q-Taylor theorems, polynomial expansions, and interpolation of entire functions *

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Abstract

We establish q-analogues of Taylor series expansions in special polynomial bases for functions where $\ln M(r; f)$ grows like $\ln^2 r$. This solves the problem of constructing such entire functions from their values at $[aq^n + q^{-n}/a]/2$, for $0 < q < 1$. Our technique is constructive and gives an explicit representation of the sought entire function. Applications to q-series identities are given.

Running Title: q-interpolation

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1 Introduction

Two important problems in complex function theory are the problems of expanding a function in a series of polynomials and the interpolation problem of finding an entire function from its values on a given sequence $\{x_n\}$, $x_n \to \infty$ as $n \to \infty$. The polynomial expansion problem has a long history. J. M. Whittaker [17], [18] introduced the concept of basic sets of polynomials where the polynomials are ordered but not necessarily according to

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their degrees and all degrees are present. A more recent treatment of Whittaker's approach is in the interesting monograph [14] by B. H. Makar. Boas and Buck [4] restricted the polynomials to having a generating function of a special type which guarantees that $p_n(x)$ has precise degree n, $n = 0, 1, \cdots$. As a result of the specialization imposed by Boas and Buck, they have been able to obtain more refined results than those which hold for general basic sets of polynomials.

In this paper we solve the interpolation problem for the sequence $\{x_{2n}\}$,

$$x_n = [aq^{n/2} + q^{-n/2}/a]/2, \quad 0 < q < 1, \ 0 < a < 1, \tag{1.1}$$

for entire functions f satisfying

$$\limsup_{r \to +\infty} \frac{\ln M(r; f)}{\ln^2 r} = c, \tag{1.2}$$

for a particular c which depends upon q. Here $M(r; f)$ is [3]

$$M(r; f) = \sup\{|f(z)| : |z| \le r\}. \tag{1.3}$$

In the process of solving this problem we also solve the expansion problem of entire functions in two specific bases of polynomials, namely $\{\phi_n(x; a)\}$ and $\{\rho_n(x)\}$ defined in (1.9)–(1.10) and the coefficients in the expansion on $\{\phi_n(x; a)\}$ involve function evaluations at $\{x_{2n}\}$. In the case of $\{\rho_n(x)\}$, the interpolation points are

$$u_n = i(q^{n/2} - q^{-n/2})/2, \quad n = \cdots, -1, 0, 1, \cdots. \tag{1.4}$$

Carlson's theorem [3] states that an entire function f of order one and type less than π is uniquely determined by the sequence $\{f(n) : n = 0, 1, \ldots\}$. Moreover if $f(x)$ is entire of order 1 and type $< \ln 2$ then

$$f(x) = \sum_{n=0}^{\infty} \binom{x}{n} (\Delta^n f)(0), \tag{1.5}$$

and the series converges uniformly on compact subsets of the complex x-plane, [3, Theorem 9.10.7]. In the above $(\Delta f)(x) = f(x+1) - f(x)$. Another representation was obtained by Ramanujan in his first notebooks, where he wrote

$$\int_0^{\infty} x^{s-1} \sum_{k=0}^{\infty} f(k)(-x)^k dx = \frac{\pi}{\sin \pi s} f(-s). \tag{1.6}$$

Hardy [8, (11.2A), p. 186] proved (1.6) by contour integration and pointed out that it holds under the assumptions in Carlson's theorem. Therefore Ramanujan's formula (1.6) provides a constructive proof of Carlson's theorem by showing how to construct the function f from $\{f(n)\}$. Therefore, in some sense, our formulas are closer in spirit to (1.5). An interesting question is to find the analogue of Ramanujan's formula (1.6).

Ramis [15] defined an entire function f to have a q-exponential growth of order k and a finite type if there exist real numbers K, α, $K > 0$, such that

$$|f(x)| < K|x|^{\alpha} \exp\left(\frac{k \ln^2 |x|}{2 \ln^2 q}\right).$$

Thus functions satisfying (1.2) are of q-exponential growth of order $2c \ln^2 q$.

Two of our main results are Theorems 3.1 and 3.3 which are stated and proved in §3. Before we can state our results we need to explain the notation used, which is mainly from [1] and [7]. The q-shifted factorials are

$$(1.7)\quad (a;q)_0 := 1, \quad (a;q)_n := \prod_{k=1}^{n} (1 - aq^{k-1}), \quad n = 1, 2, \ldots, \text{ or } \infty,$$

while the multiple q-shifted factorials are defined by

$$(1.8)\qquad (a_1, a_2, \ldots, a_k; q)_n := \prod_{j=1}^{k} (a_j; q)_n.$$

The bases of polynomials we are interested in are defined by

$$(1.9)\qquad \phi_n(\cos\theta; a) = (ae^{i\theta}, ae^{-i\theta}; q)_n = \prod_{k=0}^{n-1} [1 - 2axq^k + a^2 q^{2k}],$$

$$(1.10)\qquad \rho_n(\cos\theta) = (1 + e^{2i\theta})(-q^{2-n} e^{2i\theta}; q^2)_{n-1} e^{-in\theta}.$$

The motivation for considering these special bases is our desire to establish Taylor-like series where the Askey-Wilson operator plays the role of $\frac{d}{dx}$ and these polynomials play the role of monomials. The basis $\{\phi_n(x;a)\}$ was introduced in the Askey-Wilson memoir [2] but the basis $\{\rho_n(x)\}$ does not seem to have been considered before we inroduced them in [11].

We now define the Askey-Wilson operator $\mathcal{D}_q$ introduced in [2]. Given a function f we set $\breve{f}(e^{i\theta}) := f(x)$, $x = \cos\theta$, that is

$$(1.11)\qquad \breve{f}(z) = f((z + 1/z)/2), \quad z = e^{i\theta}.$$

In other words we think of $f(\cos\theta)$ as a function of $e^{i\theta}$. In this notation the Askey-Wilson divided difference operator $\mathcal{D}_q$ is defined by

$$(1.12)\qquad (\mathcal{D}_q f)(x) = \frac{\breve{f}(q^{1/2}e^{i\theta}) - \breve{f}(q^{-1/2}e^{i\theta})}{(q^{1/2} - q^{-1/2})\ i\ \sin\theta}, \quad x = \cos\theta.$$

For example with $f(x) = 4x^3 - 3x$, $\breve{f}(z) = [z^3 + z^{-3}]/2$, and

$$\mathcal{D}_q f(x) = \frac{q^{3/2} - q^{-3/2}}{q^{1/2} - q^{-1/2}}\,(4x^2 - 1).$$

It is a fact that $\mathcal{D}_q$ reduces the degree of a polynomial by one and

$$(1.13)\qquad \lim_{q\to 1} \mathcal{D}_q f(x) = \frac{d}{dx} f(x),$$

at the points where f is differentiable. Furthermore in the calculus of the Askey-Wilson operator the basis $\{\phi_n(x;a) : n \geq 0\}$ plays the role played by the monomials $\{(1 - 2ax + a^2)^n : n \geq 0\}$ in the differential and integral calculus.

Note that although we use $x = \cos\theta$, θ is not necessarily real but $e^{\pm i\theta}$ are defined as

$$e^{\pm i\theta} = x \pm \sqrt{x^2 - 1},$$

and the branch of the square root is taken such that $\sqrt{x^2-1} \approx x$ as $x \to \infty$. This makes $|e^{-i\theta}| \leq |e^{i\theta}|$, with equality if and only if $x \in [-1, 1]$.

