Variants of the Rogers-Ramanujan Identities

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Abstract

We evaluate several integrals involving generating functions of continuous q-Hermite polynomials in two different ways. The resulting identities give new proofs and generalizations of the Rogers-Ramanujan identities. Two quintic transformations are given, one of which immediately proves the Rogers-Ramanujan identities without the Jacobi triple product identity. Similar techniques lead to new transformations for unilateral and bilateral series. The quintic transformations lead to curious identities involving primitive 5th roots of unity which are then extended to primitive pth roots of unity for odd p.

Running Title: Rogers-Ramanujan Identities

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1. Introduction. The Rogers-Ramanujan identities

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} = \frac{1}{(q,q^4;q^5)_\infty}, \tag{1.1}$$

$$\sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q;q)_n} = \frac{1}{(q^2,q^3;q^5)_\infty}, \tag{1.2}$$

play a central role in the theory of partitions [6] and [3]. (We have used the standard notation found in [19].) Rogers used the q-Hermite and q-ultraspherical polynomials to prove (1.1) and (1.2). He did not realize that these polynomials are orthogonal polynomials. (This was established in the 1970's, see [11], [12].) In this paper we use the orthogonality of the q-Hermite and q-ultraspherical polynomials to prove the Rogers-Ramanujan identities and give some new generalizations.

The idea is to evaluate integrals involving q-Hermite polynomials in two different ways, then equate the results. The proof of the Rogers-Ramanujan identities given in §2 is not fundamentally different from Rogers'. He found a constant term in a certain expansion without using integration. However the integrals

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do provide motivation and a different viewpoint in this theory. For example a very natural integral gives (3.4), which generalizes the Rogers-Ramanujan identities. Along the way explicit polynomials of Schur also immediately appear, see (3.5) and (3.6). Another natural integral gives the inverse to (3.4), Theorem 3.1.

Another byproduct of the integrals are new transformation formulas which generalize the Rogers-Ramanujan identities, for example Theorem 5.1 and (6.3). Of special note is Theorem 7.1, a quintic transformation which immediately proves the Rogers-Ramanujan identities without using the Jacobi triple product identity. A second quintic transformation is given in Theorem 7.2. Special cases of the second quintic transformation give curious looking formulas involving primitive fifth roots of unity, see (7.11) and (7.12). In §8, Proposition 8.1, we give a common extension of these formulas to primitive pth roots of unity for odd p. Section 9 discusses Schur's involution and the numerators and denominators of the Rogers-Ramanujan continued fraction.

We believe the contribution of this work to the old and rich subject of partition identities is the realization that partition identities follow from evaluating integrals in two different ways. The technical details of proofs employ orthogonality of functions, theory of q-series, and explicit knowledge of the connection coefficients between q-ultraspherical polynomials with different parameters. It has been known for a long time that connection coefficients can be used to prove partition identities. For example Andrews [4], and earlier Andrews and Askey [9], explored the role played by connection coefficient problems in partition identities. Rogers' original proof utilized his evaluation of the connection coefficients of the q-ultraspherical polynomials.

For the rest of this section we recall facts about the q-Hermite and q-ultraspherical polynomials. The q-Hermite polynomials $H_n(x|q)$ may be defined by the the generating function

$$\sum_{n=0}^{\infty} \frac{H_n(x|q)}{(q;q)_n} t^n = \frac{1}{(te^{i\theta}, te^{-i\theta}; q)_\infty}, \quad x = \cos\,\theta. \tag{1.3}$$

or the explicit formula

$$H_n(\cos\,\theta|q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q e^{-i(n-2k)\theta}. \tag{1.4}$$

Their orthogonality relation is

$$\frac{(q;q)_\infty}{2\pi} \int_0^\pi H_n(\cos\theta|q) H_m(\cos\theta|q) (e^{2i\theta}, e^{-2i\theta}; q)_\infty \, d\theta = \delta_{mn} (q;q)_n. \tag{1.5}$$

The q-ultraspherical polynomials have the explicit representation

$$C_n(\cos\theta; \beta|q) = \sum_{k=0}^{n} \frac{(\beta;q)_k (\beta;q)_{n-k}}{(q;q)_k (q;q)_{n-k}} \, e^{-i(n-2k)\theta}, \tag{1.6}$$

which implies the special and limiting cases

$$C_n(x;0|q) = \frac{H_n(x|q)}{(q;q)_n}, \quad \lim_{\beta\to\infty} \beta^{-n} C_n(x;\beta|q) = (-1)^n q^{n(n-1)/2} \frac{H_n(x|q^{-1})}{(q;q)_n}. \tag{1.7}$$

Rogers established the following connection formula for the q-ultraspherical polynomials, [11], [19],

$$C_n(x;\gamma|q) = \sum_{k=0}^{[n/2]} \frac{\beta^k(\gamma/\beta;q)_k(\gamma;q)_{n-k}}{(q;q)_k(q\beta;q)_{n-k}} \frac{(1-\beta q^{n-2k})}{(1-\beta)} C_{n-2k}(x;\beta|q). \tag{1.8}$$

Therefore (1.8) implies the connection coefficient relationship

$$H_n(x|q^{-1}) = \sum_{s=0}^{[n/2]} \frac{q^{-s(n-s)}(q;q)_n}{(q;q)_s(q;q)_{n-2s}} H_{n-2s}(x|q). \tag{1.9}$$

We also note that the q-ultraspherical polynomials have the generating function

$$\sum_{n=0}^{\infty} C_n(\cos\theta;\beta|q)t^n = \frac{(\beta te^{i\theta}, \beta te^{-i\theta};q)_\infty}{(te^{i\theta}, te^{-i\theta};q)_\infty}, \tag{1.10}$$

which, in view of (1.7), implies (1.3) and the generating function

$$\sum_{n=0}^{\infty} \frac{H_n(\cos\theta|q^{-1})}{(q;q)_n} q^{\binom{n}{2}}(-t)^n = (te^{i\theta}, te^{-i\theta};q)_\infty. \tag{1.11}$$

The orthogonality relation for the q-ultraspherical polynomials is, [11], [19],

$$\begin{aligned} \int_0^\pi C_m(\cos\theta;\beta|q)C_n(\cos\theta;\beta|q) \frac{(e^{2i\theta}, e^{-2i\theta};q)_\infty}{(\beta e^{2i\theta}, \beta e^{-2i\theta};q)_\infty} d\theta \\ = \frac{2\pi(\beta, q\beta;q)_\infty}{(q,\beta^2;q)_\infty} \frac{(1-\beta)(\beta^2;q)_n}{(1-\beta q^n)(q;q)_n} \delta_{mn}. \end{aligned} \tag{1.12}$$

2. The Rogers-Ramanujan identities. In this section we reinterpret Rogers' proof of the Rogers-Ramanujan identities as an integral evaluation. The sum side evaluates an integral $I(t,q)$ using q-Hermite orthogonality (1.5), while the product side evaluates $I(t,q)$ for special choices of t using the usual Fourier orthogonality.

Our integral $I(t,q)$ below is a q-analogue of the easily established

$$I(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-xt+t^2/2} e^{-x^2/2} dx = e^{t^2}.$$

We see that $I(t)$ is the integral, with respect to the Hermite weight, of the inverse of the Hermite generating function. For the q-analogue, we integrate the inverse of the q-Hermite generating function against the q-Hermite weight

$$I(t,q) = \frac{(q;q)_\infty}{2\pi} \int_0^\pi (te^{i\theta}, te^{-i\theta}, e^{2i\theta}, e^{-2i\theta};q)_\infty \, d\theta. \tag{2.1}$$

Proof of (1.1). From the generating function (1.11), the connection coefficient formula (1.9), and the orthogonality relation (1.5), we have

$$
\begin{aligned}
(2.2) \qquad I(t,q) &= \sum_{l=0}^{\infty}(-t)^l q^{\binom{l}{2}} \sum_{s=0}^{[l/2]} \frac{q^{s(s-l)}}{(q;q)_s(q;q)_{l-2s}}\delta_{l-2s,0} \\
&= \sum_{n=0}^{\infty} \frac{q^{n^2-n}}{(q;q)_n} t^{2n}.
\end{aligned}
$$

It is clear that

$$\lim_{q\to 1^-} I(t\sqrt{1-q},q) = I(t) = e^{t^2}.$$

For the product side choose $t = \sqrt{q}$, and expand the infinite products by the Jacobi triple product identity

$$(2.3) \qquad (q, \sqrt{q}e^{i\theta}, \sqrt{q}e^{-i\theta}; q)_\infty = \sum_{k=-\infty}^{\infty} (-1)^k q^{k^2/2} e^{-ik\theta},$$

and

$$(2.4) \qquad (q, e^{2i\theta}, e^{-2i\theta}; q)_\infty = \sum_{j=-\infty}^{\infty} (-1)^j q^{\binom{j+1}{2}} e^{2ij\theta}(1-e^{2i\theta}).$$

Since the integrand in $I(t,q)$ is an even function of θ, we integrate on $[-\pi,\pi]$, and use the Fourier orthonormality of $\{e^{in\theta}/\sqrt{2\pi} : -\infty < n < \infty\}$ to find

$$(2.5) \qquad I(\sqrt{q},q) = \frac{1}{(q;q)_\infty}\sum_{j=-\infty}^{\infty}(-1)^j q^{(5j^2+j)/2} = \frac{(q^5,q^3,q^2;q^5)_\infty}{(q;q)_\infty} = \frac{1}{(q,q^4;q^5)_\infty},$$

where the Jacobi triple product identity (2.3) was used in the last step. Now (2.2) and (2.5) establish (1.1) and the proof is complete.

