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q-Analogues of Euler's Odd = Distinct theorem

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Abstract Two q-analogues of Euler's theorem on integer partitions with odd or distinct parts are given. A q-lecture hall theorem is given.

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1 Introduction

Euler's Odd = Distinct theorem is

Theorem *The number of integer partitions of N into odd parts equals the number of integer partitions of N into distinct parts.*

This is an elementary theorem [1, p. 5] which may be easily proven from generating functions. In [9, Definition 9.3] a t-analogue of the q-binomial coefficient is defined, whose combinatorial interpretation involves partitions whose part sizes are polynomials in a positive integer q. Thus it is natural to look for a q-analogue of Euler's Odd = Distinct theorem, where the part sizes are also polynomials in a positive integer q. In this paper we give two such theorems. The results are very simple, but the statements are appealing and may hint at a larger theory.

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2 q-analogues

It is well-known that Euler's theorem follows from the $a_n = n$ case of the following easy proposition, which is implicit in [1, p. 5] and [1, Theorem 8.4]. A proof of Proposition 2 which generalizes Proposition 1 is given in Sect. 5.

Proposition 1 *Let $\{a_n\}_{n\geq 1}$ be a sequence of distinct positive integers. If $a_{2n}/a_n = m_n$ is an integer for all n, then the number of integer partitions of N into parts of size $\{a_{2n+1}\}_{n\geq 0}$ is equal to the number of integer partitions of N into parts of size $\{a_n\}_{n\geq 1}$, where a_n has multiplicity at most $m_n - 1$.*

The first q-analogue of Euler's Odd = Distinct theorem uses

$$a_n = [n]_q = \frac{1-q^n}{1-q} = 1 + q + \cdots + q^{n-1},$$

which satisfies

$$a_{2n}/a_n = [2n]_q/[n]_q = q^n + 1.$$

If q is a positive integer, it is clear that the hypotheses in Proposition 1 are fulfilled.

Theorem 1 *Let q be a positive integer. The number of integer partitions of N into q-odd parts $[2n+1]_q$ is equal to the number of integer partitions of N into parts $[n]_q$ of multiplicity at most q^n.*

Note that Euler's Odd = Distinct theorem is the $q = 1$ case of Theorem 1.

Another q-analogue of n is given by

$$\frac{q^n - q^{-n}}{q - q^{-1}},$$

which may be written as a quotient of sines, but is not an integer for positive integers q.

Integrability can be obtained using Chebyshev polynomials. Recall that Chebyshev polynomials of the first and second kinds satisfy the three-term recurrence relation

$$p_{n+1}(x) = 2xp_n(x) - p_{n-1}(x), \quad n \geq 1. \tag{2.1}$$

The polynomials of the first kind $T_n(x)$ have the initial conditions $T_0(x) = 1$, $T_1(x) = x$, while those of the second kind $U_n(x)$ have $U_0(x) = 1$, $U_1(x) = 2x$.

The Chebyshev polynomials have explicit trigonometric expressions

$$U_n(x) = \frac{\sin((n+1)\theta)}{\sin\theta}, \qquad T_n(x) = \cos(n\theta), \quad x = \cos\theta.$$

Note that $U_n(x)$ is a quotient of sines, and $U_{n-1}(1) = n$.

If q is a positive integer, define

$$\{n\}_q = U_{n-1}((1+q)/2),$$

which is a polynomial in q. The recurrence relation (2.1) shows that $\{n\}_q$ is an integer, as is $T_n((1+q)/2))$. Since $U_{n-1}(1) = n$, $\{n\}_q$ may be considered as another q-analogue of n. It does not have positive coefficients as a polynomial in q, for example,

$$\{4\}_q = q^3 + 3q^2 + q - 1.$$

Next we check that $\{n\}_q$ are distinct, in fact this sequence is increasing. Let $x = (1+q)/2 > 1$ and note that (2.1) for $U_n(x)$ may be rewritten as