The action of $\mathcal{D}_q$ on the bases in (1.9)–(1.11) is given by

$$(1.14)\qquad \mathcal{D}_q \phi_n(x;a) = -\frac{2a(1-q^n)}{1-q}\phi_{n-1}(x; aq^{1/2}),$$

$$(1.15)\qquad \mathcal{D}_q \rho_n(x) = 2q^{(1-n)/2}\frac{1-q^n}{1-q}\,\rho_{n-1}(x).$$

As already mentioned the values of an entire function f on the nonnegative integers determine f, [3], when f is of order one and type less than π. On the other hand $f(z) = \sin\pi z$ is order 1 and type π and vanishes at all the integers, so type π is a cut off point. A similar situation occurs for the interpolation points $\{x_{2k} : k = 0, 1, \ldots\}$. The function $\phi_\infty(x;a)$ vanishes at all the points x_{2k}, so if $\{f(x_{2k}) : k \geq 0\}$ determine an entire function f uniquely then f is expected to grow slower that $\phi_\infty(x;a)$. It turned out that when c in (1.2) is $< 1/(2\ln q^{-1})$ then f can be interpolated from $\{f(x_{2k})\}$

and f has a polynomial expansion in $\{\phi_n(x;a)\}$. For $f(x) = \phi_\infty(x;a)$, $c = 1/(2\ln q^{-1})$, so the barrier, which corresponds to type π, is $1/(2\ln q^{-1})$. This will be proved in §3.

One purpose of this work is to extend the following theorem from polynomials to entire functions. Theorem 1.1 combines results from [10] and [11].

Theorem 1.1. *Let $f(x)$ be a polynomial and assume that x_n is defined by (1.1). Then*

$$f(x) = \sum_{k=0}^{\infty} f_k \phi_k(x;a),$$

with

$$f_k = \frac{(q-1)^k}{(2a)^k (q;q)_k} q^{-k(k-1)/4} (\mathcal{D}_q^k f)(x_k).$$

In addition we have

$$f(x) = \sum_{k=0}^{\infty} f_k \rho_k(x),$$

where

$$f_k = \frac{q^{(k^2-k)/4}(1-q)^k}{2^k (q;q)_k} (\mathcal{D}_q^k f)(0).$$

The idea in our extension of Theorem 1.1 to entire functions f is to first expand $1/(y-x)$ in $\{\phi_n(x;a)\}$ and $\{\rho_n(x)\}$ then use Cauchy's theorem to expand entire functions in the same bases. The expansion of the Cauchy kernel is in Theorem 2.1. The expansion of entire functions is established in §3.

It is interesting to note that although none of the series in Theorem 1.1 converge to $f(x)$ when $f(x) = 1/(y-x)$, the expansions with an additional term can be used via Cauchy's theorem to expand slow growing entire functions in the bases under consideration.

In Section 4 we rewrite Theorem 3.1 in the form of a Mittag-Leffler expansion, see (4.2). This expansion turns out to be very useful in studying summation theorems for basic hypergeometric series. Some examples are give in Section 5. Section 6 contains concluding remarks and the evaluation of c in (1.2) for the q-exponential function $\mathcal{E}_q(z;\alpha)$. We have only included few examples of the implications of the material derived here and we avoided including technical special functions results, which will appear in a more specialized publication.

One interesting byproduct of our results is the following version of a formula of Cooper [5]

$$\begin{aligned}(1.16)\quad \mathcal{D}_q^n f(x) \\ &= \frac{2^n q^{n(1-n)/4}}{(q^{1/2}-q^{-1/2})^n} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{q^{k(n-k)} z^{2k-n} \breve{f}(q^{(n-2k)/2} z)}{(q^{1+n-2k} z^2; q)_k (q^{2k-n+1} z^{-2}; q)_{n-k}},\end{aligned}$$

where $z = e^{i\theta}$, $x = \cos\theta$, and

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}}, \tag{1.17}$$

is the q-analogue of the binomial coefficient.

The key tool used to establish the expansion of the Cauchy kernel is the theory of basic hypergeometric functions. An interesting open problem is to find a purely complex variable proof of this expansion or of the expansions of entire functions. For convenience we include the definition of a basic hypergeometric series

$$\begin{aligned}{}_r\phi_s\left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix} \,\middle|\, q,\ z \right) &= {}_r\phi_s(a_1, \dots, a_r; b_1, \dots, b_s; q, z) \\ &= \sum_{n=0}^{\infty} \frac{(a_1, \dots, a_r; q)_n}{(q, b_1, \dots, b_s; q)_n} z^n \left(-q^{(n-1)/2}\right)^{n(s+1-r)}.\end{aligned} \tag{1.18}$$

Finally we use the Bailey notation

$$\begin{aligned}&W(a^2; a_1, \cdots, a_r; q, z) \\ &:= {}_{r+3}\phi_{r+2}\left(\begin{matrix} a^2, qa, -qa, a_1, \dots, a_r \\ a, -a, qa^2/a_1, \dots, qa^2/a_r \end{matrix} \,\middle|\, q,\ z \right).\end{aligned} \tag{1.19}$$

The ϕ function in (1.19) is called very well-poised.

2 Expansions of the Cauchy kernel

In this section we expand the Cauchy kernel $1/(x-y)$ in terms of $\{\phi_n(x; a)\}$ and $\{\rho_n(x)\}$. This is done in Theorem 2.1. The Cauchy kernel expansion is then used in Theorems 2.2 and 2.3 to expand entire functions in the same bases with coefficients represented by contour integrals. These integral representations are analogues of the Cauchy formulas.

Let

$$\psi_n(x;a) := 1/(ae^{i\theta}, ae^{-i\theta}; q)_n. \tag{2.1}$$

It is easy to prove that

$$\mathcal{D}_q \psi_n(x;a) = \frac{2a(1-q^n)}{1-q} \psi_{n+1}(x; aq^{-1/2}). \tag{2.2}$$

It is clear that

$$\frac{1}{\cos\phi - \cos\theta} = 2e^{i\phi}\psi_1(\cos\theta; e^{i\phi}). \tag{2.3}$$

Theorem 2.1. *The Cauchy kernel* $1/(x-y)$ *has the expansion*

$$\frac{1}{y-x} = \frac{1}{y-x}\frac{\phi_\infty(x;a)}{\phi_\infty(y;a)} - 2a\sum_{n=0}^{\infty} \frac{\phi_n(x;a)}{\phi_{n+1}(y;a)} q^n,$$

for all y *such that* $y \neq x$, *and* $\phi_\infty(y;a) \neq 0$. *The above expansion also holds if* $y = y_0$, $\phi_\infty(y_0;a) = 0$, *but* $x \neq y_0$ *in the sense that the left-hand side at* $y = y_0$ *equals the limit of the right-hand side as* $y \to y_0$. *Moreover the expansion of the Cauchy kernel in* $\{\rho_n\}$ *is*

$$\begin{aligned}\frac{1}{y-x} \quad &= \quad \frac{x}{y^2-x^2}\frac{(-qe^{2i\theta}, -qe^{-2i\theta}; q^2)_\infty}{(-qe^{2i\phi}, -qe^{-2i\phi}; q^2)_\infty} + \frac{y}{y^2-x^2}\frac{(-e^{2i\theta}, -e^{-2i\theta}; q^2)_\infty}{(-e^{2i\phi}, -e^{-2i\phi}; q^2)_\infty} \\ &\quad + 4\sum_{n=0}^{\infty} \frac{y\rho_n(x)}{[(1-q^n)^2 + 4y^2q^n]\rho_n(y)} q^n,\end{aligned}$$

provided that $y \neq x$ *and* $\rho_n(y) \neq 0$ *for all* n, $n = 0, 1, \ldots$.