Proof of (1.2). The other Rogers-Ramanujan identity is proven by choosing $t = q$ and writing the integrand as $(e^{i\theta}, qe^{-i\theta}, qe^{2i\theta}, e^{-2i\theta}; q)_\infty(1+e^{i\theta})$. The rest of the proof is similar to the proof of (1.1) and is omitted.

3. A generalization of the Rogers-Ramanujan identities. In this section we generalize the integral $I(t,q)$ of §2. What results are two generalizations of the Rogers-Ramanujan identities, (3.4) and Theorem 3.1, which are inverses of each other. Polynomials that Schur considered in his work on the Rogers-Ramanujan continued fraction naturally appear. Moreover one is led to the finite forms of the Rogers-Ramanujan identities due to Andrews [2].

Consider the integral

$$(3.1) \qquad I_m(t,q) = \frac{(q;q)_\infty}{2\pi}\int_0^\pi H_m(\cos\theta|q)\,(te^{i\theta}, te^{-i\theta}, e^{2i\theta}, e^{-2i\theta}; q)_\infty\, d\theta.$$

Clearly $I_0(t,q) = I(t,q)$. As in the proof of (1.1) using q-Hermite orthogonality we find that $I_m(t,q)$ is given by

$$I_m(t,q) = \sum_{l=0}^{\infty} (-t)^l q^{\binom{l}{2}} \sum_{s=0}^{[l/2]} \frac{q^{s(s-l)}}{(q;q)_s (q;q)_{l-2s}} \delta_{l-2s,m}.$$

Hence

$$I_m(t,q) = (-t)^m q^{\binom{m}{2}} \sum_{n=0}^{\infty} \frac{q^{n^2-n}}{(q;q)_n} (t^2 q^m)^n. \tag{3.2}$$

Again choosing $t = \sqrt{q}$, using (1.4), (2.3), and (2.4) we find

$$I_m(\sqrt{q}) = \frac{(-1)^m q^{m^2/2}}{(q;q)_\infty} \sum_{s=0}^{m} \begin{bmatrix} m \\ s \end{bmatrix}_q q^{2s(s-m)} (q^5, q^{3+4s-2m}, q^{2-4s+2m}; q^5)_\infty, \tag{3.3}$$

so that

$$\sum_{n=0}^{\infty} \frac{q^{n^2+mn}}{(q;q)_n} = \frac{1}{(q;q)_\infty} \sum_{s=0}^{m} \begin{bmatrix} m \\ s \end{bmatrix}_q q^{2s(s-m)} (q^5, q^{3+4s-2m}, q^{2-4s+2m}; q^5)_\infty. \tag{3.4}$$

Note that the terms $4s-2m \equiv 2 \pmod 5$ in (3.4) vanish. On the other hand if $4s-2m \equiv 0, 4 \pmod 5$ in (3.4), the infinite products may be rewritten as a multiple of the Rogers-Ramanujan product $1/(q,q^4;q^5)_\infty$, while $4s - 2m \equiv 1,\ 3 \pmod 5$ leads to a multiple of $1/(q^2,q^3;q^5)_\infty$. Thus the left side of (3.4) is a linear combination of these two functions, which we shall write as

$$\sum_{n=0}^{\infty} \frac{q^{n^2+mn}}{(q;q)_n} = \frac{(-1)^m q^{-\binom{m}{2}} a_m(q)}{(q,q^4;q^5)_\infty} + \frac{(-1)^{m+1} q^{-\binom{m}{2}} b_m(q)}{(q^2,q^3;q^5)_\infty}, \tag{3.5}$$

for some Laurent polynomials $a_m(q)$ and $b_m(q)$. It is clear from (3.4) that explicit formulas for $a_m(q)$ and $b_m(q)$ can be given using sums of q-binomial coefficients $\begin{bmatrix} m \\ s \end{bmatrix}_q$, for example

$$a_{2m}(q) = \sum_s q^{10s^2-2s-m} \begin{bmatrix} 2m \\ m+5s \end{bmatrix}_q - q^{10s^2+2s-m} \begin{bmatrix} 2m \\ m-1-5s \end{bmatrix}_q.$$

However a more elegant expression for $a_{2m}(q)$ may be found using the q-Pascal triangle relationships

$$\begin{bmatrix} 2m \\ m+5s \end{bmatrix}_q = \begin{bmatrix} 2m-1 \\ m-1+5s \end{bmatrix}_q + q^{m+5s} \begin{bmatrix} 2m-1 \\ m+5s \end{bmatrix}_q,$$

$$\begin{bmatrix} 2m \\ m-1-5s \end{bmatrix}_q = \begin{bmatrix} 2m-1 \\ m-1-5s \end{bmatrix}_q + q^{m+1+5s} \begin{bmatrix} 2m-1 \\ m-2-5s \end{bmatrix}_q,$$

namely

$$a_{2m}(q) = \sum_{\lambda} (-1)^{\lambda} q^{\lambda(5\lambda-3)/2} \begin{bmatrix} 2m-1 \\ \lfloor \frac{2m+1-5\lambda}{2} \rfloor \end{bmatrix}_q .$$

Similarly we establish explicit formulas for $a_{2m+1}(q)$, $b_{2m}(q)$, and $b_{2m+1}(q)$. From these representations one can then obtain the general formulas

$$a_m(q) = \sum_{\lambda} (-1)^{\lambda} q^{\lambda(5\lambda-3)/2} \begin{bmatrix} m-1 \\ \lfloor \frac{m+1-5\lambda}{2} \rfloor \end{bmatrix}_q , \quad b_m(q) = \sum_{\lambda} (-1)^{\lambda} q^{\lambda(5\lambda+1)/2} \begin{bmatrix} m-1 \\ \lfloor \frac{m-1-5\lambda}{2} \rfloor \end{bmatrix}_q . \tag{3.6}$$

These polynomials were considered by Schur [22], [8], [3] as numerators and denominators of the Rogers-Ramanujan continued fraction.

The left side of (3.5) is the generating function for partitions with difference at least two whose smallest part is at least $m+1$. Andrews [2] gave a polynomial generalization of the Rogers-Ramanujan identities by showing that

$$a_m(q) = \sum_j q^{j^2+j} \begin{bmatrix} m-j-2 \\ j \end{bmatrix}_q , \quad b_m(q) = \sum_j q^{j^2} \begin{bmatrix} m-j-1 \\ j \end{bmatrix}_q . \tag{3.7}$$

They have the following combinatorial interpretations: $a_m(q)$ ($b_m(q)$) is the generating function for partitions with difference at least 2 whose largest part is at most $m-2$ and whose smallest part is at least 2 (1). The representations in (3.7) also makes it easy to determine the large m asymptotics of $a_m(q)$ and $b_m(q)$, hence express the Rogers-Ramanujan continued fraction as a quotient of two infinite series.

Andrews' proof of the relationships (3.7) consists of first showing that the left-hand side l_m of (3.6) satisfy the recurrence relation $l_m - l_{m+1} = q^{m+1} l_{m+2}$. This implies that $a_{m+2} = a_{m+1} + q^m a_m$ and $b_{m+2} = b_{m+1} + q^m b_m$, which the alternate forms (3.7) satisfy. The initial conditions are $a_0 = 1 = b_1$, $a_1 = b_0 = 0$.