$$(U_n(x) - U_{n-1}(x)) - (U_{n-1}(x) - U_{n-2}(x)) = (2x-2)U_{n-1}(x).$$

The right side is positive, since $x > 1$ and $U_{n-1}(x) > 0$. (All of $U_{n-1}(x)$'s zeros lie in $[-1, 1]$, $U_{n-1}(1) = n > 0$.) Thus $U_n(x) - U_{n-1}(x)$ is increasing and positive because $U_1(x) - U_0(x) = 2x - 1 > 1$. We conclude that $\{n+1\}_q > \{n\}_q$.

Our second q-analogue of Euler's Odd = Distinct theorem uses Proposition 1 with $a_n = \{n\}_q$. In this case, with $(1+q)/2 = x = \cos\theta$,

$$\{2n\}_q/\{n\}_q = \frac{\sin(2n\theta)}{\sin(n\theta)} = 2\cos(n\theta) = 2T_n((1+q)/2),$$

is an integer.

Theorem 2 *Let q be a positive integer. The number of integer partitions of N into q-odd parts $\{2n+1\}_q$ is equal to the number of integer partitions of N into parts $\{n\}_q$ of multiplicity at most $2T_n((1+q)/2) - 1$.*

Again since $T_n(1) = 1$, Euler's Odd = Distinct theorem is the $q = 1$ case of Theorem 2.

3 The Glaisher bijection

An explicit bijection for Euler's theorem was given by Glaisher [8, p. 12]. Given a partition into distinct parts, any part of size $o \times 2^k$, where o is odd, is replaced by 2^k copies of the part o. Thus an odd part o will have multiplicity given by a sum of powers of 2. Since each non-negative integer m has a unique base 2 expansion, this map is a bijection. In this section we give a version of this bijection for Proposition 1. O'Hara's Algorithm B [7] also applies to Proposition 1, and gives the bijection below.

Fix a partition λ with part sizes a_n of multiplicity at most $m_n - 1$. Fix an odd integer o. Let the part a_{o2^k} in λ have multiplicity c_k, so $0 \le c_k \le m_{o2^k} - 1$. Replace each part a_{o2^k} by $m_{o2^{k-1}} m_{o2^{k-2}} \cdots m_o = a_{o2^k}/a_o$ parts of size a_o. Each part a_o now has a multiplicity which is given as a sum of these numbers, namely

$$\text{multiplicity of} \quad a_o = c_0 + \sum_{k=1}^{\infty} c_k m_{o2^{k-1}} m_{o2^{k-2}} \cdots m_o,$$
$$\text{where } 0 \le c_k \le m_{o2^k} - 1.$$

We must show that each non-negative integer m may be uniquely expressed in the above form

$$m = c_0 + \sum_{k=1}^{\infty} c_k m_{o2^{k-1}} m_{o2^{k-2}} \cdots m_o, \quad \text{where } 0 \le c_k \le m_{o2^k} - 1.$$

This follows from

$$m_o - 1 + \sum_{k=1}^{K} (m_{o2^k} - 1) m_{o2^{k-1}} m_{o2^{k-2}} \cdots m_o = m_{o2^K} m_{o2^{K-1}} \cdots m_o - 1.$$

For Theorem 1 the Glaisher map replaces each part $[2^k o]_q$ by $[2^k]_{q^o}$ parts of size $[o]_q$. For Theorem 2 the Glaisher map replaces each part $\{2^k o\}_q$ by

$$2^k T_o((1+q)/2) T_{2o}((1+q)/2) \cdots T_{2^{k-1}o}((1+q)/2)$$

parts of size $\{o\}_q$.

4 Lecture Hall results

The lecture hall theorem [2, Theorem 1.1] states that the number of integer partitions of N into odd parts $1, 3, \ldots, 2k-1$ is equal to the number of integer partitions $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_k$ of N satisfying

$$\frac{\lambda_1}{k} \ge \frac{\lambda_2}{k-1} \ge \cdots \ge \frac{\lambda_k}{1} \ge 0.$$

A q-analogue of this theorem could possibly use q-analogues of both the odd integers $1, 3, \ldots, 2k-1$, and the denominators of the inequalities $1, 2, \ldots, k$.

