REALIZATIONS OF REGULAR ABSTRACT POLYHEDRA OF TYPES $\{3,6\}$ AND $\{6,3\}$

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ABSTRACT. This paper classifies and gives methods for computing the irreducible realizations of the abstract polyhedra corresponding to regular maps of type $\{3,6\}$ and $\{6,3\}$. A complete list of irreducible realizations is given for polyhedra of type $\{3,6\}$.

RÉSUMÉ. Cet article classifie et donne des méthodes pour calculer les réalisations irréductibles des polyédres abstraits correspondant aux cartes réguliéres de type $\{3,6\}$ et $\{6,3\}$. Une liste compléte des réalisations irréductibles des polyédres de type $\{3,6\}$ est donnée.

1. INTRODUCTION

A regular abstract polyhedron is a poset with certain symmetry properties. By considering the permutation action of its automorphism group on an appropriate set, one can geometrically realize the abstract polyhedron in Euclidean space. Each irreducible representation of this group gives an irreducible realization of the regular abstract polyhedron. In this paper we carry out this program for four infinite families of regular abstract polyhedra.

These four families are $\{3,6\}_{(b,0)}$, $\{3,6\}_{(b,b)}$, $\{6,3\}_{(b,0)}$ and $\{6,3\}_{(b,b)}$. A list of the distinct irreducible realizations of $\{3,6\}_{(b,0)}$ is given in Section 3; results for $\{3,6\}_{(b,b)}$ are presented in the following section. There are infinitely many irreducible realizations of the abstract polyhedra $\{6,3\}_{(b,0)}$ and $\{6,3\}_{(b,b)}$. A listing of the distinct symmetry group actions on these realizations is given in Sections 5 and 6.

2. DEFINITIONS

An *abstract polyhedron* is a graded poset $\mathcal{P}$ with ranks $\{-1,0,1,2,3\}$ having the four properties listed below (see [11]). The rank zero elements of $\mathcal{P}$ are called *vertices* of $\mathcal{P}$; this set is denoted by $\mathcal{P}_0$. Elements of rank one are *edges*, denoted $\mathcal{P}_1$, and the rank two elements $\mathcal{P}_2$ are referred to as *faces* of $\mathcal{P}$. A *flag* of $\mathcal{P}$ is a maximal totally ordered subset of $\mathcal{P}$. We say two flags are *adjacent* when they differ by just one face.

1. $\mathcal{P}$ is a finite ranked poset with unique minimum and maximum elements whose ranks are -1 and 3, respectively.
2. Each flag of $\mathcal{P}$ must contain five elements; in particular, each flag contains the -1 and 3 faces of $\mathcal{P}$.

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3. The polyhedron $\mathcal{P}$ must be *strongly flag connected*; for every pair of flags ϕ and ψ there must be a chain of flags $\phi = \phi_0, \phi_1, ..., \phi_k = \psi$ such that ϕ_i and ϕ_{i+1} are adjacent, with $\phi \cap \psi \subset \phi_i$ for each i.
4. For all i, if F is an i face and G is an $i+2$ face of $\mathcal{P}$, there are exactly two faces H such that $F < H < G$.

Abstract polyhedra are a special case of the more general *abstract polytopes* discussed in detail in [11], [10] and [9].

We now define the notion of regularity for abstract polyhedra $\mathcal{P}$. Choose some base-flag Φ in $\mathcal{P}$. If there exist three automorphisms ρ_0, ρ_1 and ρ_2 of $\mathcal{P}$ such that each ρ_i fixes all but the i^{th} face of Φ, the automorphism group $\mathcal{A}(\mathcal{P})$ of $\mathcal{P}$ will be flag-transitive, and we say that $\mathcal{P}$ is a *regular abstract polyhedron* [11].

A *realization* of an abstract regular polyhedron is a collection of points called vertices V in some Euclidean space $\mathbb{R}^d$ which has the property that there is a surjection $\beta : \mathcal{P}_0 \mapsto V$ such that the action of each element of $\mathcal{A}(\mathcal{P})$ induces an isometry of V. Since this isometry can be extended to the entire space (uniquely, iff $\text{Aff}(V) = \mathbb{R}^d$), this gives us a representation of the group $\mathcal{A}(\mathcal{P})$ on the vector space $\mathbb{R}^d$ [10, Theorem 6]. Note that the vertices of a realization need not be centered at the origin.

We say that the realization is *vertex faithful* or simply *faithful* if the map β is bijective. One example of a faithful realization of a polyhedron is the *simplex realization*. This is obtained by sending each vertex of $\mathcal{P}$ to one of the orthogonal unit basis vectors of $\mathbb{R}^n$, where $n = |\mathcal{P}_0|$ (see [10]). The corresponding representation of $\mathcal{A}(\mathcal{P})$ is called the *simplex representation*. The *trivial realization* is the non-faithful realization in which all vertices of the realization coincide at a single point.

For convenience, we refer to appropriate segments joining elements of V as edges of the realization of $\mathcal{P}$, and consider the faces of the realization of $\mathcal{P}$ to be described by appropriate unions of edges of the realization. Note that these "faces" are not necessarily planar.

The edges of $\mathcal{P}$ are a special case of the more general notion of *diagonals* of $\mathcal{P}$. These are described by unordered pairs $\{v, w\}$ of vertices of $\mathcal{P}$. The *diagonal classes* $\{\Lambda_1, ..., \Lambda_j\}$ of $\mathcal{P}$ are the equivalence classes of the diagonals under the action of $\mathcal{A}(\mathcal{P})$. The *diagonal vector* $\{0, \delta_1, ..., \delta_r\}$ of a realization of $\mathcal{P}$ is given by squares of distances between vertices in diagonals of each class; $\delta_i = |\beta(v) - \beta(w)|^2$, where $\{v, w\} \in \Lambda_i$ [10]. Two non-trivial realizations are defined to be *equivalent* if their diagonal vectors are the same up to a scalar multiple.