Our next result is an inverse relation to (3.4). Consider the integral

$$J(k) := \frac{(q;q)_\infty}{2\pi} \int_0^\pi (\sqrt{q}e^{i\theta}, \sqrt{q}e^{-i\theta}, e^{2i\theta}, e^{-2i\theta}; q)_\infty U_k(\cos\theta) d\theta,$$

where $U_k(\cos\theta) = H_k(\cos\theta|0)$ is the Chebyshev polynomial of the 2nd kind. We can expand the Chebyshev polynomials in terms of the q-Hermite polynomials (one way is to use (7.2) below)

$$U_k(x) = \sum_{j=0}^{[k/2]} (-1)^j q^{\binom{j+1}{2}} \begin{bmatrix} k-j \\ j \end{bmatrix}_q H_{k-2j}(x|q),$$

so (3.2) implies

$$J(k) = \sum_{j=0}^{[k/2]} (-1)^{k+j} q^{(k-2j)^2/2+\binom{j+1}{2}} \begin{bmatrix} k-j \\ j \end{bmatrix}_q \sum_{s=0}^{\infty} \frac{q^{s^2+s(k-2j)}}{(q;q)_s}. \tag{3.8}$$

The value of $J(k)$ may be found directly using Fourier orthogonality, (2.3), (2.4),

$$U_k(\cos\theta) = \frac{e^{-ik\theta} - e^{i(k+2)\theta}}{1 - e^{2i\theta}}$$

and the Jacobi triple product identity

$$J(k) \quad = \quad \frac{(-1)^k q^{k^2/2}}{(q;q)_\infty}(q^5, q^{3-2k}, q^{2+2k}; q^5)_\infty .$$

Theorem 3.1 *The following identity holds*

$$\frac{(q^5, q^{3-2k}, q^{2+2k}; q^5)_\infty}{(q;q)_\infty} = \sum_{j=0}^{[k/2]} (-1)^j q^{2j(j-k)+j(j+1)/2} \begin{bmatrix} k-j \\ j \end{bmatrix}_q \sum_{s=0}^{\infty} \frac{q^{s^2+s(k-2j)}}{(q;q)_s}. \tag{3.9}$$

Observe that Theorem 3.1 provides an infinite family of extensions to the two Rogers-Ramanujan identities, because the cases $k = 0, 1$ yield (1.1) and (1.2) respectively.

Carlitz [16] had results closely related to (3.4) and Theorem 3.1.

4. Multisum versions of the Rogers-Ramanujan identities. The analytic generalization of the Rogers-Ramanujan identities to modulus $2k+3$ is a k-fold sum [3, p. 111], [5]

$$\frac{(q^{2k+3}, q^{k+1-b}, q^{k+2+b}; q^{2k+3})_\infty}{(q;q)_\infty} = \sum_{s_1, \cdots, s_k \geq 0} \frac{q^{s_1^2 + \cdots + s_k^2 + s_{k-b+1} + \cdots + s_k}}{(q;q)_{s_1 - s_2} \cdots (q;q)_{s_{k-1} - s_k} (q;q)_{s_k}}, \tag{4.1}$$

where b is an integer, $0 \leq b \leq k$. These may also be proven by evaluating an integral two different ways: using q-Hermite orthogonality and Fourier orthogonality. In this section we sketch two proofs of (4.1). Each uses a connection coefficient problem for polynomials related to the q-Hermite. Our proof uses integration, while the other proof (due to Bressoud for $b = 0$), evaluates polynomials.

Both proofs use the Laurent polynomials in z defined by

$$H_{2n}(z, a|q) = \sum_{s=-n}^{n} \begin{bmatrix} 2n \\ n-s \end{bmatrix}_q q^{as^2} z^s. \tag{4.2}$$

These polynomials are related to q-Hermite polynomials by

$$H_{2n}(e^{2i\theta}, 0|q) = H_{2n}(\cos\theta|q) \quad \text{and} \quad H_{2n}(e^{2i\theta}, 1|q) = q^{n^2} H_{2n}(\cos\theta|q^{-1}).$$

The connection coefficient result that we use was given by Bressoud [14]

$$\frac{H_{2n}(z, a|q)}{(q;q)_{2n}} = \sum_{s=0}^{n} \frac{q^{s^2}}{(q;q)_{n-s}} \frac{H_{2s}(z, a-1|q)}{(q;q)_{2s}}. \tag{4.3}$$

Note that if $a = 1$, (4.3) is (1.9), which was the key result used in §2. We have, upon iterating (4.3) $k+1$ times,

$$(4.4)\qquad H_{2n}(z,k+1+a|q)=\sum_{s_1,\cdots,s_{k+1}\geq 0}\frac{(q;q)_{2n}q^{s_1^2+\cdots+s_k^2}H_{2s_{k+1}}(z,a|q)}{(q;q)_{n-s_1}(q;q)_{s_1-s_2}\cdots(q;q)_{s_k-s_{k+1}}(q;q)_{2s_{k+1}}}.$$

Bressoud chooses $a=1/2$, whence $H_{2s}(z,1/2|q)=(-q^{1/2}z,-q^{1/2}/z;q)_s$, and choosing $z=-q^{-1/2}$ forces $H_{2s_{k+1}}(z,a|q)=0$ for $s_{k+1}>0$. Then the $n\to\infty$ limit of (4.4) becomes (4.1) for $b=0$.

The integral version of this proof is to consider $a=0$, force $s_{k+1}=0$ by integrating with respect to the q-Hermite measure, and then let $n\to\infty$. This operation immediately gives the right side of (4.1). The product side is found by evaluating the integral

$$(4.5)\qquad \begin{aligned}&\frac{1}{2\pi}\int_0^\pi (q^{2k+2},-q^{k+1}e^{2i\theta},-q^{k+1}e^{-2i\theta};q^{2k+2})_\infty(e^{2i\theta},e^{-2i\theta};q)_\infty d\theta\\ =\;&\frac{(q^{2k+3},q^{k+1},q^{k+2};q^{2k+3})_\infty}{(q;q)_\infty}\end{aligned}$$

from Fourier orthogonality. Here we used

$$(4.6)\qquad \lim_{n\to\infty}H_{2n}(z,k+1|q)=(q^{2k+2},-q^{k+1}z,-q^{k+1}/z;q^{2k+2})_\infty/(q;q)_\infty.$$

A small variation allows both proofs for $1\leq b\leq k$. The following identity is easily established

$$(4.7)\qquad \sum_{s=-n}^{n}\begin{bmatrix}2n\\ n-s\end{bmatrix}_q w^{s^2-s}q^{\binom{s}{2}}(-1)^s=q^n\sum_{s=-n}^{n}\begin{bmatrix}2n\\ n-s\end{bmatrix}_q w^{s^2-s}q^{\binom{s+1}{2}}(-1)^s.$$

For Bressoud's proof of (4.1), rewrite (4.7) as

$$(4.8)\qquad H_{2n}(-q^{-c},c|q)=q^nH_{2n}(-q^{1-c},c|q),$$

for any c. If we apply (4.3) k times to $H_{2n}(-q^{-3/2},k+3/2|q)$, we have an expansion in terms of $H_{2s_k}(-q^{-3/2},3/2|q)$. Then use (4.8) to switch to $q^{s_k}H_{2s_k}(-q^{-1/2},3/2|q)$, and apply (4.3) once more to force $s_{k+1}=0$. This is the $b=1$ case of (4.1). For the general b case we apply (4.3) $k-b+1$ times, to $H_{2n}(-q^{-1/2-b},k+3/2|q)$ then alternatively apply (4.8) and (4.3) b times.

For the integration proof we slightly modify the integrand and note that if $z=e^{2i\theta}$

$$(4.9)\qquad \frac{(q;q)_\infty}{\pi}\int_0^\pi z^k(qe^{2i\theta},e^{-2i\theta};q)_\infty d\theta=q^{\binom{k}{2}}(-1)^k.$$

This time rewrite (4.7) as

$$(4.10)\qquad \int_0^\pi H_{2n}(zq^{-c},c|q)(qe^{2i\theta},e^{-2i\theta};q)_\infty d\theta=q^n\int_0^\pi H_{2n}(zq^{1-c},c|q)(qe^{2i\theta},e^{-2i\theta};q)_\infty d\theta.$$

For the $b = 1$ case, we begin by integrating $H_{2n}(z/q, k+1|q)$. Apply (4.3) k times to evaluate the integral of $H_{2s_k}(z/q, 1|q)$. Then apply (4.10) to switch to the integral of $q^{s_k} H_{2s_k}(z, 1|q)$, and finally apply (4.3) to force $s_{k+1} = 0$. The general b case is proven as before, integrating $H_{2n}(z/q^b, k+1|q)$.

