PROOF OF A MONOTONICITY CONJECTURE

T. Prellberg and D. Stanton

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Abstract. A monotonicity conjecture of Friedman, Joichi and Stanton is established.

In [2] the following monotonicity conjecture was made.

Conjecture. *If $n \geq 3$ is an odd integer, then*

$$\frac{1-q}{\prod_{i=n}^{2n-1}(1-q^i)} + q$$

has non-negative power series coefficients.

The purpose of this note is prove the Conjecture.

The conjecture has been established for prime values of n by Andrews [1], and for $n \leq 99$, using a computer proof (see [2], [4]). The proof given here relies upon an identity for the rational function of the conjecture, which is our Lemma. A similar identity was found by Andrews [1] to establish the case when n is prime.

Recall the notation

$$(a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1}), \quad [n]_q = (1-q^n)/(1-q),$$

and

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}}.$$

Lemma. *If n is a non-negative integer, then*

$$\frac{1-q}{(q^n;q)_n} + q = \frac{1}{1-q^{4n^2-6n+2}}\left(1 - q^{4n^2-6n+3} + \sum_{m=0}^{n-2} q^{(n+m)(2m+1)}\frac{(q^{n-1};q^{-1})_m}{(q^2;q)_{2m}}\right.$$
$$\left. + \sum_{m=0}^{n-3} q^{2(n+m+1)(m+1)}\frac{(q^{n-1};q^{-1})_{m+1}}{(q^2;q)_{2m+1}}\right).$$

Proof of the Conjecture. We may assume the Lemma and take $n \geq 5$. We show that the individual terms of the Lemma inside the parentheses have non-negative coefficients, and that q^{4n^2-6n+3} also occurs.

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First we show that the mth term in each of the two sums has non-negative coefficients. If $m = 0$ the term in the second sum is $q^{2(n+1)}(1 - q^{n-1})/(1 - q^2)$, which is non-negative since n is odd, while the term in the first sum is q^n.

So we take $1 \leq m \leq n - 3$ and first consider the second sum. If $2m + 2 \geq n - 1$, then

$$\frac{(q^{n-1}; q^{-1})_{m+1}}{(q^2; q)_{2m+1}} = \frac{1}{(q^n; q)_{2m+3-n}(q^2; q)_{n-m-3}}$$

which clearly has non-negative coefficients. Next suppose that $2m + 2 < n - 1$ and let

$$2^s \leq m + 1 \leq 2^{s+1} - 1$$

for some positive integer s. Note that $2^{s+1} \leq 2m + 2 < n - 1$. Then

$$\frac{(q^{n-1}; q^{-1})_{m+1}}{(q^2; q)_{2m+1}} = \frac{1}{[n]_q} \begin{bmatrix} n \\ 2^{s+1} \end{bmatrix}_q \frac{1}{(q^{2^{s+1}+1}; q)_{2m+2-2^{s+1}}(q^{n-2^{s+1}+1}; q)_{2^{s+1}-m-2}}$$

We now appeal to the fact [1, Th. 2], [3, Prop. 2.5.1] that

$$\frac{1}{[n]_q} \begin{bmatrix} n \\ k \end{bmatrix}_q$$

has non-negative coefficients if $1 < k < n$ and $GCD(n, k) = 1$, to obtain non-negativity of the mth term since n is odd.

For the first sum a similar proof applies. For $1 \leq m \leq n - 2$ we have

$$\frac{(q^{n-1}; q^{-1})_m}{(q^2; q)_{2m}} = \frac{1}{(q^n; q)_{2m+2-n}(q^2; q)_{n-m-2}}, \text{ if } 2m + 1 \geq n - 1,$$

while for $2m + 1 < n - 1$ we let $2^s < m + 1 \leq 2^{s+1}$ to obtain

$$\frac{(q^{n-1}; q^{-1})_m}{(q^2; q)_{2m}} = \frac{1}{[n]_q} \begin{bmatrix} n \\ 2^{s+1} \end{bmatrix}_q \frac{1}{(q^{2^{s+1}+1}; q)_{2m+1-2^{s+1}}(q^{n-2^{s+1}+1}; q)_{2^{s+1}-m-1}}.$$

Finally we must show that the term q^{4n^2-6n+3} does appear in the sum. The $m = n - 2$ term of the first sum is

$$\frac{q^{4n^2-10n+6}}{(q^n; q)_{n-2}},$$

and a q^{4n-3} does appear due to the denominator factors of $(1-q^n)$ and $(1-q^{2n-3})$.

Proof of the Lemma. The lemma is equivalent to

$$\begin{aligned} \frac{1}{(q^n; q)_n} =& 1 + \sum_{m=0}^{n-1} q^{(n+m)(2m+1)} \frac{(q^{n-1}; q^{-1})_m}{(q; q)_{2m+1}} \\ &+ \sum_{m=0}^{n-2} q^{2(n+m+1)(m+1)} \frac{(q^{n-1}; q^{-1})_{m+1}}{(q; q)_{2m+2}}, \end{aligned} \tag{1}$$

because the $m=n-1$ term of the first sum and the $m=n-2$ term of the second sum do sum to

$$\frac{q^{4n^2-6n+2}}{(q^n;q)_n}.$$

To prove (1), the q-binomial theorem implies

$$\begin{aligned}\frac{1}{(q^n;q)_n} =&1+\sum_{j=1}^{\infty}\frac{(q^n;q)_j}{(q;q)_j}q^{nj} = 1+\sum_{j=1}^{\infty}\frac{(q^{n+1};q)_{j-1}}{(q;q)_j}\left(q^{nj}-q^{n(j+1)}\right)\\ =&1+\frac{q^n}{1-q}+(1-q^{n-1})\sum_{j=2}^{\infty}\frac{(q^{n+1};q)_{j-2}}{(q;q)_j}q^{(n+1)j}\\ =&1+\frac{q^n}{1-q}+(1-q^{n-1})\frac{q^{2(n+1)}}{(q;q)_2}+(1-q^{n-1})\sum_{j=3}^{\infty}\frac{(q^{n+1};q)_{j-2}}{(q;q)_j}q^{(n+1)j}.\end{aligned}$$

