Three statistics on lattice paths

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Abstract

Three statistics on lattice paths are considered: the major index, the major+lesser index, and a new statistic called 001-110. Three generating functions for lattice paths which lie between two parallel lines are given. Moreover, a conjectured decomposition of the lattice paths implies stronger generating function results.

1 Introduction

It is well-known [1, 13] that integer partitions may be studied from the point of view of lattice paths, which themselves are words of 0's and 1's. Under a natural bijection, the simplest statistic on words, the inversion number (inv), corresponds to the integer being partitioned. However, several other statistics on words are known to have the same distribution as inv, for example maj. In this paper we consider two statistics on words, $001-110$, and $maj + comaj$, along with maj. The $001-110$ statistic has the same generating function as inv or maj (see Theorem 2.2), while $maj + comaj$ does not (see Theorem 2.3). A simple bijection proving that inv, maj, and $001 - 110$ are equidistributed is given in §2.

These three statistics (maj, $maj + comaj$, and $001-110$) have some remarkable properties related to a special set of lattice paths. These lattice paths move generally east along a grid of parallel lines. The three statistics have a conjectured multiplicative decomposition (see Conjecture 3.2), which allows for explicit alternating sums for each of the three generating functions ((5),(7), and (9)).

A special case of these lattice paths was considered by Andrews et al. [3], who also found an alternating sum for the generating function of integer partitions with certain hook difference conditions. This is equivalent to finding the generating function for the $001 - 110$ statistic, see (7).

Borwein conjectured [2] the positivity of the coefficients of similar alternating sums. The only difference is that his sums have a modified quadratic power of q. A natural idea [4] to prove this conjecture is to weight the paths by an appropriate statistic, thus yielding the Borwein alternating sum. In this paper we give the modification of $001 - 110$ to maj and $maj + comaj$, although neither settles the Borwein conjecture.

In §4-6 we give explicit bijections which prove a positive generating function for each of the three statistics. Some remarks and a limiting case are given in §7.

All notation (for q-binomial coefficients, q-shifted factorials) is consistent with that found in [8].

2 q-weights of lattice paths

In this section we explicitly give three bijections from the set of integer partitions to lattice paths. These three bijections, which are defined similarly, prove that $inv = maj$, $inv = 001 - 110$, and

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a statistic of Burge is equidistributed with *inv*. We also state MacMahon's generating function for $maj + comaj$, see Theorem 2.3.

First we set some notation. Let $R(m,n)$ denote the region $\{(x,y)|0 \le x \le m, 0 \le y \le n\}$ in R^2. $R(m,n)$ can be regarded as a union of mn unit squares indexed by (i,j), where i and j are integers satisfying $0 \le i \le m-1$, $0 \le j \le n-1$. Each unit square in a rectangle is indexed by the coordinates of its lower-left corner. For a given (i,j), the *upward hook* in an $m \times n$ rectangle determined by the square (i,j) is the set $\{(i,l) : j \le l < n\} \cup \{(k,j) : i \le k < m\}$.

Let $\pi = \pi_1\pi_2\cdots\pi_l$ be a partition whose Ferrers diagram lies inside an $n \times m$ rectangle. This implies $l \le n$ and $\pi_1 \le m$. The *Frobenius notation* of the partition π is

$$\begin{bmatrix} a_1 & a_2 & \cdots & a_k \\ b_1 & b_2 & \cdots & b_k \end{bmatrix},$$

where $a_i = \pi_i - i$ and $b_i = |\{j : j > i, \text{ and } \pi_j \ge i\}|$. We associate to π the subset of

$$\{(a_1,b_1),(a_2,b_2),\ldots,(a_k,b_k)\}$$

of unit squares in $R(m,n)$. We can represent the subset corresponding to π by marking the squares in $R(m,n)$. Note that each upward hook contains at most one marked square, since a_i's (b_i's resp.) are decreasing. Conversely, any configuration of marked squares will correspond to a partition, if each upward hook contains at most one marked square.