One new ingredient, a change of base from q to q^2, is needed to prove the even modulus identities of Bressoud. Such a change of base is investigated more generally with Bailey pairs [15]. New Rogers-Ramanujan identities result.

5. Further generalizations of the Rogers-Ramanujan identities. In this section we evaluate another integral in two different ways. We use q-ultraspherical orthogonality and Fourier orthogonality. The integral generalizes $I(\sqrt{q}, q)$, thus we find a transformation which generalizes the Rogers-Ramanujan identities.

Let

$$I(\beta,\gamma,q) = \frac{(q, q/\beta^2, q, \gamma^2; q)_\infty}{2\pi(q/\beta, q/\beta, \gamma, q\gamma; q)_\infty} \int_0^\pi \frac{(\sqrt{q}e^{i\theta}, \sqrt{q}e^{-i\theta}, e^{2i\theta}, e^{-2i\theta}; q)_\infty}{(\sqrt{q}e^{i\theta}/\beta, \sqrt{q}e^{-i\theta}/\beta, \gamma e^{2i\theta}, \gamma e^{-2i\theta}; q)_\infty} d\theta. \tag{5.1}$$

Clearly the case $\gamma = 0$, $\beta \to \infty$ was considered in §2.

From the generating function for the q-ultraspherical polynomials (1.10) we obtain

$$\begin{aligned} I(\beta,\gamma,q) &= \frac{(q, q/\beta^2, q, \gamma^2; q)_\infty}{2\pi(q/\beta, q/\beta, \gamma, q\gamma; q)_\infty} \\ &\quad \times \sum_{n=0}^\infty \frac{q^{n/2}}{\beta^n} \int_0^\pi C_n(\cos\theta; \beta|q) \frac{(e^{2i\theta}, e^{-2i\theta}; q)_\infty}{(\gamma e^{2i\theta}, \gamma e^{-2i\theta}; q)_\infty} d\theta. \end{aligned} \tag{5.2}$$

Using the Rogers connection coefficient formula for the q-ultraspherical polynomials, (1.8), we rewrite $I(\beta,\gamma,q)$ in the form

$$I(\beta,\gamma,q) = \frac{(q, q/\beta^2; q)_\infty}{(q/\beta, q/\beta; q)_\infty} \, {}_2\phi_1(\beta, \beta/\gamma; q\gamma; q, q\gamma/\beta^2). \tag{5.3}$$

For the Fourier orthogonality, we use Ramanujan's ${}_1\psi_1$ evaluation [19, (II.28)]

$${}_1\psi_1(a; c; q, z) = \frac{(q, c/a, az, q/az; q)_\infty}{(c, q/a, z, c/az)_\infty}, \tag{5.4}$$

instead of the Jacobi triple product identity. The sum (5.4) holds for $|c/a| < |z| < 1$. We apply (5.4) with $a = \beta$, $c = q/\beta$, $z = \sqrt{q}e^{i\theta}/\beta$, and $a = 1/\gamma$, $c = \gamma$, $z = \gamma e^{2i\theta}$ to find

$$I(\beta,\gamma,q) = \frac{1}{4\pi} \sum_{m,n=-\infty}^\infty \frac{(\beta; q)_n}{(q/\beta; q)_n} \frac{(1/\gamma; q)_m}{(\gamma; q)_m} \left(\frac{\sqrt{q}}{\beta}\right)^n \gamma^m \int_0^{2\pi} e^{in\theta + 2im\theta}(1 - e^{-2i\theta}) d\theta.$$

Thus

$$I(\beta,\gamma,q) = \sum_{m=-\infty}^\infty \frac{(\beta; q)_{2m}}{(q/\beta; q)_{2m}} \left(\frac{q}{\beta^2}\right)^m \frac{(1/\gamma; q)_m}{(\gamma; q)_m} \gamma^m. \tag{5.5}$$

This leads to Theorem 5.1.

Theorem 5.1 *If* $|\gamma| < 1$ *and* $|\beta| > \sqrt{q}$, *then*

$$\tag{5.6} \begin{aligned} &\frac{(q, q/\beta^2; q)_\infty}{(q/\beta, q/\beta; q)_\infty} \; {}_2\phi_1(\beta, \beta/\gamma; q\gamma; q, q\gamma/\beta^2) = \\ &= {}_5\psi_5 \left(\begin{matrix} \sqrt{\beta}, & -\sqrt{\beta}, & \sqrt{q\beta}, & -\sqrt{q\beta}, & 1/\gamma \\ \sqrt{q/\beta}, & -\sqrt{q/\beta}, & q/\sqrt{\beta}, & -q/\sqrt{\beta}, & \gamma \end{matrix} \,\middle|\, q, q\gamma/\beta^2 \right). \end{aligned}$$

The assumptions in Theorem 5.1 can relaxed to $|q\gamma/\beta^2| < 1$ by analytic continuation.

It is interesting to note that the limiting case $\beta \to \infty$ of Theorem 5.1 gives a generating function for the integrals $I_m(\sqrt{q}, q)$. This is the case since as $\beta \to \infty$ then (5.3) and (5.5) give, after replacing m by $-m$, the relationship

$$(q;q)_\infty \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q, q\gamma; q)_n} = \sum_{m=-\infty}^{\infty} \frac{\gamma^m (q/\gamma; q)_m}{(q\gamma; q)_m} q^{2m^2},$$

which is equivalent to

$$(q;q)_\infty \sum_{n=0}^{\infty} \frac{q^{n^2}(q^{n+1}\gamma; q)_\infty}{(q;q)_n} = \sum_{m=-\infty}^{\infty} q^{2m^2} (-1)^m \, (q^{m+1}\gamma; q)_\infty \, (q^{-m}\gamma; q)_m .$$

Theorem 5.1 is a well-disguised special case of the well-poised ${}_2\phi_1$ transformation [19, (3.4.7)]. To see this write the ${}_5\psi_5$ as

$$\begin{aligned} &1 + \left\{ \sum_{m=1}^{\infty} + \sum_{m=-1}^{-\infty} \right\} \frac{(\beta; q)_{2m} (1/\gamma; q)_m}{(q/\beta; q)_{2m} (\gamma; q)_m} (q\gamma/\beta^2)^m \\ &= 1 + \sum_{m=1}^{\infty} \frac{(\beta; q)_{2m}}{(q/\beta; q)_{2m}} \left(\frac{q\gamma}{\beta^2} \right)^m \left[\frac{(1/\gamma; q)_m}{(\gamma; q)_m} + \frac{(q/\gamma; q)_m}{(q\gamma; q)_m} \right] \\ &= 1 + \sum_{m=1}^{\infty} \frac{(\beta; q)_{2m} (1/\gamma; q)_m}{(q/\beta; q)_{2m} (q\gamma; q)_m} (1 + q^m) \left(\frac{q\gamma}{\beta^2} \right)^m . \end{aligned}$$

This can be easily shown to be a multiple of a ${}_2\phi_1$ via the limiting case $u \to 1^-$ of [19, (3.4.7)] with the choices $x = u/\gamma, a = \beta, b = \beta/\gamma$.

6. An integral evaluation. In this section we consider a generalization of Theorem 5.1, (6.3), by introducing a Chebyshev polynomials of the second kind in the integrand. In other words we consider the integral

$$\tag{6.1} \begin{aligned} J(\beta, \gamma, k) \; &= \; \frac{(q, q, q/\beta^2, \gamma^2; q)_\infty}{2\pi \, (q/\beta, q/\beta, \gamma, q\gamma; q)_\infty} \\ &\times \int_0^\pi \frac{(\sqrt{q}e^{i\theta}, \sqrt{q}e^{-i\theta}, e^{2i\theta}, e^{-2i\theta}; q)_\infty}{(\sqrt{q}e^{i\theta}/\beta, \sqrt{q}e^{-i\theta}/\beta, \gamma e^{2i\theta}, \gamma e^{-2i\theta}; q)_\infty} U_k(\cos\theta) \, d\theta. \end{aligned}$$

Using (5.4), the Fourier orthogonality evaluation of $I(\beta,\gamma,q)$ of §4 implies

$$J(\beta,\gamma,k)=\frac{(\beta;q)_k}{(q/\beta;q)_k}\left(\frac{q}{\beta^2}\right)^{k/2}\sum_{n=-\infty}^{\infty}\frac{(\beta q^k;q)_{2n}(1/\gamma;q)_n}{(q^{k+1}/\beta;q)_{2n}(\gamma;q)_n}\left(\frac{q\gamma}{\beta^2}\right)^n. \tag{6.2}$$