First note that $\rho_n(x)/\rho_n(y)$ is uniformly bounded if y is not a zero of $(-e^{2i\phi}, -e^{-2i\phi}; q)_\infty$, since we have

$$\begin{aligned}\lim_{N\to\infty} \frac{\rho_{2N}(x)}{\rho_{2N}(y)} &= \frac{(-e^{2i\theta}, -e^{-2i\theta}; q^2)_\infty}{(-e^{2i\phi}, -e^{-2i\phi}; q^2)_\infty} \\ \lim_{N\to\infty} \frac{\rho_{2N+1}(x)}{\rho_{2N+1}(y)} &= \frac{x(-qe^{2i\theta}, -qe^{-2i\theta}; q^2)_\infty}{y(-qe^{2i\phi}, -qe^{-2i\phi}; q^2)_\infty}.\end{aligned}$$

Hence the second series in Theorem 2.1 converges absolutely and uniformly for x and y in compact sets. Moreover Theorem 2.1 provides a motivation to expand the Cauchy kernel in $\{\phi_n(x;a)\}$ and $\{\rho_n(x)\}$. The respective expansions may not converge to the Cauchy kernel, so we evaluate the difference explicitly using the theory of basic hypergeometric functions [1], [7].

Proof. With $x = \cos\theta, y = \cos\phi$ we get

$$\begin{aligned}
&\frac{(q-1)^k q^{k(1-k)/4}}{(2a)^k (q;q)_k}\, \mathcal{D}_q^k (y-x)^{-1}\Bigg|_{x=x_k} \\
&\quad = \frac{2(-1)^k q^{k(1-k)/2} e^{i(k+1)\phi}}{a^k (ae^{i\phi}, q^{-k}e^{i\phi}/a; q)_{k+1}} = \frac{-2aq^k}{\phi_{k+1}(\cos\phi; a)},
\end{aligned}$$

after a simple calculation. Thus the sum on the right-hand side of the first formula in Theorem 2.1 is

$$\begin{aligned}
&-2a \sum_{k=0}^{\infty} \frac{(ae^{i\theta}, ae^{-i\theta}; q)_k}{(ae^{i\phi}, ae^{-i\phi}; q)_{k+1}}\, q^k \\
&\quad = \frac{-2a}{1-2ay+a^2}\, {}_3\phi_2\left(\begin{matrix} q, ae^{i\theta}, ae^{-i\theta} \\ qae^{i\phi}, \ qae^{-i\phi} \end{matrix} \,\Bigg|\, q, q \right).
\end{aligned} \tag{2.4}$$

Applying the transformation [7, (III.9)] we see that the last expression is given by

$$\begin{aligned}
&-\frac{2a}{(1-q)(1-ae^{i\phi})} \\
&\quad \times {}_3\phi_2\left(\begin{matrix} q, qe^{i(\phi-\theta)}, qe^{i(\phi+\theta)} \\ qae^{i\phi}, \quad q^2 \end{matrix} \,\Bigg|\, q, ae^{-i\phi} \right) \\
&= \frac{2e^{i\phi}}{(1-e^{i(\phi-\theta)})(1-e^{i(\phi+\theta)})} \\
&\quad \times \left[1 - {}_2\phi_1\left(\begin{matrix} e^{i(\phi-\theta)}, e^{i(\phi+\theta)} \\ ae^{i\phi} \end{matrix} \,\Bigg|\, q, ae^{-i\phi} \right) \right].
\end{aligned}$$

The ${}_2\phi_1$ is summable by the q-analogue of Gauss' theorem [7, (II.8)] and its sum is $(ae^{i\theta}, ae^{-i\theta}; q)_\infty / (ae^{i\phi}, ae^{-i\phi}; q)_\infty$. The result is the first expansion in Theorem 2.1.

To prove the second expansion we evaluate the sum on the right-hand side of the second formula in Theorem 2.1. After the application of (1.10) the sum we are concerned with is found to be

$$2 \sum_{n=0}^{\infty} \frac{e^{i(n+1)\phi}(1+e^{2i\theta})(-q^{2-n}e^{2i\theta}; q^2)_{n-1}}{e^{in\theta}(iq^{-n/2}e^{i\phi}, -iq^{-n/2}e^{i\phi}; q)_{n+1}}. \tag{2.5}$$

We next sum over n even and over n odd. The even sum is

$$
\begin{aligned}
&\frac{2e^{i\phi}}{1+e^{2i\phi}} \sum_{n=0}^{\infty} \frac{(-e^{2i\theta}, -e^{-2i\theta}; q^2)_n q^{2n}}{(iqe^{i\phi}, iqe^{-i\phi}, -iqe^{i\phi}, -iqe^{-i\phi}; q)_n} \\
&= \frac{2e^{i\phi}}{1+e^{2i\phi}} \, {}_3\phi_2 \left(\begin{matrix} q^2, -e^{2i\theta}, -e^{-2i\theta} \\ -q^2 e^{2i\phi}, \; -q^2 e^{-2i\phi} \end{matrix} \,\middle|\, q^2, q^2 \right) \\
&= \frac{2e^{3i\phi}(1+e^{-2i\phi})}{(1-e^{2i(\phi+\theta)})(1-e^{2i(\phi-\theta)})} \\
&\quad \times \left[1 - \, {}_2\phi_1 \left(\begin{matrix} e^{2i(\phi-\theta)}, e^{2i(\phi+\theta)} \\ -e^{2i\phi} \end{matrix} \,\middle|\, q^2, -e^{-2i\phi} \right) \right],
\end{aligned}
$$

where the transformation [7, (III.9)] was applied in the last step. Again Gauss' theorem [7, (II.8)] sums the ${}_2\phi_1$ and we see that the even sum is

$$
\frac{y}{y^2-x^2} \left[1 - \frac{(-e^{2i\theta}, -e^{-2i\theta}; q^2)_\infty}{(-e^{2i\phi}, -e^{-2i\phi}; q^2)_\infty} \right]. \tag{2.6}
$$

The odd sum can be similarly handled and can be simplified to

$$
\frac{x}{y^2-x^2} \left[1 - \frac{(-qe^{2i\theta}, -qe^{-2i\theta}; q^2)_\infty}{(-qe^{2i\phi}, -qe^{-2i\phi}; q^2)_\infty} \right]. \tag{2.7}
$$

Equating (2.5) to the sum of (2.6) and (2.7) gives the second part of the theorem and the proof is complete. □

Theorem 2.2. *Let f be analytic in a bounded domain D and let C be a contour within D and x belong to the interior of C. If the distance between C and the set of zeros of $\phi_\infty(x; a)$ is positive then*

$$
\begin{aligned}
f(x) \;=\; & \frac{\phi_\infty(x;a)}{2\pi i} \oint_C \frac{f(y)}{y-x} \frac{dy}{\phi_\infty(y;a)} \\
& - \frac{a}{\pi i} \sum_{n=0}^{\infty} q^n \phi_n(x;a) \oint_C \frac{f(y)\,dy}{\phi_{n+1}(y;a)}.
\end{aligned}
$$

Proof. It is clear that

$$
\phi_n(x;a)/\phi_{n+1}(y;a) \to \phi_\infty(x;a)/\phi_\infty(y;a)
$$

uniformly in y on compact subsets not intersecting the set of zeros of $\phi_\infty(y;a)$. Thus we can multiply the first expansion in Theorem 2.1 by $f(y)$ and integrate with respect to y and interchange integration and summation. The result then follows from Cauchy's theorem. □

Theorem 2.2 gives the q-analogue of expanding $f(x)$ around $x = (a + 1/a)/2$. From the theory of functions we know that if $f(a) = 0$ and $f^{(j)}(a) = 0$ for $1 \le j \le m-1$, then the Taylor series starts with the term $f^{(m)}(a)(x-a)^m/m!$. This feature continues to hold but we have define a q-analogue of a multiple zero.