Given two realizations $P \subset \mathbb{R}^p$ and $Q \subset \mathbb{R}^q$ of a polyhedron $\mathcal{P}$ with vertex sets $V = \beta_P(\mathcal{P}_0)$ and $W = \beta_Q(\mathcal{P}_0)$, we obtain a third realization $P\#Q \subset \mathbb{R}^p \times \mathbb{R}^q$ called the *blend* $P\#Q$ by defining $\beta_{P\#Q}(p) = (\beta_P(p), \beta_Q(p))$. A realization is said to be *irreducible* or *pure* if it is not the blend of two non-trivial realizations. A realization of $\mathcal{P}$ is pure exactly when the representation of $\mathcal{A}(\mathcal{P})$ on $\text{Aff}(V)$ is irreducible [10, p. 47].

Define the *Wythoff space* W_G of a representation G of $\mathcal{A}(\mathcal{P})$ to be the subspace of points fixed by the action of both ρ_1 and ρ_2 (see [10]). The dimension of the Wythoff space of an irreducible representation equals the multiplicity of that representation in the simplex representation.

From any representation G of $\mathcal{A}(\mathcal{P})$, we can obtain a realization of $\mathcal{P}$ by applying *Wythoff's construction*. Given a point $w \in W_G$, we define a set V of vertices and a map $\beta : \mathcal{P}_0 \to V$ by $\beta(v) = w$, $\beta(g(v)) = g(w)$, for some fixed $v \in V$. If

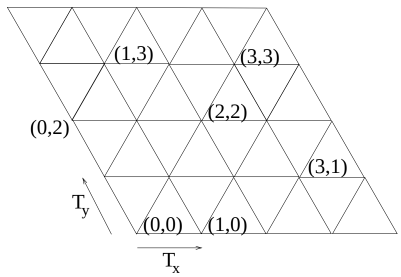

FIGURE 1. Net for $\{3,6\}_{(4,0)}$

$dim(W_G) > 1$, different choices of p and q in W_G can yield inequivalent realizations P and Q of $\mathcal{P}$. The linear combination $\lambda P + \mu Q$ is defined to be the realization determined by applying Wythoff's construction to $\lambda p + \mu q$ (see [10]).

In the case that all the irreducible components G of the simplex representation of $\mathcal{P}$ have $w_G = 1$, we can enumerate the distinct irreducible realizations of the simplex realization of $\mathcal{P}$. By Theorems 9 and 10 of [10], we know we can reconstruct any realization of $\mathcal{P}$ by scaling and taking blends of these irreducible component realizations.

If $\overline{w}$ is the dimension of the Wythoff space of the simplex realization, then:

$$\sum_G w_G^2 = \overline{w}, \text{ and } \sum_G w_G d_G = |\mathcal{P}_0|,$$

where the sum is over the distinct irreducible orthogonal representations G of $\mathcal{A}(\mathcal{P})$ and d_G is the dimension of representation G [10, Theorem 17].

In the case of polyhedra of type $\{3,6\}$, we will have $w_G = 1$ for each G. Using the fact that w_G equals the multiplicity of an irreducible representation G in the simplex realization, we shall conclude that these representations are irreducible and inequivalent. Our observations are confirmed by the fact that $\sum_G w_G d_G = |\mathcal{P}_0|$. Here, Wythoff's construction generates a complete list of irreducible realizations up to scalar multiples.

In the case of polyhedra of type $\{6,3\}$, $w_G = 2$ does occur. We generate a list of component representations and determine that the list is complete and its elements are irreducible by comparison with the case $\{3,6\}$ and by confirming $\sum_G w_G d_G = |\mathcal{P}_0|$. Because Wythoff's construction can produce inequivalent realizations from the same representation, it is impossible to provide a complete list of irreducible realizations.

3. Irreducible Realizations of $\{3,6\}_{(b,0)}$

In this section we consider the regular abstract polyhedra $\{3,6\}_{(b,0)}$. Consider the tiling of the plane by triangles depicted in Figure 1. Identify opposite edges of the parallelogram in Figure 1 as in a torus. Then the edges and triangles of this tiling are the one and two faces of $\{3,6\}_{(4,0)}$.

We can also define it in terms of the translational symmetries of the tiling labeled T_x and T_y. Taking the quotient of the triangle tiling by the symmetry group generated by T_x^4 and T_y^4 yields the abstract polyhedron $\{3,6\}_{(4,0)}$. The -1 face of $\{3,6\}_{(4,0)}$ corresponds to the empty set, the 3-face corresponds to the entire polyhedron, and the ordering on the poset is given by inclusion.

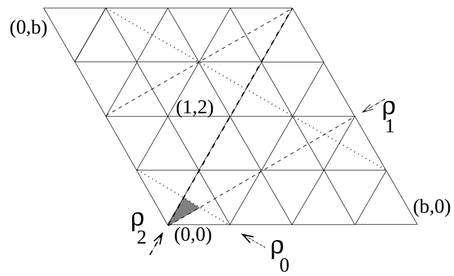

FIGURE 2. Generators of the automorphism group of $\{3,6\}_{(4,0)}$

In the example above, replacing 4 by a positive integer b yields the regular abstract polyhedron $\{3,6\}_{(b,0)}$. The rhombus in Figure 1 has four equilateral triangles along each edge; identifying opposite edges of this rhombus gives the regular abstract polyhedron $\{3,6\}_{(b,0)}$. Identifying opposite sides of a rhombus with b equilateral triangles along an edge gives us the regular abstract polyhedron $\{3,6\}_{(b,0)}$, which has $2b^2$ faces, $3b^2$ edges, and b^2 vertices [5].