We now evaluate $J(\beta,\gamma,k)$ using q-ultraspherical orthogonality. Use the generating function (1.10), the fact $C_k(x;q|q)=U_k(x)$, the connection coefficient formula (1.8) and the orthogonality relation (1.12) to find

$$\begin{aligned}J(\beta,\gamma,k) &= \frac{(q,q/\beta^2,q,\gamma^2;q)_\infty}{2\pi(q/\beta,q/\beta,\gamma,q\gamma;q)_\infty}\\ &\quad\times\sum_{n=0}^{\infty}\frac{q^{n/2}}{\beta^n}\int_0^\pi C_n(\cos\theta;\beta|q)C_k(\cos\theta;q|q)\frac{(e^{2i\theta},e^{-2i\theta};q)_\infty}{(\gamma e^{2i\theta},\gamma e^{-2i\theta};q)_\infty}d\theta\\ &= \frac{(q,q/\beta^2;q)_\infty}{(q/\beta,q/\beta;q)_\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{[k/2]}\frac{q^{r-s+k/2}\gamma^{r+s}}{\beta^{2j+k-2s}}\frac{(\beta/\gamma;q)_r(\beta;q)_{k-2s+r}}{(q;q)_r(q\gamma;q)_{r+k-2s}}\\ &\quad\times\frac{(1-\gamma q^{k-2s})}{(1-\gamma)}\frac{(q/\gamma;q)_s(q;q)_{k-s}(\gamma^2;q)_{k-2s}}{(q;q)_s(q\gamma;q)_{k-s}(q;q)_{k-2s}},\end{aligned}$$

which establishes the expansion formula

$$\begin{aligned}&\frac{(\beta;q)_k}{(q/\beta;q)_k}\sum_{n=-\infty}^{\infty}\frac{(\beta q^k;q)_{2n}(1/\gamma;q)_n}{(q^{k+1}/\beta;q)_{2n}(\gamma;q)_n}\left(\frac{q\gamma}{\beta^2}\right)^n\\ &=\frac{(q,q/\beta^2;q)_\infty}{(q/\beta,q/\beta;q)_\infty}\sum_{s=0}^{[k/2]}\frac{(1-\gamma q^{k-2s})}{(1-\gamma)}\frac{(\gamma\beta^2/q)^s(\beta,\gamma^2;q)_{k-2s}}{(q,q\gamma;q)_{k-2s}}\\ &\quad\times\frac{(q/\gamma;q)_s(q;q)_{k-s}}{(q;q)_s(q\gamma;q)_{k-s}}\,{}_2\phi_1\left(\begin{matrix}\beta q^{k-2s},\beta/\gamma\\ \gamma q^{k-2s+1}\end{matrix}\,\middle|\,q,q\gamma/\beta^2\right).\end{aligned} \tag{6.3}$$

In other words

$$\begin{aligned}&{}_5\psi_5\left(\begin{matrix}q^{k/2}\sqrt{\beta}, & -q^{k/2}\sqrt{\beta}, & q^{(k+1)/2}\sqrt{\beta}, & -q^{(k+1)/2}\sqrt{q\beta}, & 1/\gamma\\ q^{(k+1)/2}/\sqrt{\beta}, & -q^{(k+1)/2}/\sqrt{\beta}, & q^{1+k/2}/\sqrt{\beta}, & -q^{1+k/2}/\sqrt{\beta}, & \gamma\end{matrix}\,\middle|\,q,q\gamma/\beta^2\right)\\ &=\frac{(q/\beta;q)_k\,(q,q/\beta^2;q)_\infty}{(\beta;q)_k\,(q/\beta,q/\beta;q)_\infty}\sum_{s=0}^{[k/2]}\frac{(1-\gamma q^{k-2s})}{(1-\gamma)}\frac{(\gamma\beta^2/q)^s(\beta,\gamma^2;q)_{k-2s}}{(q,q\gamma;q)_{k-2s}}\\ &\quad\times\frac{(q/\gamma;q)_s(q;q)_{k-s}}{(q;q)_s(q\gamma;q)_{k-s}}\,{}_2\phi_1\left(\begin{matrix}\beta q^{k-2s},\beta/\gamma\\ \gamma q^{k-2s+1}\end{matrix}\,\middle|\,q,q\gamma/\beta^2\right).\end{aligned}$$

We believe the identity (6.4) to be new.

7. Two quintic transformations. One may ask if the Rogers-Ramanujan identities follow immediately from a quintic transformation for a basic hypergeometric series. One would also hope for a proof thereby of the Rogers-Ramanujan identities which does not use the Jacobi triple product identity. In this section we give in Theorem 7.1 such a result. It is again motivated by an integral with q-Hermite polynomials. Another integral of a variant of q-Hermite polynomials leads to yet another quintic transformation and is in Theorem 7.2.

Theorem 7.1 *We have*

$$\sum_{n=0}^{\infty}\frac{q^{n^2}(qf)^{2n}}{(q;q)_n} = \frac{(f^4q^5;q)_\infty}{(f^4q^5,f^6q^{10};q^5)_\infty(f^2q^3;q)_\infty} \times\sum_{n=0}^{\infty}\frac{1-f^6q^{10n+5}}{1-f^6q^5}\frac{(f^6q^5,f^4q^{10};q^5)_n}{(q^5,f^2;q^5)_n}\frac{(f^2;q)_{5n}}{(f^4q^6;q)_{5n}}q^{5\binom{n}{2}}(-f^4q^{10})^n. \tag{7.1}$$

$$= \frac{(f^4q^9,f^2q^5,f^4q^6;q^5)_\infty}{(f^2q^3;q)_\infty}{}_3\phi_2\left(\begin{matrix}f^2q^2,f^2q^3,f^2q^5\\ f^4q^9,f^4q^6\end{matrix}\,\middle|\,q^5,f^2q^5\right).$$

$$= \frac{(f^4q^8,f^2q^6,f^4q^6;q^5)_\infty}{(f^2q^3;q)_\infty}{}_3\phi_2\left(\begin{matrix}f^2q,f^2q^3,f^2q^4\\ f^4q^8,f^4q^6\end{matrix}\,\middle|\,q^5,f^2q^6\right).$$

Observe that the Rogers-Ramanujan identities (1.1) and (1.2) correspond to the special cases $f = q^{-1}$ and $f = q^{-1/2}$ in the last two forms of (7.1). Thus Theorem 7.1 implies the Rogers-Ramanujan identities without using the Jacobi triple product identity [7].

Our proof of Theorem 7.1 relies on the connection coefficient formula

$$\frac{H_n(x|q)}{(q;q)_n} = \sum_{k=0}^{[n/2]}\frac{q^k(1-q^{n-2k+1})}{(q;q)_k(q;q)_{n-k+1}}\sum_{j=0}^{[n/2]-k}\frac{(-1)^jp^{\binom{j+1}{2}}(p;p)_{n-2k-j}}{(p;p)_j}\frac{H_{n-2k-2j}(x|p)}{(p;p)_{n-2k-2j}}, \tag{7.2}$$

which follows from a two-fold application of (1.8) combined with the observation that $C_n(x;q|q) = U_n(x)$, hence is independent of q. This observation is due to Bressoud [13, (4.11)].

Proof of Theorem 7.1. Consider the integral

$$S(t) := \frac{(qt^2;q)_\infty(p;p)_\infty}{2\pi}\int_0^\pi\frac{(e^{2i\theta},e^{-2i\theta};p)_\infty}{(te^{i\theta},te^{-i\theta};q)_\infty}\,d\theta. \tag{7.3}$$

Expand the integrand as a power series in t, for $|t| < 1$, using (1.3) then apply (7.2) and the orthogonality relation (1.5). The result is that

$$S(t) = \sum_{N=0}^{\infty}t^{2N}\sum_{j=0}^{N}\sum_{k=0}^{j}\frac{q^k(1-q^{2j-2k+1})(-1)^{j-k}p^{\binom{j-k+1}{2}}}{(q;q)_k(q;q)_{2j-k+1}}\frac{q^{\binom{N-j+1}{2}}(-1)^{N-j}}{(q;q)_{N-j}}.$$

If j is replaced by $j+k$ the resulting k-sum is evaluable by a special case of the terminating ${}_2\phi_1$ summation theorem [19, (II.6)] to obtain

$$S(t) = \sum_{N=0}^{\infty}t^{2N}\sum_{j=0}^{N}\frac{q^{(N-j)(N+j+2)}p^{\binom{j+1}{2}}(-1)^j}{(q;q)_{2j}(q;q)_{N-j}(q^{2j+2};q)_{N-j}}. \tag{7.4}$$

Choosing $p = q^5$, the j-sum in (7.4) is evaluable by a limiting case of the very-well poised ${}_6\phi_5$ summation theorem [19, (II.21)] to obtain

$$S(t) = \sum_{N=0}^{\infty}\frac{q^{N^2+2N}t^{2N}}{(q;q)_N}.$$

This is the left side of Theorem 7.1, if $t^2 = f$. The first equality is obtained using the Nassrallah-Rahman ${}_8\phi_7$ representation of the integral [19, (6.3.7)]. The ${}_3\phi_2$ versions in (7.1) follow from (3.2.11) (or (3.8.9)) and (III.9) in [19]. The evaluation of the integral in (7.3) as a multiple of a ${}_3\phi_2$ when $p = q^5$ was proved combinatorially in [20].

