is zero if the first argument is negative. Thus, taking the $i=k-1$ factor in (KOH'), we see that

$$2n-2(k-1)-\sum_{j=0}^{k-2}2(k-1-j)d_{k-j}\ge 0,$$

which is equivalent to

$$\sum_{j=2}^{k}(j-1)d_j\ge k-1-n,$$

or

$$k=\sum_{j=1}^{k}jd_j\ge k-1-n+\text{number of parts of }\lambda.$$

The final inequality implies that λ has at most $n+1$ parts, which is (**). Clearly $\alpha(\lambda)\ge 0$ holds unless λ has $n+1$ distinct parts, in which case $\alpha(\lambda)=-1$. In this case the $i=k-1$ factor in (KOH') is alternating ($G(0,1)=1$) without the factor of $(1-q)$, so it enough to prove that $\alpha(\lambda)+1\ge 0$. □

Remarks: To prove Theorem 1 we need only the the $n\to\infty$ case of (KOH). John Stembridge rediscovered an identity of Hall which implies this result

$$(\text{JS})\qquad \begin{bmatrix} n+k\\ k\end{bmatrix}_q=\sum_{d\vdash k}q^{2n(d)}\begin{bmatrix} n+1\\ d_1,\dots,d_k\end{bmatrix}_q.$$

Then George Andrews observed that (JS) is nothing but an iteration of q-Vandermonde. Subsequently John Stembridge and Jim Joichi gave bijections that prove (JS). Their proofs are closely related to [1].

If nk is odd, Theorem 2 cannot hold, because the leading term has the wrong sign. The exponent in Theorem 2 is not always best possible: $G(11,6)(1-q)^2$ alternates in sign.

Ron Evans has made the following related conjecture. He has verified it for $a=1$ from Theorem 2.

Conjecture. *Let n, k, and a be nonnegative integers, with $k > 3$ and a odd. Let $G(n, k, a)$ be defined by (GP), with q^a replacing q in the numerator. Then the coefficients of $G(n, k, a)(1-q)^{[(k+1)/2]}$ alternate in sign if nk is even, and the coefficients of $G(n, k, a)(1-q)^{[(k+1)/2]}/(1-q^2)$ alternate in sign if nk is odd.*

Some other remarks about (KOH) can be found in [7].

References

1. George Andrews, *Partitions and Durfee dissection*, Amer. J. Math. **101** (1979), 735–742.
2. I. G. Macdonald, *Symmetric functions and Hall polynomials*, Clarendon Press, Oxford, 1979.
3. Andrew Odlyzko, *On differences of the partition function*, Acta Arithmetica **49** (1988), 237-254.
4. Kathleen M. O'Hara, *Unimodality of Gaussian coefficients: a constructive proof, research announcement,* (to appear).
5. ______, *Unimodality of Gaussian coefficients:a constructive proof*, J. Comb. Th. A (to appear).
6. Doron Zeilberger, *Kathy O'Hara's constructive proof of the unimodality of the Gaussian polynomials*, Amer. Math. Monthly (to appear).
7. ______, *A one-line high school algebra proof of the unimodality of the Gaussian polynomials* $\begin{bmatrix} n \\ k \end{bmatrix}_q$ *for* $k < 20$, Proceedings of the IMA Workshop on q-series, Vol. 18, Springer (to appear).