Our goal is to describe the irreducible realizations of the polyhedra in this family. To this end, we shall decompose the simplex representation of $\{3,6\}_{(b,0)}$ in $\mathbb{C}^{b^2}$ into its irreducible parts.

It is relatively simple to find basis vectors $\{f_{l,m}\} \subset \mathbb{C}^{b^2}$, each of which is an eigenvector of T_x and T_y. The orbits of these vectors under the action of $\mathcal{A}(\{3,6\}_{(b,0)})$ are bases of the irreducible realization spaces of $\{3,6\}_{(b,0)}$. A change of coordinates will allow us to consider these as realizations over $\mathbb{R}$, and to apply the theorems mentioned in the previous section.

Recall that $T_x^b = T_y^b = I$ when these elements of the automorphism group act on the simplex realization. Hence, the eigenvalues of T_x and T_y must be b^{th} roots of unity. If we use the labeling of the vertices $v_{j,k}$ of $\{3,6\}_{(b,0)}$ indicated in Figure 1, a simple calculation shows that the vectors $\{f_{l,m}\}_{0\leq l,m<b}$ defined by:

$$f_{l,m} = \sum_{0\leq j,k<b} e^{(lj+mk)\frac{2\pi i}{b}} v_{j,k}$$

are simultaneous eigenvectors of T_x and T_y with eigenvalues $e^{-l\frac{2\pi i}{b}}$ and $e^{-m\frac{2\pi i}{b}}$, respectively.

We can use this information to compute the eigenvalues of the adjacency matrix A_1 of the edge graph of $\{3,6\}_{(b,0)}$.

THEOREM 1. *The adjacency matrix A_1 of the one-skeleton of the abstract polyhedron $\{3,6\}_{(b,0)}$ is given by:*

$$A_1 = T_x + T_xT_y + T_y + T_x^{-1} + (T_xT_y)^{-1} + T_y^{-1}.$$

The multiset of eigenvalues of this matrix is:

$$\left\{2\cos(\frac{2\pi l}{b}) + 2\cos(\frac{2\pi m}{b}) + 2\cos(\frac{2\pi(l+m)}{b}) | 0\leq l<b, 0\leq m<b\right\}$$

$$= \left\{8\cos(\frac{\pi l}{b})\cos(\frac{\pi m}{b})\cos(\frac{\pi(l+m)}{b}) - 2 | 0\leq l<b, 0\leq m<b\right\}.$$

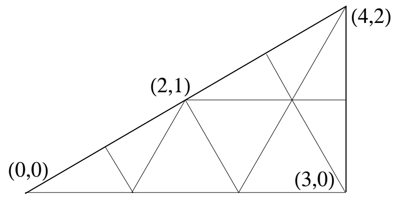

FIGURE 3. Fundamental region for $\{3,6\}_{(6,0)}$

The transformations T_x and T_y originate in translational symmetries of $\{3,6\}$. Figure 2 indicates a convenient choice of the generating "reflections" ρ_0, ρ_1 and ρ_2 of $\mathcal{A}(\{3,6\}_{(4,0)})$. These automorphisms act as follows:

$$\rho_0 : v_{j,k} \mapsto v_{1-k,1-j}, \quad \rho_1 : v_{j,k} \mapsto v_{j,j-k} \quad \text{and} \quad \rho_2 : v_{j,k} \mapsto v_{k,j}.$$

Note that $T_x = \rho_1\rho_0(\rho_1\rho_2)^2$ and $T_y = \rho_2\rho_1\rho_0\rho_1\rho_2\rho_1$. Also,

$$\rho_0(f_{l,m}) = e^{(l+m)\frac{2\pi i}{b}} f_{-m,-l}, \ \rho_1(f_{l,m}) = f_{l+m,-m}, \text{ and } \rho_2(f_{l,m}) = f_{m,l}. \tag{1}$$

A subspace of $\mathbb{C}^{b^2}$ is invariant under the action of $\mathcal{A}(\{3,6\}_{(b,0)})$ exactly when it is invariant under the action of the ρ_i. Since $\rho_0 = \rho_1 T_x(\rho_2\rho_1)^2$ and $T_y = \rho_2^{-1}T_x\rho_2$, it is equivalent to require that the subspace be invariant under the action of T_x, ρ_1 and ρ_2. In particular, the spaces spanned by the orbits of the vectors $f_{l,m}$ under the action of ρ_1 and ρ_2 will be fixed by $\mathcal{A}(\{3,6\}_{(b,0)})$. For arbitrary l and m, this orbit is:

$$\left\{ \begin{array}{cccc} f_{l,m}, & f_{-l,-m}, & f_{l+m,-m}, & f_{-l-m,m}, \\ f_{-l,l+m}, & f_{l,-l-m}, & f_{m,-l-m}, & f_{-m,l+m}, \\ f_{-l-m,l}, & f_{l+m,-l}, & f_{-m,-l}, & f_{m,l} \end{array} \right\}. \tag{2}$$

For each choice of l and m these vectors span a space $V_{l,m}$ on which the representation of $\mathcal{A}(\{3,6\}_{(b,0)})$ is irreducible; we will prove this by the method outlined in Section 2. Note that for some values of l and m, $\dim(V_{l,m}) < 12$.