Another quintic transformation may be found by considering the polynomials

$$\hat{H}_n(\cos(\theta)|q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{k(n-k)} e^{i(n-2k)\theta}$$

whose generating function is

$$\sum_{n=0}^{\infty} \frac{\hat{H}_n(\cos\theta|q)}{(q;q)_n} t^n = \frac{(t^2;q)_\infty}{(te^{i\theta}, te^{-i\theta};q)_\infty}.$$

Theorem 7.2 *We have the quintic transformation*

$$\begin{aligned}
\text{(7.5)} \qquad \sum_{n=0}^{\infty} \frac{q^{5n^2} t^{2n}}{(q^5;q^5)_n} &= \frac{(t^4;q^5)_\infty}{(t^2;q^5)_\infty (\omega t^{4/5}, \omega^4 t^{6/5};q)_\infty} \\
&\quad \times \sum_{n=0}^{\infty} \frac{(\omega^4 t^{6/5}/q, t^{2/5}, \omega t^{2/5}, \omega^2 t^{2/5}, \omega^4 t^{2/5};q)_n}{(q, \omega^4 t^{4/5}, \omega^3 t^{4/5}, \omega^2 t^{4/5}, t^{4/5};q)_n} \\
&\quad \times \frac{(1-\omega^4 t^{6/5} q^{2n-1})}{(1-\omega^4 t^{6/5}/q)} (-\omega)^n q^{\binom{n}{2}} t^{4n/5}, \\
&= \frac{(t^4;q^5)_\infty (t^{2/5};q)_\infty}{(t^2;q^5)_\infty (t^{4/5}, \omega t^{4/5}, \omega^4 t^{4/5};q)_\infty} {}_3\phi_2 \left(\begin{matrix} t^{2/5}, \omega t^{2/5}, \omega^4 t^{2/5} \\ \omega^2 t^{4/5}, \omega^3 t^{4/5} \end{matrix} \middle| \, q, t^{2/5} \right).
\end{aligned}$$

where ω is a primitive fifth root of unity.

We sketch the proof of Theorem 7.2. Analogous to the proof of Theorem 7.1, when $\hat{H}_{2n}(\cos(5\theta)|q^5)$ is expanded in terms of the q-Hermite polynomials, the constant term is $q^{5n^2}(q^{5n+5};q^5)_n$. This is equivalent to the integral

$$\text{(7.6)} \qquad \frac{(t^2;q^5)_\infty (q;q)_\infty}{2\pi} \int_0^\pi \frac{(e^{2i\theta}, e^{-2i\theta};q)_\infty}{(te^{5i\theta}, te^{-5i\theta};q^5)_\infty} \, d\theta = \sum_{n=0}^{\infty} \frac{q^{5n^2} t^{2n}}{(q^5;q^5)_n}.$$

The integral in (7.6) is a special case of the Nassrallah-Rahman integral. In this special case the integral is a multiple of a ${}_3\phi_2$, as was proved in [20] combinatorially. This result gives the equality between the left-hand side and the extreme right-hand side. To see that the middle term in (7.5) equals the extreme right-hand side apply (3.2.11) in [19]. Another way to see the same thing is to observe that the integral is a limiting case of an ${}_8W_7$ function. For example [19, (6.3.7), (III.23)] implies

$$\begin{aligned}
&\sum_{n=0}^{\infty} \frac{q^{5n^2} t^{2n}}{(q^5;q^5)_n} \\
&= \frac{(t^4;q^5)_\infty}{(t^2;q^5)_\infty (\omega t^{4/5}, \omega^4 t^{6/5};q)_\infty} \\
&\quad \times \lim_{g \to 0} {}_8W_7(\omega^4 t^{6/5}/q; t^{2/5}, \omega t^{2/5}, \omega^2 t^{2/5}, \omega^4 t^{2/5}, t/g; q, \omega g t^{-1/5}).
\end{aligned}$$

This is equivalent to the first equality in (7.5). The second equality in (7.5) follows from (3.2.11) in [19].

If $t^2 = 1, q^5$ in (7.5) we can use the Rogers-Ramanujan identities to find two unusual results:

$$\tag{7.7} \frac{2}{(\omega, \omega^4; q)_\infty} \left[1 + \frac{1}{2} \sum_{n=1}^{\infty} \frac{(\omega^4/q, \omega; q)_n}{(q, \omega^3; q)_n} \frac{(1-\omega^4 q^{2n-1})}{(1-\omega^4/q)} (-\omega)^n q^{\binom{n}{2}} \right]$$

$$= \frac{1}{(q^5, q^{20}; q^{25})_\infty}$$

$$\tag{7.8} = \frac{1}{(\omega, \omega^4; q)_\infty} \lim_{r \to 1^-} {}_3\phi_2 \left(\begin{matrix} r, \omega r, \omega^4 r \\ \omega^2 r^2, \omega^3 r^2 \end{matrix} \middle| \, q, t^{2/5} \right),$$

and

$$\tag{7.9} \frac{1}{(1-q^5)(\omega q^2, \omega^4 q^3; q)_\infty}$$

$$\times \left[\sum_{n=0}^{\infty} \frac{(\omega^4 q^2, \omega q^2; q)_n (1-q^5)}{(q, \omega^3 q; q)_n (1-q^{5n+5})} \frac{(1-\omega^4 q^{2n+2})}{(1-\omega^4 q^2)} (-\omega)^n q^{\binom{n}{2}+2n} \right]$$

$$= \frac{1}{(q^{10}, q^{15}; q^{25})_\infty}$$

$$\tag{7.10} = \frac{(\omega q; q)_\infty}{(1-q^5)(\omega^2 q^2, \omega^3 q^2, \omega q^2; q)_\infty} {}_3\phi_2 \left(\begin{matrix} \omega q, \omega^2 q, q \\ \omega^4 q^2, q^2 \end{matrix} \middle| \, q, \omega q \right),$$

respectively. To get (7.10) we set $t^{2/5} = \omega q$ in (7.5).

We can rewrite (7.7) by writing the quantity in square brackets as

$$\frac{1}{2} + \frac{1}{2} \lim_{n \to \infty} {}_5\phi_4 \left(\begin{matrix} \omega^4/q, \sqrt{q}\omega^2, -\sqrt{q}\omega^2, \omega, q^{-n} \\ \omega^2/\sqrt{q}, -\omega^2/\sqrt{q}, \omega^3, q^n \omega^4 \end{matrix} \middle| \, q, q^n \omega \right).$$

The ${}_5\phi_4$ is summed by (II.20) in [19] and the result is the curious formula

$$\tag{7.11} \frac{1}{(\omega, \omega^4; q)_\infty} + \frac{1}{(\omega^2, \omega^3; q)_\infty} = \frac{1}{(q^5, q^{20}; q^{25})_\infty}.$$

The identities (7.11) also follows from (7.8) by writing the ${}_3\phi_2$, as $r \to 1^-$ as

$$1 + (1-r) \left[\sum_{n=1}^{\infty} \frac{(\omega, \omega^4; q)_n \, (q; q)_{n-1}}{(q, \omega^2, \omega^3; q)_n} r^n \right] + O(1-r).$$

By Abel's summability the ${}_3\phi_2$ tends to $1 + (\omega, \omega^4; q)_\infty / (\omega^2, \omega^3; q)_\infty$, which leads to (7.11). A direct proof of an extension of formula (7.11) to odd moduli is given in Proposition 8.1.

