PROOF OF A MONOTONICITY CONJECTURE

T. Prellberg and D. Stanton

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Abstract. If $p_k(P)$ is the number of integer partitions of $k \geq 1$ whose parts lie in P, it is shown that $p_k(P)$ is an increasing function of k for $P = \{n, n+1, \cdots, 2n-1\}$, where $n \geq 3$ is odd. This completes the classification of all such monotonic P with $min(P) \neq 2, 3$, or 5.

In [2] the following monotonicity conjecture was made.

Conjecture. *If $n \geq 3$ is an odd integer, then*

$$\frac{1-q}{\prod_{i=n}^{2n-1}(1-q^i)} + q$$

has non-negative power series coefficients.

The purpose of this note is prove the conjecture.

The conjecture has been established for prime values of n by Andrews [1], and for $n \leq 99$, using a computer proof (see [2], [4]). The proof given here relies upon an identity for the rational function of the conjecture, which is our Lemma. A similar identity was found by Andrews [1] to establish the case when n is prime.

The conjecture states that if n is odd, the number of integer partitions of k with part sizes $n, n+1, \cdots, 2n-1$ is an increasing function of k for $k \geq 1$. A general form of this monotonicity question for part sizes belonging to a set P was considered in [2]. A classification of all such P whose minimum value is odd and at least 7 was given in [2, Theorem 3.5], assuming the validity of the conjecture. The analogous statement when the minimum value of P is even and at least 4 is given in [2, Theorem 3.6]. Establishing the conjecture completes this classification problem if the minimum value of P is not equal to 2,3 or 5.

Recall the notation

$$(a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1}), \quad [n]_q = (1-q^n)/(1-q),$$

and

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}}.$$

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Lemma. *If $n \geq 2$ is an integer, then*

$$\frac{1-q}{(q^n;q)_n}+q=\frac{1}{1-q^{4n^2-6n+2}}\Bigg(1-q^{4n^2-6n+3}+\sum_{m=0}^{n-2} q^{(n+m)(2m+1)}\frac{(q^{n-1};q^{-1})_m}{(q^2;q)_{2m}} + \sum_{m=0}^{n-3} q^{2(n+m+1)(m+1)}\frac{(q^{n-1};q^{-1})_{m+1}}{(q^2;q)_{2m+1}}\Bigg).$$

Proof of the Conjecture. We may assume the Lemma and take $n \geq 5$. We show that the individual terms of the Lemma inside the parentheses have non-negative coefficients, and that q^{4n^2-6n+3} also occurs.

First we show that the mth term in each of the two sums has non-negative coefficients. If $m=0$ the term in the second sum is $q^{2(n+1)}(1-q^{n-1})/(1-q^2)$, which is non-negative since n is odd, while the term in the first sum is q^n.

So we take $1 \leq m \leq n-3$ and first consider the second sum. If $2m+2 \geq n-1$, then

$$\frac{(q^{n-1};q^{-1})_{m+1}}{(q^2;q)_{2m+1}}=\frac{1}{(q^n;q)_{2m+3-n}(q^2;q)_{n-m-3}}$$

which clearly has non-negative coefficients. Next suppose that $2m+2 < n-1$ and let

$$2^s \leq m+1 \leq 2^{s+1}-1$$

for some positive integer s. Note that $2^{s+1} \leq 2m+2 < n-1$. Then

$$\frac{(q^{n-1};q^{-1})_{m+1}}{(q^2;q)_{2m+1}}=\frac{1}{[n]_q}\begin{bmatrix} n \\ 2^{s+1}\end{bmatrix}_q \frac{1}{(q^{2^{s+1}+1};q)_{2m+2-2^{s+1}}(q^{n-2^{s+1}+1};q)_{2^{s+1}-m-2}}$$

We now appeal to the fact [1, Th. 2], [3, Prop. 2.5.1] that

$$\frac{1}{[n]_q}\begin{bmatrix} n \\ k\end{bmatrix}_q$$

has non-negative coefficients if $1<k<n$ and $GCD(n,k)=1$, to obtain non-negativity of the mth term since n is odd.

For the first sum a similar proof applies. For $1 \leq m \leq n-2$ we have

$$\frac{(q^{n-1};q^{-1})_m}{(q^2;q)_{2m}}=\frac{1}{(q^n;q)_{2m+2-n}(q^2;q)_{n-m-2}},\ \text{if } 2m+1 \geq n-1,$$

while for $2m+1 < n-1$ we let $2^s < m+1 \leq 2^{s+1}$ to obtain

$$\frac{(q^{n-1};q^{-1})_m}{(q^2;q)_{2m}}=\frac{1}{[n]_q}\begin{bmatrix} n \\ 2^{s+1}\end{bmatrix}_q \frac{1}{(q^{2^{s+1}+1};q)_{2m+1-2^{s+1}}(q^{n-2^{s+1}+1};q)_{2^{s+1}-m-1}}.$$

Finally we must show that the term q^{4n^2-6n+3} does appear in the sum. The $m=n-2$ term of the first sum is

$$\frac{q^{4n^2-10n+6}}{(q^n;q)_{n-2}},$$

and a q^{4n-3} does appear due to the denominator factors of $(1-q^n)$ and $(1-q^{2n-3})$.