Definition. Let $x = (a + 1/a)/2$ be a zero of $f(x)$. We say that it has q-multiplicity m if

$$(2.8)\ f(z_k) = 0, 1 \le k \le m-1, \text{ and } f(z_m) \neq 0, \quad z_k := \frac{1}{2}(aq^k + q^{-k}/a).$$

Similarly x pole of f has q-multiplicity m if x is a zero of $1/f$ with q-multiplicity m.

It must be emphacized that the above definition is completely analogous to the definition of a multiple zero in difference equations in Hartman [9]. With this definition one can see that if $(a + 1/a)/2$ is a zero of f of q-multiplicity m then the terms corresponding to $n = 0, 1, \cdots, m-1$ in the sum in Lemma 2.2 vanish and the sum starts from $n = m$.

Theorem 2.3. *Let f be analytic in a bounded domain D and let C be a contour within D and x is interior to C. If the contour C is at a positive distance from the set $\{\pm i(q^{n/2} - q^{-n/2})/2; n = 0, 1, \ldots\}$, then*

$$\begin{aligned} f(x) &= \frac{2x}{\pi i}\oint_C \frac{yf(y)}{y-x}\, \frac{(-qe^{i(\theta+\phi)}, -qe^{i(\theta-\phi)}, -qe^{i(\phi-\theta)}, -qe^{-i(\theta+\phi)}; q)_\infty}{(-q, -q; q)_\infty\,(-e^{2i\phi}, -e^{-2i\phi}; q)_\infty}\, dy \\ &\quad + \frac{2}{\pi i}\sum_{n=0}^{\infty} \rho_n(x) q^n \oint_C \frac{y\, f(y)\, dy}{[(1-q^n)^2 + 4y^2q^n]\rho_n(y)}, \end{aligned}$$

with $x = \cos\theta$ and $y = \cos\phi$.

Proof. The proof is very similar to the proof of Theorem 2.2. The only step requiring justification is the identity

$$(2.9)\qquad \begin{aligned} &\frac{x}{y^2-x^2}\,\frac{(-qe^{2i\theta}, -qe^{-2i\theta}; q^2)_\infty}{(-qe^{2i\phi}, -qe^{-2i\phi}; q^2)_\infty} + \frac{y}{y^2-x^2}\,\frac{(-e^{2i\theta}, -e^{-2i\theta}; q^2)_\infty}{(-e^{2i\phi}, -e^{-2i\phi}; q^2)_\infty} \\ &= \frac{4xy}{y-x}\,\frac{(-qe^{i(\theta+\phi)}, -qe^{i(\theta-\phi)}, -qe^{i(\phi-\theta)}, -qe^{-i(\theta+\phi)}; q)_\infty}{(-q, -q; q)_\infty\,(-e^{2i\phi}, -e^{-2i\phi}; q)_\infty}. \end{aligned}$$

The proof of (2.9) uses the relationships [16, Chapter 21]

$$\begin{aligned} \vartheta_2(z) &= 2Gq^{1/4}\cos z(-q^2e^{2iz}, -q^2e^{-2iz}; q^2)_\infty \\ &= \frac{Gq^{1/4}}{2\cos z}(-e^{2iz}, -e^{-2iz}; q^2)_\infty, \\ \vartheta_3(z) &= G(-qe^{2iz}, -qe^{-2iz}; q^2)_\infty, \end{aligned}$$

and the product formulas in Exercise 3, page 488 in Whittaker and Watson [16]. The notations $\vartheta_j := \vartheta_j(0)$, $G := (q^2;q^2)_\infty$ [16] were used. We omit the details. $\square$

We record the following equivalent form of the representation of f in Theorem 2.3

$$
\begin{aligned}
(2.10)\qquad f(x) \;=\; & \frac{x}{2\pi i}\oint_C \frac{f(y)}{y^2-x^2}\,\frac{(-qe^{2i\theta},-qe^{-2i\theta};q^2)_\infty}{(-qe^{2i\phi},-qe^{-2i\phi};q^2)_\infty}\,dy \\
& +\frac{1}{2\pi i}\oint_C \frac{y\,f(y)}{y^2-x^2}\,\frac{(-e^{2i\theta},-e^{-2i\theta};q^2)_\infty}{(-e^{2i\phi},-e^{-2i\phi};q^2)_\infty}\,dy \\
& +\frac{2}{\pi i}\sum_{n=0}^{\infty}\rho_n(x)q^n\oint_C \frac{y\,f(y)\,dy}{[(1-q^n)^2+4y^2q^n]\rho_n(y)}.
\end{aligned}
$$

3 Expansions of entire functions

In this section we establish expansion theorems for entire functons of q-exponential growth. The expansions are in terms of the bases $\{\phi_n(x;a)\}$ and $\{\rho_n(x)\}$.

Observe that $M(r;\phi_\infty(x;a)) = \phi_\infty(-r;a)$, since $a>0$. Hence with

$$(3.1)\quad r_m = [aq^{m+\delta}+a^{-1}q^{-(m+\delta)}]/2, \quad -1<\delta<0, \quad m=0,1,\ldots$$

we find

$$
\begin{aligned}
& M(r_n;\phi_\infty(x;a)) = \phi_\infty(-r_n;a) \\
(3.2)\qquad & = (-q^{-n-\delta};q)_n\,(-q^{-\delta},-a^2q^{n+\delta};q)_\infty \\
& = q^{[\delta^2-n-(n+\delta)^2]/2}(-q^{\delta+1};q)_n(-q^{-\delta},-a^2q^{n+\delta};q)_\infty
\end{aligned}
$$

so that

$$\lim_{n\to\infty}\frac{\ln M(r_n;\phi_\infty)}{\ln^2 r_n} = \frac{1}{2\ln q^{-1}}.$$

The fact that $\phi_\infty(x;a)$ vanishes at x_{2n} for all n motivates our next theorem.

Theorem 3.1. *Any entire function f satisfying (1.2) with $c<1/(2\ln q^{-1})$ has an convergent expansion*

$$f(x) = \sum_{k=0}^{\infty} f_k\phi_k(x;a).$$

Moreover any such f is uniquely determined by its values on $\{x_{2n} : n\geq 0\}$.

Note that $(\mathcal{D}_q^k f)(x_k)$ is a linear combination of $f(x_0), \cdots, f(x_{2k})$, so that the coefficients f_k in Theorem 1.1 also depend on the points $\{x_{2n} : n \geq 0\}$.

The proof of Theorem 3.1 relies on a lemma which we now state and prove.

Lemma 3.2. *Let f be entire and satisfy the condition in Theorem 3.1. Then*

$$\lim_{n\to\infty} \oint_{|y|=r_n} \frac{f(y)}{y-x} \frac{dy}{\phi_\infty(y;a)} = 0.$$

Moreover, the same conclusion holds if

$$\lim_{n\to\infty} q^{n(n+2\delta+1)/2} \sup\{|f(r_n e^{i\theta}| : 0 \leq \theta < 2\pi\} = 0.$$

Proof. It is clear that $\inf\{|\phi_\infty(y;a)| : |y| = r\} = |\phi_\infty(r;a)|$. Hence for $|y| = r_n$, we have

$$\begin{aligned} |\phi_\infty(y;a)| &\geq |(q^{-n-\delta};q)_n| (q^{-\delta}, a^2 q^{n+\delta};q)_\infty \\ &= q^{-n(n+2\delta+1)/2} \, (q^{\delta+1};q)_n \, (q^{-\delta}, a^2 q^{n+\delta};q)_\infty \end{aligned}$$

Therefore

$$\begin{aligned} \ln M(r_n; f(y)/\phi_\infty(y;a)) &\leq \frac{1}{2}[n + (n+\delta)^2] \ln q + \ln M(r_n;f) + O(1) \\ &= \ln M(r_n;f) - \frac{1}{2}\frac{\ln^2 r_n}{\ln q^{-1}} + O(\ln r_n), \end{aligned}$$

and the lemma follows. □

Instead of proving the expansion in Theorem 3.1 in the basis $\{\phi_n(x;a)\}$ we shall prove the following equivalent result.