Another curious identity arises from (7.9) or (7.10). First rewrite the left-hand side of (7.9) as

$$\frac{(1+\omega^2)(1-q^5)^{-1}}{(\omega q^2, \omega^4 q^2; q)_\infty} \sum_{n=0}^{\infty} \frac{(\omega q, \omega^2 q, \omega^4 q; q)_n}{(\omega^2 q^2, \omega^3 q^2, q^2; q)_n} \frac{(1-\omega^4 q^{2n+2})}{(1+\omega^2)} (-\omega q)^n q^{\binom{n+1}{2}},$$

which simplifies to

$$\frac{(1-q)(1-\omega^2 q)(1-\omega^3 q)(1+\omega^2)}{(1-q^5)(1-\omega)(1-\omega^4)(\omega q^2,\omega^4 q^2;q)_\infty(-\omega q)}$$
$$\times\left[-1+\lim_{m\to\infty}{}_4\phi_3\left(\begin{matrix}\omega^4,-\omega^2 q,\omega,q^{-m}\\ -\omega^2,q\omega^3,q^m\omega^4\end{matrix}\;\middle|\;q,q^{m+1}\omega\right)\right].$$

The ${}_4\phi_3$ can be summed by the q-analogue of Dixon's theorem [19, (II.14)] and the result when combined with (7.9) is the second curious identity

$$\frac{\omega^4 q^{-1}}{(1-\omega)(1-\omega^2)}\left[\frac{1}{(\omega q,\omega^4 q;q)_\infty}-\frac{1}{(\omega^2 q,\omega^3 q;q)_\infty}\right]=\frac{1}{(q^{10},q^{15};q^{25})_\infty}. \tag{7.12}$$

The above identity also follows from (7.10) by noting that the ${}_3\phi_2$ is

$$\frac{(1-q)(1-\omega^4 q)}{(1-\omega)(1-\omega^2)\omega q}\left[{}_2\phi_1\left(\begin{matrix}\omega,\omega^2\\ \omega^4 q,\end{matrix}\;\middle|\;q,\omega q\right)-1\right]. \tag{7.13}$$

The q-Gauss theorem [19, (II.8)] sums the ${}_2\phi_1$ and establishes (7.12).

We now relate the results of this section to Schur's polynomials of §3.

Corollary 7.3 *For $m=1,2,\ldots$, the transformation*

$$\frac{(q^{10m};q^5)_\infty}{(q^{5m};q^5)_\infty(\omega q^{2m},\omega^4 q^{3m};q)_\infty}$$
$$\times\sum_{n=0}^{\infty}\frac{(\omega^4 q^{3m-1},q^m,\omega q^m,\omega^2 q^m,\omega^4 q^m;q)_n}{(q,\omega^4 q^{2m},\omega^3 q^{2m},\omega^2 q^{2m},q^{2m};q)_n}$$
$$\times\frac{(1-\omega^4 q^{3m+2n-1})}{(1-\omega^4 q^{3m-1})}(-\omega)^n q^{\binom{n}{2}}q^{2mn}$$
$$=\frac{(q^{10m};q^5)_\infty(q^m;q)_\infty}{(q^{5m};q^5)_\infty(q^{2m},\omega q^{2m},\omega^2 q^{2m};q)_\infty}{}_3\phi_2\left(\begin{matrix}q^m,\omega q^m,\omega^4 q^m\\ \omega^2 q^{2m},\omega^3 q^{2m}\end{matrix}\;\middle|\;q,q^m\right)$$
$$=\frac{(-1)^m q^{-5\binom{m}{2}}a_m(q^5)}{(q^5,q^{20};q^{25})_\infty}+\frac{(-1)^{m+1}q^{-5\binom{m}{2}}b_m(q^5)}{(q^{10},q^{15};q^{25})_\infty},$$

holds, where ω is a primitive fifth root of unity.

Proof. Put $t=q^{5m/2}$ in (7.5) and apply (3.5).

Similarly the choice $f=q^{-1+m/2}$ in Theorem 7.1 and (3.5) give

$$\frac{(q^{2m+1};q)_\infty}{(q^{2m+1},q^{3m+4};q^5)_\infty(q^{m+1};q)_\infty} \tag{7.14}$$
$$\times\sum_{n=0}^{\infty}\frac{1-q^{3m+10n-1}}{1-q^{3m-1}}\frac{(q^{3m-1},q^{2n+6};q^5)_n}{(q^5,q^{m-2};q^5)_n}\frac{(q^{m-2};q)_{5n}}{(q^{2m+2};q)_{5n}}q^{5\binom{n}{2}}(-1)^n q^{(2m+6)n}.$$
$$=\frac{(q^{2m+1};q)_\infty(q^{m+1};q^5)_\infty}{(q^{m+1};q)_\infty(q^{2m+1},q^{2m+2},q^{2m+3};q^5)_\infty}{}_3\phi_2\left(\begin{matrix}q^{m+5},q^{m+1},q^{m+2}\\ q^{2m+5},q^{2m+4}\end{matrix}\;\middle|\;q^5,q^{m+1}\right)$$
$$=\frac{(-1)^m q^{-\binom{m}{2}}a_m(q)}{(q,q^4;q^5)_\infty}+\frac{(-1)^{m+1}q^{-\binom{m}{2}}b_m(q)}{(q^2,q^3;q^5)_\infty}.$$

8. Remarks. One may translate the Fourier orthogonality of the integrals into constant term identities. For example $I(\sqrt{q}, q)$, which gives the Rogers-Ramanujan identities, becomes

$$CT(1/z, qz; q)_\infty (qz^2, q/z^2; q^2)_\infty = 2\frac{(q^2, q^3, q^5; q^5)_\infty}{(q;q)^2_\infty}, \tag{8.1}$$

where $CTf(z)$ is the constant term in the Laurent expansion of $f(z)$ around $z = 0$. Of course (8.1) is immediate from the Jacobi triple identity. Do other root systems also give well-known theorems in this way?

It is remarkable that the j-sum in (7.4) factors for six other values, $p = q^{7/2}, q^{5/2}, q^4, q^3, q^2, q^1$. The $p = q^{7/2}$ case should give a q^{14} transformation similar to Theorem 7.1, while the $p = q^{5/2}$ case will be a transformation on q^{10}.

Recall that the polynomials $a_m(q)$ and $b_m(q)$ of §3 are linear independent solutions of a recurrence relation. Al-Salam and Ismail [1] studied the more general recurrence relation

$$y_{m+1} = x(1 + aq^m)y_m - bq^{m-1}y_{m-1}.$$

They gave an explicit formula for the solution $U_m(x; a, b)$ satisfying $U_0(x; a, b) = 1$ and $U_1(x; a, b) = x(1+a)$. They also showed that the numerators of the corresponding continued J-fraction are $(1+a)U_{m-1}(x; aq, bq)$. The a_m's and b_m's are embedded in the family of the U_m's since $b_m(q) = U_{m-1}(1; 0, -q)$ and $a_m(q) = U_{m-2}(1; 0, -q^2)$. Furthermore $U_m(x; a, b)$ and $U_m(x; aq, bq)$ are linear independent solutions of the above recurrence relation. For properties of the polynomials $U_n(x; a, b)$, and the associated continued fraction, see [1].

We now come to the extension of (7.11) and (7.12).

Proposition 8.1 *If $p \geq 5$ is odd, ω is a primitive pth root of unity, and $j \leq (p-1)/2$ is a fixed positive integer, then*

$$\frac{1}{p}\sum_{m=1}^{(p-1)/2} \omega^{-mj}(1-\omega^m)(1-\omega^{m(2j-1)}) \prod_{k=1, k\neq m}^{(p-1)/2} \frac{1}{(\omega^k q, \omega^{p-k}q; q)_\infty} \tag{8.2}$$

$$= (-1)^j q^{\binom{j}{2}} \prod_{1\leq a\leq p-1, a\neq (p\pm(2j-1))/2} \frac{1}{(q^{ap}; q^{p^2})_\infty}$$

$$= \sum_{m=1}^{(p-1)/2} \frac{\omega^{m(1-j)}(1-\omega^{m(2j-1)})}{(\omega^m - 1)} \prod_{k=1, k\neq m}^{(p-1)/2} \frac{1}{(\omega^k, \omega^{p-k}; q)_\infty}. \tag{8.3}$$

Proof. Let

$$G(z) = (q, z, q/z; q)_\infty = \sum_{n=-\infty}^{\infty} q^{\binom{n}{2}}(-z)^n.$$

Then

$$\frac{1}{p}\sum_{m=0}^{p-1} G(z\omega^m) = (q^{p^2}, q^{(p^2-p)/2}z^p, q^{(p^2+p)/2}/z^p; q^{p^2})_\infty. \tag{8.4}$$

If $z = q^j$, the $m = 0$ term of (8.4) vanishes. Dividing both sides of (8.4) by $(q^p; q^p)_\infty$ and combining the m and $p - m$ terms yields the result since $p = \prod_{m=1}^{p-1}(1 - \omega^m)$.