We will use the marked squares of π to define three bijections from partitions to lattice paths.

We set some notation for lattice paths. A *lattice path* $L = (c_1, c_2, \ldots, c_{m+n})$ from $(0,0)$ to (m,n) is a sequence of m 0's and n 1's. We let 0 represent a unit horizontal step and 1 a unit vertical step. We let $L_{(m,n)}$ denote the set of lattice paths from $(0,0)$ to (m,n).

A unit square (i,j) is called a *corner cell* of L, if two sides of (i,j) coincide with L. A corner cell (i,j) of L is called a 01-*corner cell* of L, if $c_{i+j+1} = 0$ and $c_{i+j+2} = 1$. A corner cell (i,j) of L is called a 10-*corner cell* of L, if $c_{i+j+1} = 1$ and $c_{i+j+2} = 0$. A 01-corner cell (i,j) of L is called *exposed* if $c_{i+j} = 0$. A 10-corner cell (i,j) of L is called *exposed*, $c_{i+j} = 1$. We prepend 0 to L, so that an initial 01 is exposed. A corner cell, 01- or 10-, which is not exposed, is called *unexposed*.

Definition 2.1 *Let* $\pi = \begin{bmatrix} a_1 & a_2 & \cdots & a_k \\ b_1 & b_2 & \cdots & b_k \end{bmatrix}$ *be a partition whose Ferrers diagram lies inside an* $n \times m$ *rectangle. We define lattice paths* $\psi_1(\pi)$ *and* $\psi_2(\pi)$ *as follows:*

1. $\psi_1(\pi)$ *is the unique lattice path* L *in* $R(m,n)$ *from* $(0,0)$ *to* (m,n) *whose* 10*-corner cells are* $\{(a_1,b_1),(a_2,b_2),\ldots,(a_k,b_k)\}$.
2. $\psi_2(\pi)$ *is the unique lattice path* L *in* $R(m,n)$ *from* $(0,0)$ *to* (m,n) *whose exposed corner cells,* 01*- or* 10*-, are* $\{(a_1,b_1),(a_2,b_2),\ldots,(a_k,b_k)\}$.

The map ψ_1 can be constructed by drawing two rays on each marked square: one easterly, emanating from the north edge of the 10-corner cell; and one southernly, emanating from the west edge of the 10-corner cell. The lattice path L is obtained by taking a union of these line segments, see Figure 1.

The fact that ψ_2 is well-defined requires some thought. Consider the example of $\psi_2(\pi)$ in Figure 1. The only occurrences of 110 or 001 in the word for L must occur at the marked cells. A path always has a 0 prepended, so starting at the origin L must alternate $1010\cdots$. Once this staircase path (as in Figure 1) reaches the horizontal line of the south edge of the first marked square, it must move across, appending $00\cdots01$, making the first marked square an exposed 01-corner cell. If the first marked square were above the diagonal, the path would move up when it reaches the vertical line of the west edge of the marked square, appending $11\cdots10$, an exposed 10-corner cell. As the path moves to the next marked square, staircases are again required, followed by the appropriate $00\cdots01$ or $11\cdots10$.

It is well-known [1] that the generating function for partitions inside an $m \times n$ rectangle is given by

$$\sum_{\pi \in R(m,n)} q^{|\pi|} = \begin{bmatrix} m+n \\ m \end{bmatrix}_q.$$

In terms of the Frobenius notation one has $|\pi| = (a_1 + b_1 + 1) + \cdots + (a_k + b_k + 1)$. If $\psi_1(\pi) = L = (c_1, c_2, \ldots, c_{m+n})$, the 10-corner cells (i,j) yield $c_{i+j+1} = 1$, $c_{i+j+2} = 0$, a descent in position $i+j$. In terms of the major index maj [6], we have

$$|\pi| = maj(L),$$

thus ψ_1 is a bijection from inv to maj proving that

$$\sum_{L \in L_{(m,n)}} q^{maj(L)} = \begin{bmatrix} m+n \\ m \end{bmatrix}_q.$$

The bijection ψ_2 similarly proves a new statistic is Mahonian. From the definition of the exposed cells, we count descents 10 when preceded by a 1, (110), and we count ascents 01 when preceded by a 0, (001). We call this statistic the $001-110$ *statistic*,

$$001 - 110(L) = \sum_{i \geq 1,\ (c_{i-1}, c_i, c_{i+1}) = (1,1,0) \text{ or } (0,0,1)} i.$$

A list of all lattice paths from (0,0) to (2,2) is given below (including the prepended 0), with their $001-110$ statistics.