The Wythoff subspace $W_{l,m}$ of $V_{l,m}$ is spanned by the sum of the vectors of (2) and has dimension $w_{l,m} = 1$. This fact will continue to hold once we have converted to representations over $\mathbb{R}$, so we need only show that the number of different $V_{l,m}$ equals $\overline{w}$ to see that $\sum_G w_G^2 = \overline{w}$. But because all the $W_{l,m}$ are one dimensional, $\overline{w}$ is just the number of diagonal classes of $\{3,6\}_{(b,0)}$ (see [10]). We now describe a bijection between diagonal classes and choices of l and m which yield distinct representation spaces $V_{l,m}$.

To determine the different diagonal classes, fix $v_{0,0}$ and study its relationship to the other $v_{j,k}$. If $(v_{0,0}, v_{j,k})$ is in diagonal class Λ and $v_{j',k'}$ is sent to $v_{j,k}$ by some action of ρ_1 and ρ_2, then $(v_{0,0}, v_{j',k'}) \in \Lambda$. We can classify the different diagonal classes by the vertices in the smallest region that completely covers $\{3,6\}_{(b,0)}$ when acted on by ρ_1 and ρ_2. Such a region is shown in Figure 3.

A bijection between the spaces $V_{l,m}$ and the vertices $v_{j,k}$ shown in this fundamental region is given by $\Omega : f_{l,m} \mapsto v_{l+m,l}$, where the subscripts are interpreted modulo b. Although the roles of ρ_1 and ρ_2 are reversed $(\Omega(\rho_2(f_{l,m})) = \rho_1(\Omega(f_{l,m}))$ and $\Omega(\rho_1(f_{l,m})) = \rho_2(\Omega(f_{l,m})))$, the orbit of $f_{l,m}$ under the action of ρ_1 and ρ_2 will be of the same order as that of $v_{l+m,l} = \Omega(f_{l,m})$. Since a representative vertex from each orbit appears in Figure 3, the number of distinct $V_{l,m}$ is the same as the

TABLE 1. Number and dimension of realizations of $\{3,6\}_{(b,0)}$

$b \mod 6$	Dimension over $\mathbb{R}$					Total
	1	2	3	6	12	
0	1	1	1	$b-3$	$(b^2-6b+12)/12$	$(b^2+6b+12)/12$
1	1	0	0	$b-1$	$(b^2-6b+5)/12$	$(b^2+6b+5)/12$
2	1	0	1	$b-2$	$(b^2-6b+8)/12$	$(b^2+6b+8)/12$
3	1	1	0	$b-2$	$(b^2-6b+9)/12$	$(b^2+6b+9)/12$
4	1	0	1	$b-2$	$(b^2-6b+8)/12$	$(b^2+6b+8)/12$
5	1	0	0	$b-1$	$(b^2-6b+5)/12$	$(b^2+6b+5)/12$

number of vertices appearing in Figure 3. The dimension of $V_{l,m}$ is the order of the orbit of $v_{l+m,l}$; Table 1 summarizes this information.

To this point we have been working with representations of $\mathcal{A}(\{3,6\}_{(b,0)})$ over $\mathbb{C}$. In order to apply the formula $\sum w_G^2 = \overline{w}$, we must consider representations over $\mathbb{R}$. The chief obstacle to this is the fact that ρ_0 acts on the $f_{l,m}$ with complex coefficients. We circumvent this by breaking the action down into its real and imaginary parts. Notice that the coefficient of $\rho_0(f_{l,m})$ is the complex conjugate of the coefficient of $\rho_0(f_{-l,-m})$; in particular, $\rho_0(f_{l,m}+f_{-l,-m}) = a(f_{m,l}+f_{-m,-l}) - b(if_{m,l} - if_{-m,-l})$ for some $a,b \in \mathbb{R}$. If we change basis to:

$$\left\{ \begin{array}{cc} f_{l,m}+f_{-l,-m}, & if_{l,m}-if_{-l,-m}, \\ f_{l+m,-m}+f_{-l-m,m}, & if_{l+m,-m}-if_{-l-m,m}, \\ f_{-l,l+m}+f_{l,-l-m}, & if_{-l,l+m}-if_{l,-l-m}, \\ f_{m,-l-m}+f_{-m,l+m}, & if_{m,-l-m}-if_{-m,l+m}, \\ f_{-l-m,l}+f_{l+m,-l}, & if_{-l-m,l}-if_{l+m,-l}, \\ f_{m,l}+f_{-m,-l}, & if_{m,l}-if_{-m,-l} \end{array} \right\}, \tag{3}$$

the ρ_i act on the basis vectors with real coefficients. Each of our original complex representations breaks down into equivalent "real" and "imaginary" representations of $\mathcal{A}(\{3,6\}_{(b,0)})$ over $\mathbb{R}$. Because the pairs are equivalent, we know that those corresponding to inequivalent $V_{l,m}$ are also inequivalent. The dimension of the Wythoff space of each of these real representations is still one. Hence this representation corresponds to a unique realization of $\{3,6\}_{(b,0)}$ which we shall call $R_{l,m}$.

We may now use the results in Table 1 to confirm that $\sum_G w_G^2 = \overline{w}$ and $\sum_G w_G d_G = b^2$, and so conclude that the $R_{l,m}$ enumerated by the vertices shown in Figure 3 are all of the irreducible orthogonal realizations of $\{3,6\}_{(b,0)}$ up to scaling. Furthermore, we can express the action of T_x and T_y on the basis described above, which allows us to calculate locations of the vertices of the $R_{l,m}$.