In this section we give a q-analogue of the lecture hall theorem in Corollary 4, which uses the integers $\{n\}_q$. We do not have a corresponding result for Theorem 1, but instead give another bijective result using inequalities of the parts in Theorem 5.

Bousquet-Mélou and Eriksson [3, Theorem 4.5] gave an infinite family of sequences $\{a_j\}$ for which there are lecture hall theorems for partitions satisfying

$$\frac{\lambda_1}{a_k} \ge \frac{\lambda_2}{a_{k-1}} \ge \cdots \ge \frac{\lambda_k}{a_1} \ge 0.$$

Their choice of a_k is a polynomial is two variables, here denoted x and y (x and y are denoted l and k in [3, Theorem 4.5]),

$$\begin{aligned} a_{2n}(x, y) &= (-1)^{n+1} x U_{n-1}(1 - xy/2), \\ a_{2n+1}(x, y) &= (-1)^n (U_n(1 - xy/2) - U_{n-1}(1 - xy/2)). \end{aligned} \tag{4.1}$$

The (x, y)-versions of the odd numbers $1, 3, \ldots, 2k-1$ are

$$\begin{aligned} &a_i(x, y) + a_{i-1}(y, x), \quad 1 \le i \le k, \text{ for } k \text{ even}, \\ &a_{i-1}(x, y) + a_i(y, x), \quad 1 \le i \le k, \text{ for } k \text{ odd}. \end{aligned} \tag{4.2}$$

A lecture hall version for the numbers $\{n\}_q$ is given by choosing $x = y = 1 + q$, in this case

$$a_n(1+q, 1+q) = U_{n-1}((1+q)/2) = \{n\}_q.$$

Theorem 3 *Let q be a positive integer. The number of integer partitions of N into parts $\{1\}_q, \{1\}_q + \{2\}_q, \dots, \{k\}_q + \{k-1\}_q$ is equal to the number of integer partitions $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \geq 0$ of N satisfying*

$$\frac{\lambda_1}{\{k\}_q} \geq \frac{\lambda_2}{\{k-1\}_q} \geq \cdots \geq \frac{\lambda_k}{\{1\}_q} \geq 0.$$

A bijection for Theorem 3 is given in [10].

The Chebyshev polynomials satisfy the trigonometric identity

$$U_{2i}(x) = U_i(2x^2 - 1) + U_{i-1}(2x^2 - 1),$$

which is equivalent to

$$\{2i+1\}_q = \{i+1\}_Q + \{i\}_Q, \quad Q = q^2 + 2q - 2.$$

This gives the following appealing version of the q-lecture hall theorem.

Corollary 4 *Let q be a positive integer and $Q = q^2 + 2q - 2$. The number of integer partitions of N into parts $\{1\}_q, \{3\}_q, \dots, \{2k-1\}_q$ is equal to the number of integer partitions $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \geq 0$ of N satisfying*

$$\frac{\lambda_1}{\{k\}_Q} \geq \frac{\lambda_2}{\{k-1\}_Q} \geq \cdots \geq \frac{\lambda_k}{\{1\}_Q} \geq 0.$$