Theorem 3.3. *The expansion formula*

$$f(x) = \sum_{n=0}^\infty q^n f_n \phi_n(x;a),$$

with

$$f_n = \sum_{k=0}^n \frac{(-1)^k q^{k(k-1)/2} (1 - a^2 q^{2k})}{(q;q)_k (q;q)_{n-k} (a^2 q^k;q)_{n+1}} f(x_{2k}),$$

holds for functions f satisfying the assumptions of Theorem 3.1.

Proof. In Theorem 2.2 we choose C to be C_m, a circle centered at $y = 0$ and radius r_m. Lemma 3.2 shows that the first integral in Theorem 2.2 is small if m is large. We split the remaining sum in Theorem 2.2 into tail terms with $n > m$, and initial terms with $n \le m$. We will show that the tail is small, leaving the initial terms. Then a residue calculation establishes the expression for f_n, because the poles of $f(y)/\phi_{n+1}(y;a)$ are at $y = x_{2k}, k = 0, 1, \cdots, n$.

Note that if $n > m$ then

$$\begin{aligned}
&\min\{|\phi_{n+1}(y)| : y \in C_m\} \\
&= |\phi_{n+1}(r_m; a)| = \left|(q^{-m-\delta}, a^2 q^{m+\delta}; q)_{n+1}\right| \\
&= (q^{-m-\delta}; q)_m \, (-1)^m \, (q^{-\delta}; q)_{n+1-m} (a^2 q^{m+\delta}; q)_{n+1} \\
&= q^{-m(m+2\delta+1)/2} (q^{\delta+1}; q)_m, \, (q^{-\delta}; q)_{n+1-m} (a^2 q^{m+\delta}; q)_{n+1} \\
&\ge q^{-m(m+2\delta+1)/2} A = q^{-((m+\delta)^2+1-\delta^2)/2} A,
\end{aligned}$$

where A is a positive constant independent of n and m. Therefore for sufficiently large m, and $y \in C_m$,

$$\ln[M(r_m; f/\phi_{n+1})] \le [c_1 + 1/(2\ln q)] \ln^2 r_m + O(m)$$

for some c_1, $c \le c_1 < 1/(2\ln q^{-1})$.

This is a uniform bound of $e^{-D(\ln r_m)^2}$, $D > 0$, for each integral for $n > m$. Since $\phi_n(x;a) \to \phi_\infty(x;a)$, there is a uniform bound B for $\phi_n(x;a)$ on compact sets. Thus the tail is bounded by

$$\sum_{n=m+1}^{\infty} B q^n e^{-D(\ln r_m)^2} \le B q^{m+1} e^{-D(\ln r_m)^2}/(1-q),$$

which is small for m large. □

For polynomials f we equate the coefficients f_n in the expansions of f in $\{\phi_n(x;a)\}$ in Theorems 3.3 and 1.1 and discover the identity

$$\begin{aligned}
\mathcal{D}_q^n f(x_n) \\
= \frac{(2a)^n q^{n(n+3)/4}}{(q-1)^n} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{(-1)^k q^{k(k-1)/2}}{(a^2 q^k; q)_{n+1}} (1 - a^2 q^{2k}) \, f(x_{2k}).
\end{aligned} \tag{3.3}$$

Since (3.3) holds for arbitrary polynomials it must hold for all continuous functions. Using the notation

$$\eta_q^{\pm 1} f(x) = \breve{f}(q^{\pm 1/2} z), \tag{3.4}$$

and noting that a, is a general parameter and $x_n = \eta^n x_0$, we can rewrite (3.3) in the form

$$\begin{aligned} \mathcal{D}_q^n f(x) \\ = \frac{(2z)^n q^{n(3-n)/4}}{(q-1)^n} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{(-1)^k q^{k(k-1)/2}\, \eta^{2k-n} f(x)}{(z^2 q^{k-n};q)_k (z^2 q^{2k+1-n};q)_{n-k}}, \end{aligned} \tag{3.5}$$

with $x = (z + z^{-1})/2$. Equation (3.5) can be shown to be equivalent to Cooper's (1.16).

Theorem 3.4. *Let f be an entire function satisfying (1.2) and assume that $c < 1/\ln q^{-1}$. Then f has the expansion*

$$f(x) = \sum_{n=0}^{\infty} f_n \rho_n(x),$$

where

$$f_n = i^n \sum_{k=0}^{n} (-1)^k \frac{(q^k + q^{n-k}) q^{(k^2+(n-k)^2)/2}}{2(q^2;q^2)_k (q^2;q^2)_{n-k}} f(u_{n-2k}),$$

and $\{u_n\}$ is given by (1.4).

For general entire functions not necessarily satisfying (1.2), we note that the property $u_j = -u_{-j}$, allows one to conclude that for even functions f, $f_{2n+1} = 0$ for all $n \geq 0$, while for odd functions f, $f_{2n} = 0$, for all $n \geq 0$, confirming that f and its formal expansion $\sum_{n=0}^{\infty} f_n \rho_n(x)$ have the same parity.

Proof. We basically repeat the proof of Theorem 3.3, with some changes in the technical details. We will use

$$\frac{\rho_{2N}(x)}{\rho_{2N}(y)} = \frac{(-e^{2i\theta}, -e^{-2i\theta};q^2)_N}{(-e^{2i\phi}, -e^{-2i\phi};q^2)_N} \tag{3.6}$$

$$\frac{\rho_{2N+1}(x)}{\rho_{2N+1}(y)} = \frac{x(-qe^{2i\theta}, -qe^{-2i\theta};q^2)_N}{y(-qe^{2i\phi}, -qe^{-2i\phi};q^2)_N}. \tag{3.7}$$

Let C_m be a circle centered at $y = 0$ with radius r_m,

$$r_m = [q^{-(m+\delta)/2} - q^{(m+\delta)/2}]/2, \quad -1 < \delta < 0, \tag{3.8}$$

and m is even. We use the form of the Cauchy kernel in (2.10) and show that the first integral in (2.10) with $C = C_m$ tends to zero as $m \to \infty$. Thus we minimize the modulus of the denominators in order to give an upper bound for the integral. The first denominator is minimized by choosing $y^2 = -r_m^2$,

$$\begin{aligned}
&\min\{|(-qe^{2i\phi}, -qe^{-2i\phi}; q^2)_\infty| : y \in C_m\} \\
&= |(q^{1-m-\delta}, q^{m+\delta+1}; q^2)_\infty| = |(q^{1-m-\delta}; q^2)_{m/2}|\ (q^{1-\delta}, q^{m+\delta+1}; q)_\infty \\
&\geq Aq^{-(m+\delta)^2/4}
\end{aligned}$$

for some positive constant A independent of m.