Given a block diagonal matrix M with matrices $\{M_1, M_2, ..., M_k\}$ on its diagonal, we write $M = (M_1|M_2|...|M_k)$. If we let $R(\theta)$ be the rotation matrix $\begin{pmatrix} cos(\theta) & -sin(\theta) \\ sin(\theta) & cos(\theta) \end{pmatrix}$ and identify T_x and T_y with the matrices describing their action, we can concisely describe the irreducible realizations of $\{3,6\}_{(b,0)}$ as follows:

THEOREM 2. *The vertices of a generic irreducible realization $R_{l,m}$ of $\{3,6\}_{(b,0)}$ are given by the orbit of the point $p = (1,0,1,0,1,0,1,0,1,0,1,0)$ under the action of the matrices:*

$$T_x = (R(\phi)|R(\phi+\psi)|R(-\phi)|R(\psi)|R(-\phi-\psi)|R(\psi)) \text{ and}$$
$$T_y = (R(\psi)|R(-\psi)|R(\phi+\psi)|R(-\phi-\psi)|R(\phi)|R(\phi)),$$

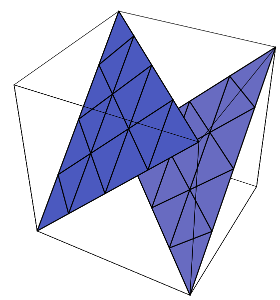

FIGURE 4. The faces of $\{3,6\}_{(4,0)}$ embedded in a cube

where $\phi = -2\pi l/b$ *and* $\psi = -2\pi m/b$. *Here* $\beta(v_{j,k}) = T_x^j T_y^k(p)$.

The matrices of the ρ_i are easily calculated from (1) and (3). In the generic case, each ρ_i acts by fixing some six dimensional subspace and reflecting its complement through the origin.

Contemplating the descriptions of T_x and T_y, we see that the vertices of $R_{l,m}$ lie on the surface of a 6-torus in $\mathbb{R}^{12}$. Since $\{3,6\}_{(b,0)}$ is topologically a 2-torus, they lie on an embedding of T^2 in T^6. Although this twelve dimensional figure is difficult to visualize, we can describe the lower dimensional realizations of $\{3,6\}_{(b,0)}$.

The realizations $R_{l,0}$ ($l \neq 0, b/2$) are six dimensional, with $T_x = (R(\phi)|R(-\phi)|I_2)$ and $T_y = (I_2|R(\phi)|R(-\phi))$. They can be visualized as follows. Consider the action of the parametric equation

$$P : (x,y,z) \to (\sin(x), \cos(x), \sin(y), \cos(y), \sin(z), \cos(z))$$

on the cube of edge length 2π shown in Figure 4; under this action, opposite faces of the cube shown there are identified to form a 3-torus. Opposite edges of the two shaded triangles shown in Figure 4 are identified to form a 2-torus embedded in that 3-torus. The shaded triangles can be subdivided into b^2 smaller triangles; in the example shown, $b = 4$. The images of the vertices, edges and faces of this subdivision under the action of P correspond to those of the realization $R_{1,0}$ of $\{3,6\}_{b,0}$. (See also [2].)

4. ABSTRACT POLYHEDRA OF TYPE $\{3,6\}_{(b,b)}$

Identifying opposite sides of the rhombus shown in Figure 5 describes the abstract regular polyhedron $\{3,6\}_{(2,2)}$, which has 3×2^2 vertices and 6×2^2 faces. In general, $\{3,6\}_{(b,b)}$ is obtained by identifying points of $\{3,6\}$ that are equivalent under the the action of the symmetry group $< T_x^{3b}, T_x^{3b}, (T_x^2 T_y)^b >$, where T_x and T_y transform $\{3,6\}$ as was described in Section 2 (see [5]).

There is a natural map from $\{3,6\}_{(b,b)}$ to $\{3,6\}_{(b,0)}$ given by identifying $T_x^b = T_y^b = 1$. In other words, $\{3,6\}_{(b,b)}$ *collapses* onto $\{3,6\}_{(b,0)}$. So, every realization of $\{3,6\}_{(b,0)}$ gives rise to a realization of $\{3,6\}_{(b,b)}$. Also, $\{3,6\}_{(3b,0)}$ collapses onto $\{3,6\}_{(b,b)}$. For example, the six dimensional realizations of $\{3,6\}_{(b,b)}$ correspond to the realizations $R_{3l,0}$ and $R_{l,l}$ described in Section 3.

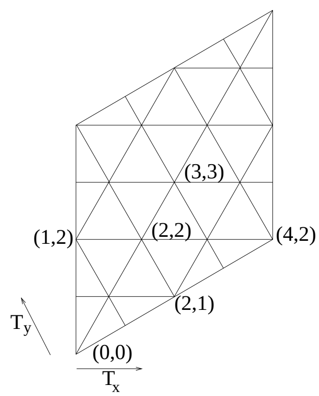

FIGURE 5. Net for $\{3,6\}_{(2,2)}$

TABLE 2. Number and dimension of realizations of $\{3,6\}_{(b,b)}$

b	Dimension over $\mathbb{R}$					Total
	1	2	3	6	12	
even	1	1	1	$2b-3$	$(3b^2-12b+12)/12$	$(3b^2+12b+12)/12$
odd	1	1	0	$2b-2$	$(3b^2-12b+9)/12$	$(3b^2+12b+9)/12$

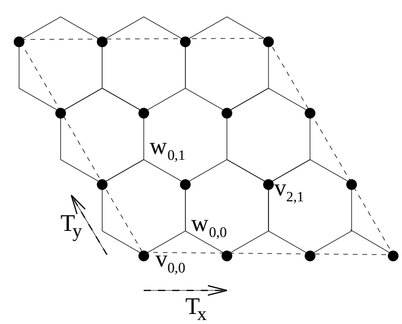

FIGURE 6. A Net for $\{6,3\}_{(3,0)}$

The chief difference between these two families of abstract polyhedra is the existence of a group element $T_x^2T_y$ with order lower than that of T_x and T_y. If we construct a set of simultaneous eigenvectors of T_x and $T_x^2T_y$ analogous to the $\{f_{l,m}\}$ of the previous section, repeating the calculations of that section yields the enumeration of irreducible realizations presented in Table 2.