The case $j = 1$ of (8.3) takes the form

$$\sum_{m=1}^{(p-1)/2} \prod_{k=1, k\neq m}^{(p-1)/2} \frac{1}{(\omega^k, \omega^{p-k}; q)_\infty} = \prod_{1\leq a\leq p-1, a\neq (p\pm(2j-1))/2} \frac{1}{(q^{ap}; q^{p^2})_\infty}, \tag{8.5}$$

and reduces to (7.11) when $p = 5$. On the other hand when $j = (p - 1)/2$ (8.2) becomes

$$\begin{aligned} &\sum_{m=1}^{(p-1)/2} \frac{[\omega^{m(p-1)/2} + \omega^{m(p+1)/2} - \omega^{m(p-3)/2} - \omega^{m(p+3)/2}]}{\prod_{k=1,k\neq m}^{(p-1)/2} (\omega^k, \omega^{p-k}; q)_\infty} \\ &= p \prod_{1\leq a\leq p-1, a\neq (p\pm(2j-1))/2} \frac{1}{(q^{ap}; q^{p^2})_\infty}, \end{aligned} \tag{8.6}$$

which reduces to (7.12) when $p = 5$ since in this case the numerator on the left-hand side of (8.6) is $(-1)^{m-1}(\omega^2 + \omega^3 - \omega - \omega^4)$.

9. Appendix. In this section we recapitulate Schur's involution which proves the Rogers-Ramanujan identities. We also restrict his involution to prove (3.5)-(3.6)

$$(q;q)_\infty \sum_{n=0}^{\infty} \frac{q^{n^2+mn}}{(q;q)_n} = (-1)^m q^{-\binom{m}{2}} \left[a_m(q)(q^5, q^2, q^3; q^5)_\infty - b_m(q)(q^5, q^1, q^4; q^5)_\infty)\right]. \tag{9.1}$$

Schur defines an involution ϕ on the set of ordered pairs of partitions (λ, μ), where λ has distinct parts, and μ has parts differing by at least two. If the sign of (λ, μ) is $(-1)^{\#\text{parts of } \lambda}$, the left side of (9.1) is the generating function of all such (λ, μ), where the smallest part of μ is at least $m + 1$. The right side of (9.1) is the generating function of the fixed points when Schur's involution ϕ is restricted to this set.

The involution ϕ is defined in stages.

For the first stage, compare the largest part of λ and the largest part of μ, find the larger of the two parts, and move that part to the other partition. This fails only if the largest parts are equal, or the largest part of λ is one more than the largest part of μ. We call these two cases A and B respectively.

We define three parameters p, q, and r: p is the smallest part of λ, q is the length of the leading run of λ (as in Franklin's involution for the pentagonal number theorem, [6, p. 10]), and r is the length of the leading "double run" of μ.

In each case the definition of ϕ depends upon inequalities between p, q, and r. In case A,

1. if $p = \min\{p, q, r\}$, apply Franklin's involution to λ,
2. if $r = \min\{p, q, r\} < p$, remove 1 from each of the first r parts of μ, and create a new smallest part of λ of size r,
3. if $q = \min\{p, q, r\} < \min\{p, r\}$, remove one from each of the q largest parts of λ, add one to the 2nd through $q + 1$ parts of μ, and move the largest part of μ to the largest part of λ.

The definition of ϕ in case B is

1. if $q=\min\{p,q,r\}<p$, apply Franklin's involution to λ,
2. if $p=\min\{p,q,r\}$, add 1 to each of the first p parts of μ, and delete the smallest part of λ of size p,
3. if $r=\min\{p,q,r\}<\min\{p,q\}$, add one to the 2nd through $r+1$ parts of λ, subtract one from each of the r largest parts of μ, and move the largest part of λ to the largest part of μ.

It is easy to see that ϕ interchanges cases (1), (2), and (3) of case A with cases (1), (2), and (3) of case B respectively.

What are the fixed points, namely where is ϕ not well-defined? For case A they occur at Franklin fixed points $((2p-1,\cdots,p),(2p-1,2p-3,\cdots,1)), p\geq 0$, and for case B at the other family of Franklin fixed points $((2p,\cdots,p+1),(2p-1,2p-3,\cdots,1)), p\geq 1$. The generating function for these fixed points is the right side of (9.1) for $m=0$, by the Jacobi triple product identity.

If we restrict to the case $m=1$, so that the smallest part of μ is at least two, the fixed points change. They are $((2p,\cdots,p+1),(2p,2p-2,\cdots,2)), p\geq 0$, $((2p+1,\cdots,p),(2p,2p-3,\cdots,2)), p\geq 1$, and $(1,\emptyset)$. Again (9.1) for $m=1$ results from the Jacobi triple product identity.

We now consider the fixed points for the general $m\geq 2$ case. Any pair $(\lambda,\emptyset)$, where the largest part of λ is $\leq m$ is fixed. The remaining fixed points are from case A(2) and case B(3): $((m+2r-1,m+2r-2,\cdots,m+r,\theta),(m+2r-1,\cdots,m+1))$, $((m+2r,m+2r-1,\cdots,m+r,\theta),(m+2r-1,\cdots,m+1))$ where θ is any partition with distinct parts, part sizes between $r+1$ and $m+r-1$.

So the generating function of the fixed points is

$$\begin{aligned} FP(m) = &\quad (q;q)_m+\sum_{r\geq 1}(-1)^r q^{5r^2/2+r(2m-1/2)}(q^{r+1};q)_{m-1} \\ &+\sum_{r\geq 1}(-1)^{r+1}q^{5r^2/2+r(2m+3/2)+m}(q^{r+1};q)_{m-1} \\ = &\quad \sum_{r\geq 0}(-1)^r q^{5r^2/2+r(2m-1/2)}(q^{r+1};q)_{m-1} \\ &+\sum_{r\geq 0}(-1)^{r+1}q^{5r^2/2+r(2m+3/2)+m}(q^{r+1};q)_{m-1}. \end{aligned} \tag{9.2}$$

If we expand $(q^{r+1};q)_{m-1}$ by the q-binomial theorem we have

$$\begin{aligned} FP(m) = &\sum_{s,r\geq 0}\begin{bmatrix} m-1\\ s\end{bmatrix}_q (-1)^{r+s}q^{5r^2/2+r(2m-1/2)+(r+1)s+\binom{s}{2}} \\ &+\sum_{s,r\geq 0}\begin{bmatrix} m-1\\ s\end{bmatrix}_q (-1)^{r+s+1}q^{5r^2/2+r(2m+3/2)+m+(r+1)s+\binom{s}{2}}. \end{aligned} \tag{9.3}$$

Upon replacing s by $m-1-s$, and r by $-m-r$ the exponent of q in the second term of (9.3) becomes the exponent of q in the first term of (9.3), thus

$$\begin{aligned} FP(m) = &\sum_{s=0}^{m-1}\begin{bmatrix} m-1\\ s\end{bmatrix}_q (-1)^s q^{\binom{s}{2}} \\ &\times\left(\sum_{r=-\infty}^{\infty}(-1)^r q^{5r^2/2+r(2m-1/2)+(r+1)s}-\sum_{r=-1}^{1-m}(-1)^r q^{5r^2/2+r(2m-1/2)+(r+1)s}\right). \end{aligned} \tag{9.4}$$

For each r between -1 and $1-m$ the s-sum of the second term in (9.4) is zero by the q-binomial theorem. If we apply the Jacobi triple product formula to the first term in (9.4) we have

$$FP(m) = \sum_{s=0}^{m-1} \begin{bmatrix} m-1 \\ s \end{bmatrix}_q (-1)^s q^{\binom{s+1}{2}} (q^5, q^{2+2m+s}, q^{3-2m-s}; q^5)_\infty . \tag{9.5}$$

Considering residue classes mod 5 one sees that (9.5) is (9.1).

Andrews and Baxter [10] also found (9.2), although they did not give the equivalent form (9.1). Kadell [21] gave an involution for the the case $m = 2$ and pointed out that his involution is different from the involution used earlier by Garsia and Milne [17], [18].

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