Similarly

$$\begin{aligned}
&\min\{|(-e^{2i\phi}, -e^{-2i\phi}; q^2)_\infty| : y \in C_m\} \\
&= |(q^{-m-\delta}, q^{m+\delta}; q^2)_\infty| = |(q^{-m-\delta}; q^2)_{m/2}|\ (q^{-\delta}, q^{m+\delta}; q)_\infty \\
&\geq Aq^{-(m+\delta)^2/4-m/2}.
\end{aligned}$$

So the sum of the first two integrals, $|I_m|$, is bounded by

$$|I_m| \leq B\ M(r_m, f)\ q^{(m+\delta)^2/4},$$

for some B independent of m. Thus

$$\ln|I_m| \leq \ln M(r_m, f) + \frac{(\ln r_m)^2}{\ln q} + O(\ln r_m)$$

which proves that $|I_m| \to 0$ as $m \to \infty$, m even.

Next we show that $\sum_{n=m+1}^{\infty} q^n I_{m,n}$ tends to zero as $m \to \infty$ and for x in compact sets, where $\{I_{m,n}\}$ are the integrals

$$I_{n,m} = \oint_{C_m} \frac{y\rho_n(x)\ f(y)\, dy}{\rho_n(y)[(1-q^n)^2 + 4y^2q^n]}. \tag{3.9}$$

For $y \in C_m$, and $n > m$, we have

$$\begin{aligned}
&|(1-q^n)^2 + 4y^2q^n| = |(1+q^ne^{2i\phi})(1+q^ne^{-2i\phi})| \\
&\quad \geq |(1-q^{n-m-\delta})(1-q^{n+m+\delta})| \\
&\geq (1-q^{1-\delta})(1-q^{1+\delta}).
\end{aligned}$$

Moreover after applying (3.6) and (3.7), we get for $y \in C_m$ and $n > m$,

$$\begin{aligned}
\left|\frac{\rho_{2n}(x)}{\rho_{2n}(y)}\right| &\leq \frac{|(-e^{2i\theta}, -e^{-2i\theta}; q^2)_n|}{|(q^{-m-\delta}, q^{m+\delta}; q^2)_n|} \\
&\leq \frac{|(-e^{2i\theta}, -e^{-2i\theta}; q^2)_n|}{|(q^{-m-\delta}; q^2)_{m/2}|(q^{-\delta}; q^2)_{n-m/2}(q^{m+\delta}; q^2)_n} \\
&\leq A_1 q^{(m+\delta)^2/4},
\end{aligned}$$

for some constant A_1, and

$$\left|\frac{\rho_{2n+1}(x)}{\rho_{2n+1}(y)}\right| \leq \frac{|x(-qe^{2i\theta}, -qe^{-2i\theta}; q^2)_n|}{|y(q^{1-m-\delta}, q^{1+m+\delta}; q^2)_n|}$$
$$\leq \frac{|x(-qe^{2i\theta}, -qe^{-2i\theta}; q^2)_n|}{|y(q^{1-m-\delta}; q^2)_{m/2}(q^{1-\delta}; q^2)_{n-m/2}(q^{1+m+\delta}; q^2)_n|}$$
$$\leq A_2 q^{(m+\delta)^2/4},$$

The constants A_1 and A_2 depend on the compact set x to which x is restricted but do not depend on y, m or n.

As before this shows that

$$\ln|I_{n,m}| \leq \ln M(f, r_m) + \frac{(\ln r_m)^2}{\ln q} + O(\ln r_m)$$
$$\leq (c_1 + 1/\ln q)(\ln r_m)^2,$$

for some c_1, $0 < c_1 < 1/\ln q^{-1}$ which shows that $\sum_{n=m+1}^{\infty} q^n |I_{n,m}|$ tends to zero as $m \to \infty$.

Next we evaluate the sum $\sum_{n=0}^{m} q^n I_{m,n}$ by residues then let $m \to \infty$. From (1.10) it follows that

$$\rho_{n+2}(y) = q^{-n}[(1-q^n)^2 + 4y^2 q^n]\, \rho_n(y),$$

hence we need to evaluate
$$\oint_{C_m} \frac{yf(y)dy}{\rho_{n+2}(y)}.$$
The poles $\{y_k\}$ of $y/\rho_{n+2}(y)$ are

$$i[q^{-k+n/2} - q^{k-n/2}]/2,$$

$k = 0, 1, \cdots, n$. Let $y_k = \cos\phi_k$, hence

$$e^{i\phi_k} = \begin{cases} iq^{-k+n/2}, & 0 \leq k \leq n/2 \\ -iq^{k-n/2}, & n/2 < k \leq n. \end{cases}$$

It is routine to find that the residue of $y/\rho_{n+2}(y)$ at $i[q^{-k+n/2} - q^{k-n/2}]/2$ is

$$i^n \frac{(-1)^k (q^{n-k} + q^k)}{8(q^2; q^2)_k (q^2; q^2)_{n-k}}\, q^{k(k-n)+n^2/2}, \quad k = 0, \cdots, n,$$

and the theorem follows. $\square$

Remark. The first part of the proof can be replaced by estimating the first integral in Theorem 2.3 directly. Let $|e^{i\theta}| \le A$ for all x in a compact set. Hence for $y \in C_m$ and fixed x, we have

$$\begin{aligned}
&\ln\left(\left|\frac{y(-qe^{i(\theta+\phi)}, -qe^{i(\theta-\phi)}, -qe^{i(\phi-\theta)}, -qe^{-i(\theta+\phi)}; q)_\infty}{(-e^{2i\phi}, -e^{-2i\phi}; q)_\infty}\right|\right) \\
&\le \ln\left((-Aq^{1-(m+\delta)/2}, -q^{1-(m+\delta)/2}/A; q)_\infty\right) \\
&\quad - \ln\left(\left|(q^{-m-\delta}; q)_\infty\right|\right) + O(1) \\
&\le \ln\left((-Aq^{1-(m+\delta)/2}, -q^{1-(m+\delta)/2}/A; q)_{m/2}\right) \\
&\quad - \ln\left(\left|(q^{-m-\delta}; q)_m\right|\right) + O(m) \\
&= \frac{m(m+2\delta)}{4}\ln q + O(m) = -\frac{\ln^2 r_m}{\ln q^{-1}} + O\left(\ln(r_m)\right).
\end{aligned}$$

Therefore the first integral on the right-hand side of in the equation in Theorem 2.3 tends to zero as $m \to \infty$.

4 A Mittag-Leffler Expansion

It is tempting to substitute for f_n in the first formula in Theorem 3.3 then rearrange the sum and find the coefficient of $f(x_{2k})$. The formal interchange of sums gives

$$\begin{aligned}
(4.1) \qquad f(x) &= \sum_{k=0}^{\infty} \frac{(-1)^k q^{k(k+1)/2}(1-a^2q^{2k})}{(q;q)_k(a^2q^k;q)_{k+1}}(ae^{i\theta}, ae^{-i\theta}; q)_k \\
&\quad \times {}_2\phi_1\left(\left.\begin{matrix} aq^k e^{i\theta}, aq^k e^{-i\theta} \\ a^2q^{2k+1}\end{matrix}\,\right|\, q, q\right) f(x_{2k}).
\end{aligned}$$

The ${}_2\phi_1$ can be summed by the q-analogue of Gauss' theorem [7, (II.8)] and its sum is $(aq^{k+1}e^{i\theta}, aq^{k+1}e^{-i\theta}; q)_\infty/(a^2q^{2k+1}, q; q)_\infty$ and (4.1) becomes

$$(4.2) \quad \frac{f(x)}{\phi_\infty(x;a)} = \sum_{k=0}^{\infty} \frac{(-1)^k q^{k(k+1)/2}(1-a^2q^{2k})}{(q;q)_k(q, a^2q^k; q)_\infty} \frac{f(x_{2k})}{1-2axq^k+a^2q^{2k}}.$$

Theorem 4.1. *Formula (4.2) holds for entire functions f satisfying (1.2) with $c < 1/(2\ln q^{-1})$.*

Proof. Let r_m be as in (3.1) and C_m be a circle centered at the origin and have radius r_m. Let x be fixed and m be large enough so that x is interior to C_m. Consider

$$I_m := \frac{1}{2\pi i} \oint_{C_m} \frac{f(y)}{\phi_\infty(y;a)} \frac{dy}{y-x}. \tag{4.3}$$

From Lemma 3.2, $I_m \to 0$ as $m \to \infty$. On the other hand

$$\begin{aligned} I_m &= \frac{f(x)}{\phi_\infty(x;a)} \\ &\quad - \sum_{k=0}^{m} \frac{(-1)^k q^{k(k+1)/2}(1-a^2q^{2k})}{(q;q)_k(q, a^2q^k;q)_\infty} \frac{f(x_{2k})}{1-2axq^k+a^2q^{2k}}, \end{aligned}$$

and the theorem follows. □

Clearly Theorem 4.1 is a Mittag-Leffler expansion.