5. ABSTRACT POLYHEDRA OF TYPE $\{6,3\}_{(b,0)}$

The combinatorial dual of the abstract polyhedron $\{3,6\}_{(b,0)}$ is $\{6,3\}_{(b,0)}$ (see [5]); identifying opposite sides of the dotted rhombus shown in Figure 6 yields the polyhedron $\{6,3\}_{(3,0)}$. Since $\{6,3\}_{(b,0)}$ has $2b^2$ vertices, the simplex representation of its automorphism group is over the space $\mathbb{C}^{2b^2}$

The generating reflections of $\mathcal{A}(\{6,3\}_{b,0})$ are indicated in Figure 7. Once again the polyhedron has "translational" symmetries T_x and T_y which are given by $T_x = \rho_0\rho_1\rho_2\rho_1\rho_0\rho_1$ and $T_y = \rho_1\rho_0\rho_1\rho_0\rho_1\rho_2$. It is evident from Figure 6 that the edge

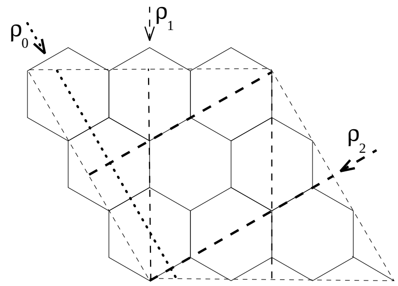

FIGURE 7. Generators of $\mathcal{A}(\{6,3\}_{(3,0)})$

graph of $\{6,3\}_{(b,0)}$ is bipartite. The symmetries ρ_2, T_x and T_y permute the vertices labeled $\{v_{j,k}\}_{0\le j,k<b}$ in exactly the same way that ρ_1, T_x and T_y permuted the vertices of $\{3,6\}_{(b,0)}$ in Section 3. The vertices $\{w_{j,k}\}_{0\le j,k<b}$ of the other half of the graph form a separate orbit under the action of T_x and T_y.

The simultaneous eigenvectors of the action of T_x and T_y on $\mathbb{C}^{2b^2}$ are $\{f_{l,m}, g_{l,m}\}$ where $0 \le l, m < b$ and $f_{l,m}$ and $g_{l,m}$ are defined as follows:

$$f_{l,m} = \textstyle\sum_{0\le j,k<b} e^{(lj+mk)\frac{2\pi i}{b}} v_{j,k}, \quad g_{l,m} = \sum_{0\le j,k<b} e^{(lj+mk)\frac{2\pi i}{b}} w_{j,k}.$$

It is not difficult to check that:

$$T_x(f_{l,m}) = e^{l\frac{2\pi i}{b}} f_{l,m}, T_y(f_{l,m}) = e^{m\frac{2\pi i}{b}} f_{l,m},$$
$$T_x(g_{l,m}) = e^{l\frac{2\pi i}{b}} g_{l,m}, \text{ and } T_y(g_{l,m}) = e^{m\frac{2\pi i}{b}} g_{l,m}.$$

Since T_x and T_y no longer send vertices to their neighbors, we cannot directly compute the eigenvalues of A_1. However, we can compute the eigenvalues of the matrix $A_2 = T_x + T_xT_y + T_y + T_x^{-1} + (T_xT_y)^{-1} + T_y^{-1}$; the computation is identical to that of Theorem 1. A simple calculation verifies that $A_1^2 = 6I + A_2$, leading us to conclude:

THEOREM 3. *The multiset of eigenvalues of the adjacency matrix of* $\{6,3\}_{(b,0)}$ *is:*

$$\left\{ \pm\sqrt{8\cos(\frac{\pi l}{b})\cos(\frac{\pi m}{b})\cos(\frac{\pi(l+m)}{b}) + 4} \;\middle|\; 0 \le l, m < b \right\}.$$

The generators ρ_i of the automorphism group of $\{6,3\}_{(b,0)}$ permute the eigenvectors $f_{l,m}$ and $g_{l,m}$ as follows:

$$\begin{aligned} \rho_0(f_{l,m}) &= g_{-l-m,m}, & \rho_0(g_{l,m}) &= f_{-l-m,m}, \\ \rho_1(f_{l,m}) &= f_{-l,l+m}, & \rho_1(g_{l,m}) &= e^{-\frac{2\pi i l}{b}} g_{-l,l+m}, \\ \rho_2(f_{l,m}) &= f_{l+m,-m}, & \rho_2(g_{l,m}) &= g_{l+m,-m}. \end{aligned}$$

Proceeding as in Section 3, we see that the following vectors form a basis for a representation $V_{l,m}$ of $\mathcal{A}(\{6,3\}_{(b,0)})$:

$$\left\{ \begin{array}{cccc} f_{l,m}, & g_{-l,-m}, & f_{l+m,-m}, & g_{-l-m,m}, \\ f_{-l,l+m}, & g_{l,-l-m}, & f_{m,-l-m}, & g_{-m,l+m}, \\ f_{-l-m,l}, & g_{l+m,-l}, & f_{-m,-l}, & g_{m,l} \end{array} \right\}.$$

In general, the Wythoff spaces of these representations are two-dimensional. Note that if we replace each $g_{p,q}$ with $f_{p,q}$ we get exactly the basis we had for the representation spaces $V_{l,m}$ of $\mathcal{A}(\{3,6\}_{(b,0)})$. This will help us to enumerate these representations.