Continuing we see that for $t\geq 0$,

$$\begin{aligned}\frac{1}{(q^n;q)_n} =&1+\sum_{m=0}^{t}q^{(n+m)(2m+1)}\frac{(q^{n-1};q^{-1})_m}{(q;q)_{2m+1}}\\ &+\sum_{m=0}^{t}q^{2(n+m+1)(m+1)}\frac{(q^{n-1};q^{-1})_{m+1}}{(q;q)_{2m+2}}\\ &+(q^{n-1};q^{-1})_{t+1}\sum_{j=2t+3}^{\infty}\frac{(q^{n+t+1};q)_{j-2t-2}}{(q;q)_j}q^{(n+t+1)j}.\end{aligned}$$

and (1) is the $t=n-1$ case.

Remarks. One may also see that the Lemma proves that the coefficients are strictly positive past q^{3n+4} for $n\geq 7$, (see [2]). The $m=1$ term of the first sum is

$$q^{3(n+1)}\frac{1+q^2+q^4+\cdots+q^{n-3}}{1-q^3},$$

which has the required property.

The equivalent form (1) of the Lemma is the $x=q^n$ special case of

$$\frac{1}{(x;q)_n}=\sum_{m=0}^{n-1}\begin{bmatrix}n+m-1\\2m\end{bmatrix}_q q^{2m^2}\frac{x^{2m}}{(x;q)_m}+\sum_{m=0}^{n-1}\begin{bmatrix}n+m\\2m+1\end{bmatrix}_q q^{2m^2+m}\frac{x^{2m+1}}{(x;q)_{m+1}}. \tag{2}$$

A generalization of (2) to any positive integer $r\geq 2$ is

$$\begin{aligned}\sum_{k=0}^{\infty}\frac{(a;q)_k}{(b;q)_k}x^k =&\sum_{t=0}^{\infty}\frac{(a;q)_{(r-1)t}(b/a;q)_t}{(b;q)_{rt}}q^{(rt-1)t-\binom{t}{2}}\frac{(-a)^t x^{rt}}{(x;q)_t}\\ &+\sum_{t=0}^{\infty}\frac{(a;q)_{(r-1)t+1}(b/a;q)_t}{(b;q)_{rt+1}}q^{rt^2-\binom{t}{2}}\frac{(-a)^t x^{rt+1}}{(x;q)_{t+1}}\\ &+\sum_{i=2}^{r-1}\sum_{t=0}^{\infty}\frac{(a;q)_{(r-1)t+i-1}(b/a;q)_{t+1}}{(b;q)_{rt+i}}q^{(rt+i-1)(t+1)-\binom{t+1}{2}}\frac{(-a)^{t+1}x^{rt+i}}{(x;q)_{t+1}}.\end{aligned} \tag{3}$$

Another identity similar to the Lemma is

$$\frac{1-q}{(q^n;q)_n}+q=\frac{1}{1-q^{n(2n-1)}}\left(1-q^{n(2n-1)+1}+\sum_{m=1}^{n-1}\begin{bmatrix} n \\ m \end{bmatrix}_q \frac{1-q}{(q^n;q)_m}q^{m(m+n-1)}\right),$$

which would also prove the Conjecture if the individual terms are non-negative.

Conjecture 1. *The power series coefficients of*

$$\begin{bmatrix} n \\ m \end{bmatrix}_q \frac{1-q}{(q^n;q)_m}$$

are non-negative

(1) *if $n>0$ is odd and $0<m<n$, or*
(2) *if $n>0$ is even and $0<m<n$ with $m \neq 2, n-2$.*

Recall [5] the Schur function $s_\lambda(x_1,\cdots,x_n)$ and the content [5, p.11, Ex. I.1.3] of a cell $x\in\lambda$. A Schur function version of Conjecture 1 is

Conjecture 2. *The power series coefficients of*

$$s_\lambda(1,q,\cdots,q^{n-1})\frac{1-q}{\prod_{x\in\lambda}1-q^{n-c(x)}}$$

are non-negative unless

(1) $\lambda=11$ *and $n>0$ is even, or*
(2) $\lambda=1^k$, $k\geq 3$ *odd*, $n=k$, *or*
(3) $\lambda=1^k$, $k\geq 3$ *even*, $n=k$ *or* $k+2$.

Conjecture 1 is the choice $\lambda=1^m$ in Conjecture 2.

References

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Institut für theoretische Physik, Technische Universität Clausthal, Arnold Sommerfeld Strasse 6, D-38678 Clausthal-Zellerfeld, Germany
E-mail address: thomas.prellberg@tu-clausthal.de

School of Mathematics, University of Minnesota, Minneapolis, MN 55455
E-mail address: stanton@math.umn.edu