Proof of the Lemma. The lemma is equivalent to

$$
\begin{aligned}
\frac{1}{(q^n;q)_n} =& 1+\sum_{m=0}^{n-1} q^{(n+m)(2m+1)}\frac{(q^{n-1};q^{-1})_m}{(q;q)_{2m+1}}\\
&+\sum_{m=0}^{n-2} q^{2(n+m+1)(m+1)}\frac{(q^{n-1};q^{-1})_{m+1}}{(q;q)_{2m+2}},
\end{aligned}
\tag{1}
$$

because the $m=n-1$ term of the first sum and the $m=n-2$ term of the second sum in (1) do sum to

$$\frac{q^{4n^2-6n+2}}{(q^n;q)_n}.$$

If these two terms are subtracted from (1), then elementary manipulations show that (1) is the Lemma.

To prove (1), the q-binomial theorem implies

$$
\begin{aligned}
\frac{1}{(q^n;q)_n} =& 1+\sum_{j=1}^{\infty}\frac{(q^n;q)_j}{(q;q)_j}q^{nj} = 1+\sum_{j=1}^{\infty}\frac{(q^{n+1};q)_{j-1}}{(q;q)_j}\left(q^{nj}-q^{n(j+1)}\right)\\
=& 1+\frac{q^n}{1-q}+(1-q^{n-1})\sum_{j=2}^{\infty}\frac{(q^{n+1};q)_{j-2}}{(q;q)_j}q^{(n+1)j}\\
=& 1+\frac{q^n}{1-q}+(1-q^{n-1})\frac{q^{2(n+1)}}{(q;q)_2}+(1-q^{n-1})\sum_{j=3}^{\infty}\frac{(q^{n+1};q)_{j-2}}{(q;q)_j}q^{(n+1)j}.
\end{aligned}
$$

Continuing we see that for $t\geq 0$,

$$
\begin{aligned}
\frac{1}{(q^n;q)_n} =& 1+\sum_{m=0}^{t} q^{(n+m)(2m+1)}\frac{(q^{n-1};q^{-1})_m}{(q;q)_{2m+1}}\\
&+\sum_{m=0}^{t} q^{2(n+m+1)(m+1)}\frac{(q^{n-1};q^{-1})_{m+1}}{(q;q)_{2m+2}}\\
&+(q^{n-1};q^{-1})_{t+1}\sum_{j=2t+3}^{\infty}\frac{(q^{n+t+1};q)_{j-2t-2}}{(q;q)_j}q^{(n+t+1)j}.
\end{aligned}
$$

and (1) is the $t=n-1$ case.

Remarks. One may also see that the Lemma proves that the coefficients are strictly positive past q^{3n+4} for $n\geq 7$, (see [2]). The $m=1$ term of the first sum is

$$q^{3(n+1)}\frac{1+q^2+q^4+\cdots+q^{n-3}}{1-q^3},$$

which has the required property.

The equivalent form (1) of the Lemma is the $x=q^n$ special case of

$$
\frac{1}{(x;q)_n} = \sum_{m=0}^{n-1}\begin{bmatrix} n+m-1\\ 2m\end{bmatrix}_q q^{2m^2}\frac{x^{2m}}{(x;q)_m}+\sum_{m=0}^{n-1}\begin{bmatrix} n+m\\ 2m+1\end{bmatrix}_q q^{2m^2+m}\frac{x^{2m+1}}{(x;q)_{m+1}}.
\tag{2}
$$

A generalization of (2) to any positive integer $r \geq 2$ is

$$\begin{aligned}
\sum_{k=0}^{\infty} \frac{(a;q)_k}{(b;q)_k} x^k = & \sum_{t=0}^{\infty} \frac{(a;q)_{(r-1)t}(b/a;q)_t}{(b;q)_{rt}} q^{(rt-1)t-\binom{t}{2}} \frac{(-a)^t x^{rt}}{(x;q)_t} \\
& + \sum_{t=0}^{\infty} \frac{(a;q)_{(r-1)t+1}(b/a;q)_t}{(b;q)_{rt+1}} q^{rt^2-\binom{t}{2}} \frac{(-a)^t x^{rt+1}}{(x;q)_{t+1}} \\
& + \sum_{i=2}^{r-1} \sum_{t=0}^{\infty} \frac{(a;q)_{(r-1)t+i-1}(b/a;q)_{t+1}}{(b;q)_{rt+i}} q^{(rt+i-1)(t+1)-\binom{t+1}{2}} \frac{(-a)^{t+1} x^{rt+i}}{(x;q)_{t+1}}.
\end{aligned} \tag{3}$$

Another identity similar to the Lemma is

$$\frac{1-q}{(q^n;q)_n} + q = \frac{1}{1-q^{n(2n-1)}} \left(1 - q^{n(2n-1)+1} + \sum_{m=1}^{n-1} \begin{bmatrix} n \\ m \end{bmatrix}_q \frac{1-q}{(q^n;q)_m} q^{m(m+n-1)} \right) ,$$

which would also prove the Conjecture if the individual terms are non-negative.

Conjecture 1. *The power series coefficients of*

$$\begin{bmatrix} n \\ m \end{bmatrix}_q \frac{1-q}{(q^n;q)_m}$$

are non-negative

(1) *if $n > 0$ is odd and $0 < m < n$, or*
(2) *if $n > 0$ is even and $0 < m < n$ with $m \neq 2, n-2$.*

References

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Institut für theoretische Physik, Technische Universität Clausthal, Arnold Sommerfeld Strasse 6, D-38678 Clausthal-Zellerfeld, Germany

E-mail address: thomas.prellberg@tu-clausthal.de

School of Mathematics, University of Minnesota, Minneapolis, MN 55455

E-mail address: stanton@math.umn.edu