It is no longer true that the different representations $V_{l,m}$ are inequivalent; in some cases the spaces $V_{l,m}$ and $V_{-l,-m}$ are distinct vector spaces that are equivalent representations. The equivalence is given by defining $T : f_{l,m} \mapsto g_{l,m}; g_{-l,-m} \mapsto f_{-l,-m}$ in the following commutative diagram:

$$\begin{array}{ccc} & T & \\ V_{l,m} & \longrightarrow & V_{l',m'} \\ g\downarrow & & g\downarrow \\ V_{l,m} & \longrightarrow & V_{l',m'} \\ & T & \end{array} \tag{4}$$

We will show that any multiplicity is at most two, and classify which spaces $V_{l,m}$ are equivalent. Suppose $V_{l,m}$ and $V_{l',m'}$ are distinct but equivalent representation spaces of $\mathcal{A}(\{6,3\}_{(b,0)})$. Then there exists a non-singular linear transformation $T : V_{l,m} \to V_{l',m'}$ such that (4) commutes for all $g \in \mathcal{A}(\{6,3\}_{(b,0)})$. In particular, the diagram holds when g is replaced by T_x or T_y. So,

$$T_xT(f_{l,m}) = TT_x(f_{l,m}) = e^{\frac{2\pi i}{b}l}T(f_{l,m});$$

similarly $T_yT(f_{l,m}) = e^{\frac{2\pi i}{b}m}T(f_{l,m})$. Hence, $V_{l',m'}$ contains a vector $T(f_{l,m})$ which is a simultaneous eigenvector of T_x and T_y on $V_{l',m'}$ with eigenvalues $e^{\frac{2\pi i}{b}l}$ and $e^{\frac{2\pi i}{b}m}$, respectively. But this implies that $T(f_{l,m})$ must be a multiple of either $f_{l,m}$ or $g_{l,m}$. Either $V_{l,m} = V_{l',m'}$ or $T(f_{l,m}) = g_{l,m}$, implying $V_{l',m'} = V_{-l,-m}$. In other words, when $V_{l,m} = V_{-l,-m}$ the representation occurs with multiplicity one in the simplex representation of $\mathcal{A}(\{3,6\}_{(b,0)})$.

The representation spaces $V_{l,m}$ fall into the following three categories:

1. $V_{l,m} \neq V_{-l,-m}$ has Wythoff dimension two and multiplicity two in the simplex representation,
2. $V_{b/3,b/3}$ has Wythoff dimension one and multiplicity one, and
3. $V_{l,m} = V_{-l,-m} = V_{k,k}$ has Wythoff dimension two and multiplicity one.

When $V_{l,m} \neq V_{-l,-m}$, the basis:

$$\left\{ \begin{array}{cc} f_{l,m} + e^{-2\pi im/b}g_{-l,-m}, & if_{l,m} - ie^{-2\pi im/b}g_{-l,-m}, \\ f_{l+m,-m} + e^{-2\pi im/b}g_{-l-m,m}, & if_{l+m,-m} - ie^{-2\pi im/b}g_{-l-m,m}, \\ f_{-l,l+m} + e^{2\pi i(l-m)/b}g_{l,-l-m}, & if_{-l,l+m} - ie^{2\pi i(l-m)/b}g_{l,-l-m}, \\ f_{m,-l-m} + e^{2\pi i(l-m)/b}g_{-m,l+m}, & if_{m,-l-m} - ie^{2\pi i(l-m)/b}g_{-m,l+m}, \\ f_{-l-m,l} + e^{2\pi il/b}g_{l+m,-l}, & if_{-l-m,l} - ie^{2\pi il/b}g_{l+m,-l}, \\ f_{-m,-l} + e^{2\pi il/b}g_{m,l}, & if_{-m,-l} - ie^{2\pi il/b}g_{m,l} \end{array} \right\}$$

provides a representation of $\mathcal{A}(\{6,3\}_{(b,0)})$ in which the ρ_i act with coefficients in $\mathbb{R}$. When $l = 0$, $m = 0$, or $l+m = 0$, this representation has dimension six. Otherwise, its dimension is twelve. In either case, the Wythoff space is two dimensional; its basis is obtained by summing the elements of each of the two orbits of the basis vectors under the action of $< \rho_1, \rho_2 >$

The space $V_{b/3,b/3} = V_{-b/3,-b/3}$ corresponds to an irreducible four dimensional realization of $\mathcal{A}(\{6,3\}_{(b,0)})$. Its Wythoff space is one dimensional and is spanned by $f_{b/3,b/3} + f_{-b/3,-b/3}$. The ρ_i act with real coefficients on the elements of the

TABLE 3. Number and dimension of irreducible representations of $\mathcal{A}(\{6,3\}_{(b,0)})$, counted without multiplicity