5 Applications

Recall that the conclusion of Lemma 3.2 holds provided that

$$\lim_{n\to\infty} M(r_n; f)\, q^{n(n+2\delta+1)/2} = 0. \tag{5.1}$$

where r_n is defined in (3.1). By examining the proof of Theorem 3.3 we see that it continues to hold under the assumption (5.1). In fact we can replace the sequence $\{r_n\}$ in (5.1) by any subsequence $\{r_{n_k}\}$.

As a first application of the above observation we let

$$g(z) = (be^{i\theta}, be^{-i\theta}; p)_\infty, \quad p \le q, \quad z := \cos\theta. \tag{5.2}$$

To verify (5.1) we employ

$$\begin{aligned} M_n(r_n; g) &\le (-|b/a|q^{-n-\delta}, -|ab|q^{n+\delta}; p)_\infty \\ &\le (-|b/a|q^{-n-\delta}, -|ab|q^{n+\delta}; q)_\infty \\ &\le |b/a|^n q^{-n(n+2\delta+1)/2}(-|b/a|q^{-\delta}, -q^{1+\delta}|ab|; q)_\infty. \end{aligned}$$

Thus (5.1) holds when $|b/a| < 1$, and (4.2) will then hold for $|b| < |a|$ and we have established the series summation

$$
\begin{aligned}
&\frac{(be^{i\theta}, be^{-i\theta}; p)_\infty}{(qae^{i\theta}, qae^{-i\theta}; q)_\infty} \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k q^{k(k+1)/2} (a^2, aq, -aq; q)_k}{(q, a, -a; q)_k (q, a^2 q; q)_\infty} \\
&\times \frac{(ae^{i\theta}, ae^{-i\theta}; q)_k}{(aqe^{i\theta}, aqe^{-i\theta}; q)_k} (abq^k, bq^{-k}/a; p)_\infty,
\end{aligned} \tag{5.3}
$$

valid for $0 < p < q$, or $p = q$ and $|b| < |a|$.

Mizan Rahman pointed out that (5.3) follows from a result of George Gasper. Gasper's formula is (5.13) on p. 68 in [6] and can be stated as

$$
\begin{aligned}
&{}_{6+2m}W_{5+2m}\left(A; B, \frac{A}{B}, d, e_1, ..., e_m, \frac{Aq^{n_1+1}}{e_1}, ..., \frac{Aq^{n_m+1}}{e_m}; q, \frac{q}{d} q^{-\sum_{j=1}^m n_j}\right) \\
&\quad = \frac{(q, Aq, Aq/Bd, Bq/d; q)_\infty}{(Bq, Aq/B, Aq/d, q/d; q)_\infty} \prod_{j=1}^m \frac{(Aq/Be_j, Bq/e_j; q)_{n_j}}{(Aq/e_j, q/e_j; q)_{n_j}}.
\end{aligned}
$$

We put

$$
A = a^2, \quad B = ae^{i\theta}, \quad d = q^{-m}, \quad e_j = aqp^{1-j}/b, n_j = 1, 1 \le j \le m.
$$

Write the ${}_{6+2m}W_{5+2m}$ as a sum over $k, k \ge 0$. The terms containing $e_1, \cdots, e_m$ contribution to the k term is

$$
\begin{aligned}
&\prod_{r=0}^{m-1} \frac{(aqp^{-r}/b, abqp^r; q)_k}{(abp^r, ap^{-r}/b; q)_k} \\
&= \prod_{r=0}^{m-1} \frac{(1 - aq^k p^{-r}/b)(1 - abp^r q^k)}{(1 - ap^{-r}/b)(1 - abp^r)} = q^{km} \frac{(abq^k, bq^{-k}/a; p)_m}{(ab, b/a; q)_m}.
\end{aligned}
$$

Now (5.3) follows by letting $m \to \infty$.

When $p = q$ in (5.3), a simple calculation using

$$
\begin{aligned}
(abq^k, bq^{-k}/a; q)_\infty &= (ab, b/a; q)_\infty \frac{(bq^{-k}/a; q)_k}{(ab; q)_k} \\
&= \frac{(-b/a)^k (aq/b; q)_k}{q^{k(k+1)/2} (ab; q)_k} (ab, b/a; q)_\infty,
\end{aligned}
$$

shows that the right-hand side of (5.1) is $(ab, b/a; q)_\infty/(q, a^2q; q)_\infty$ times a ${}_6\phi_5$ function. Thus (5.3) with $p = q$ is equivalent to

$$
(5.4)\qquad \begin{aligned}
&{}_6\phi_5\left(\begin{matrix} a^2, aq, -aq, aq/b, ae^{i\theta}, ae^{-i\theta} \\ a,\ -a,\ ab,\ aqe^{-i\theta},\ aqe^{i\theta} \end{matrix}\ \middle|\ q, \frac{b}{a}\right) \\
&\qquad = \frac{(q, a^2q, be^{i\theta}, be^{-i\theta}; q)_\infty}{(aqe^{i\theta}, aqe^{-i\theta}, ab, b/a; q)_\infty}.
\end{aligned}
$$

Formula (5.4) is the sum of a very well-poised ${}_6\phi_5$, [7, (II.20)]. The most general ${}_6\phi_5$ has four free parameters, but our (5.4) has only three free parameters.

Another application of (4.2) is to choose

$$
(5.5)\qquad f(z) = \prod_{j=1}^{m} f_j(z), \quad f_j(\cos\theta) := (b_je^{i\theta}, b_je^{-i\theta}; p_j)_\infty.
$$

Here we will only mention the case when the $p_j = p$ for all j. In this case we choose a positive integer l such that $q^{l+1} < p \le q^l$. It suffices to take $n = ls$ in (5.1) and for sufficiently large s, we get

$$
(5.6)\qquad \begin{aligned}
&M(r_{ls}; f_j) \le (-|b_j/a|q^{-ls-\delta}, -|ab_j|q^{ls+\delta}; p)_\infty \\
&\le (-|b_j/a|q^{-ls-\delta}; p)_s\,(-|b_j/a|q^{-\delta}, -|ab_j|q^{ls+\delta}; p)_\infty \\
&\le |b_j/a|^s\, q^{-s(ls+\delta)}p^{s(s-1)/2}C_j,
\end{aligned}
$$

where C_j is a constant depending only on $a, b_1, \cdots, b_m, \delta$ but not on s. With r defined through

$$
(5.7)\qquad p = q^{lr}, \quad 1 \le r < 1 + 1/l,
$$

we obtain

$$
(5.8)\qquad \begin{aligned}
&M(r_{ls}; f)q^{ls(ls+2\delta+1)/2} \\
&\le C\, q^{-(ls+\delta)sm+ls(ls+2\delta+1)/2}p^{ms(s-1)/2} \prod_{j=1}^{m} |b_j/a|^s,
\end{aligned}
$$

for some constant C. Substitute for p in (5.8) from (5.7) to see that the coefficient of s^2 in the exponent of q is nonnegative if and only if

$$
(5.9)\qquad l \ge m(2 - r).
$$

If $l = m(2 - r)$ then the coefficient of s in the exponent of q is $m[\delta + 1 - r\delta - r(1 + m(2 - r))/2]$. Thus we have established the following theorem.