One may attempt to choose x and y to find an analogue of the lecture hall theorem using the numbers $[k]_q$. For example, the choice of $x = 1 + q$, $y = 1 + q^2$ gives the inequalities

$$\frac{\lambda_1}{q[3]_q} \geq \frac{\lambda_2}{[2]_q} \geq \frac{\lambda_3}{[1]_q} \geq 0$$

and the q-analogues of $1, 3, 5$ become $1, 2 + q^2, 1 + 2q + q^2 + q^3$. The choice of $x = 1 + q^3$, $y = q + q^2$ gives analogues of $1, 3, 5$ of $[1]_q, [3]_q, q[5]_q$, and the three denominators are $1, 1 + q^3, q^5 + q^4 + q^2 + q - 1$. Neither choice of (x, y) extends to an expression involving $[k]_q$ beyond three terms. Thus for the partitions with q-odd parts $[2k-1]_q$, we do not have a lecture hall theorem from the Bousquet-Mélou and Eriksson result. Nonetheless these partitions are in bijection with a set of partitions described by other inequalities, which are given by Theorem 5. This theorem follows routinely from [4, Theorem 1], or from the explicit definition of $[2k-1]_q$.

Theorem 5 *Let q be a positive integer. The number of integer partitions of N into parts $[1]_q, [3]_q \ldots, [2k-1]_q$ is equal to the number of integer partitions $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \geq 0$ of N satisfying*

$$\lambda_i \geq \sum_{j=i+1}^{k} (q+q^2)(-q)^{j-i-1}\lambda_j, \quad 1 \leq i \leq k.$$

5 Remarks

Euler's theorem was generalized by Franklin [5] and Wilf [11, Theorem 1]: For any integer j, the number of integer partitions of N with exactly j different even parts is equal to the number of integer partitions of N in which exactly j different parts are repeated. Euler's theorem is the $j=0$ case. Such a result also applies to Proposition 1.

Proposition 2 *Suppose that the sequence $\{a_n\}_{n\geq 1}$ is given as in Proposition 1. Let $X(N)$ be the set of all integer partitions of N with allowed part sizes $\{a_n\}_{n\geq 1}$. Let j be any integer. The number of elements of $X(N)$ with exactly j different "even" parts a_e is equal to the number of elements of $X(N)$ such that a_n has multiplicity at least m_n exactly j times.*

Proof This follows from [11, Theorem 2] or by comparing the coefficient of $x^j t^N$ in the generating function

$$\prod_{n=1}^{\infty}\left(1+t^{a_n}+t^{2a_n}+\cdots+t^{a_n(m_n-1)}+\frac{xt^{a_n m_n}}{1-t^{a_n}}\right)$$
$$=\prod_{n=1}^{\infty}\frac{1+(x-1)t^{a_{2n}}}{1-t^{a_n}}=\prod_{n \text{ odd}}\frac{1}{1-t^{a_n}}\prod_{n \text{ even}}\left(1+\frac{xt^{a_n}}{1-t^{a_n}}\right). \qquad \square$$

Euler's theorem was also generalized by Glaisher [1, p. 6], who considered, for a fixed integer m, those partitions whose parts are not congruent to 0 modulo m, instead of partitions with odd parts. An analogue of Proposition 1 holds when a_{mn}/a_n is an integer. Theorems 1 and 2 will have such q-Glaisher results because both quotients are integers

$$\frac{[mn]_q}{[n]_q}=1+q^n[m-1]_{q^n}, \qquad \frac{\{mn\}_q}{\{n\}_q}=U_{m-1}(T_n((1+q)/2)).$$

We do not give a formal statement of these results.

Hickerson [6] considered partitions whose parts sizes are $[n]_q$ and gave the corresponding version of Theorem 5.

For the lecture hall theorem, $a_n(x,y)$ in (4.1) is given as a polynomial in x and y. Remarkably, this choice also satisfies the hypothesis of Proposition 1 for all integers $x, y \geq 2$,

$$\frac{a_{2n}(x,y)}{a_n(x,y)} = a_{n+1}(x,y) - a_{n-1}(x,y)$$
$$= \begin{cases} 2T_n(\sqrt{xy}/2), & \text{for } n \text{ even,} \\ 2\sqrt{x/y}\,T_n(\sqrt{xy}/2), & \text{for } n \text{ odd.} \end{cases}$$

It is not necessary to put $x = y = 1 + q$, and a bi-basic version of Euler's theorem holds.

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