$b \mod 6$	Dimension over $\mathbb{R}$						Total
	$w_G = 1$				$w_G = 2$		
	1	3	4	6	6	12	
0	2	2	1	$b-4$	$(b-2)/2$	$(b^2-6b+12)/12$	$(b^2+12b+6)/12$
1	2	0	0	$b-1$	$(b-1)/2$	$(b^2-6b+5)/12$	$(b^2+12b+11)/12$
2	2	2	0	$b-2$	$(b-2)/2$	$(b^2-6b+8)/12$	$(b^2+12b+20)/12$
3	2	0	1	$b-3$	$(b-1)/2$	$(b^2-6b+9)/12$	$(b^2+12b+3)/12$
4	2	2	0	$b-2$	$(b-2)/2$	$(b^2-6b+8)/12$	$(b^2+12b+20)/12$
5	2	0	0	$b-1$	$(b-1)/2$	$(b^2-6b+5)/12$	$(b^2+12b+11)/12$

basis:

$$\left\{ \begin{array}{ll} f_{b/3,b/3} + f_{-b/3,-b/3}, & if_{b/3,b/3} - if_{-b/3,-b/3}, \\ g_{b/3,b/3} + g_{-b/3,-b/3}, & ig_{b/3,b/3} - ig_{-b/3,-b/3} \end{array} \right\}.$$

The remaining representations $V_{l,l}$ have Wythoff dimension two, but are not irreducible as representations of $\mathcal{A}(\{6,3\}_{(b,0)})$. The representation decomposes: $V_{l,l} = W_{l,l} \oplus W'_{l,l}$, where $W_{l,l}$ and $W'_{l,l}$ are inequivalent irreducible representation spaces with Wythoff dimension one. If $\zeta = e^{-2\pi i l/b}$, the ρ_i act on the following basis of $W_{l,l}$ with coefficients in $\mathbb{R}$.

$$\left\{ \begin{array}{cc} f_{l,l} + \zeta g_{-l,-l} + f_{-l,-l} + \bar{\zeta} g_{l,l}, & if_{l,l} - i\zeta g_{-l,-l} - if_{-l,-l} + i\bar{\zeta} g_{l,l}, \\ f_{2l,-l} + \zeta g_{-2l,l} + f_{-2l,l} + \bar{\zeta} g_{2l,-l}, & if_{2l,-l} - i\zeta g_{-2l,l} - if_{-2l,l} + i\bar{\zeta} g_{2l,-l}, \\ f_{-l,2l} + g_{l,-2l} + f_{l,-2l} + g_{-l,2l}, & if_{-l,2l} - ig_{l,-2l} - if_{l,-2l} + ig_{-l,2l} \end{array} \right\}$$

The basis for $W'_{l,l}$ is similar. Note that $W_{0,0}$ is one dimensional, and $W_{b/2,b/2}$ is three dimensional.

Recall the surjection from bases of the $V_{l,m}$ to bases of the irreducible representations of $\mathcal{A}(\{3,6\}_{(b,0)})$ given by replacing the $g_{l,m}$ with $f_{l,m}$. We will use this to enumerate the irreducible representations of $\mathcal{A}(\{6,3\}_{(b,0)})$; our conclusions are presented in Table 3.

The two dimensional realization of $\{3,6\}_{(b,0)}$ corresponds to $V_{b/3,b/3}$. The one, three and six-dimensional realizations $R_{l,l}$ of $\{3,6\}_{(b,0)}$ correspond to the inequivalent pairs of one, three, and six-dimensional irreducible representations of $\mathcal{A}(\{6,3\}_{(b,0)})$ for which $V_{l,m} = V_{-l,-m} \neq V_{b/3,b/3}$. The remaining realizations of $\{3,6\}_{(b,0)}$ correspond to equivalent pairs $V_{l,m} \cong V_{-l,-m}$ of six- or twelve-dimensional irreducible representations with Wythoff dimension two.

The representations $V_{l,m} = V_{-l,-m}$ have one dimensional Wythoff spaces and are clearly irreducible. The fact that $\sum w_G d_G = 2b^2$ confirms that the realizations $V_{l,m} \neq V_{-l,-m}$ are irreducible as well. Table 3 tallies the irreducible components of the simplex representation of $\mathcal{A}(\{6,3\}_{(b,0)})$.

When the Wythoff space of a representation is two dimensional, there are uncountably many different realizations associated with the same representation [10], so it is impossible to enumerate the irreducible realizations of $\{6,3\}_{(b,0)}$. However, by taking linear combinations of realizations generated by applying Wythoff's construction to the representations described above, we can construct any realization of $\{6,3\}_{(b,0)}$.

TABLE 4. Number and dimension of irreducible representations of $\mathcal{A}(\{6,3\}_{(b,b)})$, counted without multiplicity

b	Dimension over $\mathbb{R}$						Total
	$w_G = 1$				$w_G = 2$		
	1	3	4	6	6	12	
even	2	2	1	$3b-4$	$(b-2)/2$	$(3b^2-12b+12)/12$	$(3b^2+30b+12)/12$
odd	2	0	1	$3b-3$	$(b-1)/2$	$(3b^2-12b+9)/12$	$(3b^2+30b+3)/12$

6. Realizations of $\{6,3\}_{(b,b)}$

The regular abstract polyhedron $\{6,3\}_{(b,b)}$ is the dual of $\{3,6\}_{(b,b)}$, with $6b^2$ vertices. The techniques used to determine its irreducible realizations are a combination of those presented in the previous sections. In particular, some representations of $\mathcal{A}(\{3,6\}_{(b,b)})$ have Wythoff spaces of dimension two. The distinct irreducible representations of $\mathcal{A}(\{3,6\}_{(b,b)})$ with non-trivial Wythoff spaces are enumerated in Table 4.

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