Theorem 5.1. *Formula (4.2) holds for the function f defined by (5.5) if* (i) *or* (ii) *holds, where*

$$\begin{aligned}&\text{(i)}\ \ l > m(2-r)\\ &\text{(ii)}\ l = m(2-r)\ \text{ and }\ B\, q^{1-r(1+m(2-r))/2} < |a|,\end{aligned}$$

with $B = |b_1 \cdots b_m|^{1/m}$.

The details of consequences of Theorem 5.1 will be explored elsewhere. We just mention the case $p_j = p = q^m$, so $r = 1$ and $m = l$. Thus (4.2) holds if

$$\prod_{j=1}^{m} |b_j/a| < q^{m(m-1)/2}. \tag{5.10}$$

Thus (4.2) gives

$$\begin{aligned}&\frac{\prod_{j=1}^{m}(b_j e^{i\theta}, b_j e^{-i\theta}; q^m)_\infty}{(ae^{i\theta}, ae^{-i\theta}; q)_\infty}\\ &= \sum_{k=0}^{\infty} \frac{(-1)^k q^{k(k+1)/2}(1-a^2q^{2k})}{(q;q)_k (q, a^2q^k; q)_\infty} \frac{\prod_{j=1}^{m}(b_j q^{-k}/a, ab_j q^k; q^m)_\infty}{(1-aq^k e^{i\theta})(1-aq^k e^{-i\theta})}.\end{aligned} \tag{5.11}$$

The special case $m = 2$ of (5.13) follows from [7, (III.38)]. To see this we first write upper case letters for the parameters $a, b, \cdots, q$ in [7, (III.38)]. We make the choices

$$\begin{aligned}Q = q^2, \quad A = B = e^{i\theta}, \quad C = e^{-i\theta},\\ D = q^2/b_1, \quad E = q^2/b_2, \quad F = a, \quad G = aq.\end{aligned} \tag{5.12}$$

The resulting ${}_8\psi_8$ on the left-hand side of (III.36) in [7] reduces to 1 because its numerator parameter AB is 1, while the corresponding denominator parameter ($=AQ/B$) is q^2, which the base of the ${}_8\psi_8$.

6 Remarks

In [11] we pointed out the importance of the polynomial basis

$$\phi_n(\cos\theta) = (q^{1/4}e^{i\theta}, q^{1/4}e^{-i\theta}; q^{1/2})_n, \tag{6.1}$$

in the theory of basic hypergeometric functions (q-series). We also established the q-Taylor series

$$f(x) = \sum_{k=0}^{n} f_k \phi_k(x), \tag{6.2}$$

for polynomials f, where

$$f_k = \frac{(q-1)^k}{2^k q^{k/4}(q;q)_k}(\mathcal{D}_q^k f)(\zeta_0) \tag{6.3}$$

and

$$\zeta_n = [q^{(n+1/2)/2} + q^{-(n+1/2)/2}]/2. \tag{6.4}$$

The proof of (6.2) uses

$$\mathcal{D}_q \phi_n(x) = -2q^{1/4}\frac{1-q^n}{1-q}\,\phi_{n-1}(x). \tag{6.5}$$

One can also extend (6.2) to entire functions satisfying (1.2) with $c < 1/\ln q^{-1}$ using an argument similar to what we used to prove Theorem 3.1. In fact this is essentially Theorem 3.1 because upon close examination one sees that the interpolation points used in (6.4) amount to replacing q by $q^{1/2}$.

The q-exponential function of [13] is

$$\begin{aligned} \mathcal{E}_q(\cos\theta;t) \quad = \quad & \frac{(t^2;q^2)_\infty}{(qt^2;q^2)_\infty}\sum_{n=0}^{\infty}\frac{(-it)^n}{(q;q)_n}\,q^{n^2/4} \\ & \times(-iq^{(1-n)/2}e^{i\theta}, -iq^{(1-n)/2}e^{-i\theta};q)_n. \end{aligned} \tag{6.6}$$

The function $\mathcal{E}_q(x;t)$ is entire in x for all t, $|t| < 1$. Corollary 2.5 of [11] is

$$\mathcal{E}_q(\cos\theta;t) \quad = \quad \frac{(-t;q^{1/2})_\infty}{(qt^2;q^2)_\infty}\,{}_2\phi_1\left(\begin{matrix} q^{1/4}e^{i\theta}, q^{1/4}e^{-i\theta} \\ -q^{1/2} \end{matrix}\,\middle|\, q^{1/2}, -t\right) \tag{6.7}$$

We now show that (6.7) enables us to determine the exact limiting behavior of the maximum modulus of the $\mathcal{E}_q$ function. Let $r = \cosh u$, $u > 0$. Thus (6.7) implies

$$\ln M(\cosh u;\mathcal{E}_q) \le \ln\left((-q^{1/4}e^u, -q^{1/4}e^{-u};q^{1/2})_\infty\right) + O(1),$$

as $u \to \infty$. It is clear that for any sequence of u's tending to infinity, e^u can be written in the form $q^{-(n_m+\delta_m)/2}$, with $-1/2 < \delta_m < 1/2$. From here it is not difficult to see that

$$\limsup_{r\to\infty}\frac{\ln M(r;\mathcal{E}_q)}{\ln^2 r} \le \frac{1}{\ln q^{-1}}. \tag{6.8}$$

On the other hand the sequence $r_m = [q^{-(m+1/2)/2} + q^{(m+1/2)/2}]/2$ makes

$$\text{(6.9)} \qquad \frac{(qt^2;q^2)_\infty}{(-t;q^{1/2})_\infty}\,\mathcal{E}_q(r_m;t) \;=\; {}_2\phi_1\left(\left.\begin{matrix} q^{-m/2}, q^{m+1)/2} \\ -q^{1/2} \end{matrix}\,\right|\, q^{1/2}, -t\right).$$

The right-hand side of (6.9) is a little q-Jacobi polynomial $\Phi_m^{(\alpha,\beta)}(x)$, with $\alpha = \beta = -1$ and $x = -t/q^{1/2}$, [12], hence the ${}_2\phi_1$ in (6.9) is asymptotically equal to

$$t^m q^{-m(m+1)/4}(-q^{1/2}/t;q)_\infty/(-q^{1/2};q^{1/2})_\infty,$$

by (1.5) in [12]. Therefore

$$\text{(6.10)} \qquad \limsup_{r\to\infty} \frac{\ln M(r;\mathcal{E}_q)}{\ln^2 r} \geq \lim_{m\to\infty} \frac{\ln M(r_m;\mathcal{E}_q)}{\ln^2 r_m} = \frac{1}{\ln q^{-1}}.$$

Therefore (6.8) and (6.10) establish the following theorem.

Theorem 6.1. *The maximum modulus of $\mathcal{E}_q$ has the property*

$$\limsup_{r\to\infty} \frac{\ln M(r;\mathcal{E}_q)}{\ln^2 r} = \frac{1}{\ln q^{-1}}.$$

It is worth mentioning that Theorem 6.1 shows that (6.7) does not follow from the general approach developed here. It is of interest to find a function theoretic approach to development of identities like (6.7).

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