0|0011	2
0|0101	1
0|0110	1 + 3
0|1001	3
0|1010	0
0|1100	2

Note that the generating function is $1 + q^1 + 2q^2 + q^3 + q^4 = \left[\begin{smallmatrix} 4 \\ 2 \end{smallmatrix}\right]_q$.

Theorem 2.2 *We have*

$$\sum_{L \in L_{(m,n)}} q^{001-110(L)} = \begin{bmatrix} m+n \\ m \end{bmatrix}_q.$$

Burge [5] considers a different bijection ψ_b between π and a lattice path L. The path $L = \psi_b(\pi) \in L_{(m,n)}$ in Burge's correspondence is the unique lattice path whose 10-corner cells in the upper diagonal region (the cells (i,j)'s with $i < j$) are those in $\{(a_1,b_1),(a_2,b_2),\ldots,(a_k,b_k)\}$, and whose 01-corner cells in the lower diagonal region (the cells (i,j)'s with $i \geq j$) are those in $\{(a_1,b_1),(a_2,b_2),\ldots,(a_k,b_k)\}$.

Example. Let $(m,n) = (3,2)$. If π is an empty partition, then $\psi_1(\pi) = (0,0,0,1,1)$, $\psi_2(\pi) = (1,0,1,0,0)$, $\psi_b(\pi) = (1,0,1,0,0)$. If $\pi = 1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, then $\psi_1(\pi) = (1,0,0,0,1)$, $\psi_2(\pi) = (0,1,0,1,0)$, $\psi_b(\pi) = (0,1,1,0,0)$. Note that the prepended 0 makes the square $(0,0)$ in $\psi_2(\begin{bmatrix} 0 \\ 0 \end{bmatrix}) = (0,1,0,1,0)$ an exposed 01-corner.

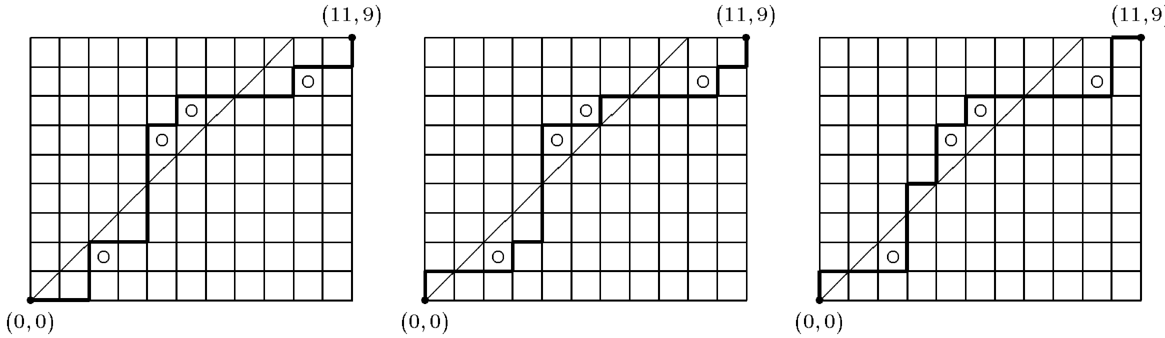

Figure 1. $(m,n)=(11,9)$, $\pi = \begin{bmatrix} 9 & 5 & 4 & 2 \\ 7 & 6 & 5 & 1 \end{bmatrix}$, $\psi_1(\pi)$, $\psi_2(\pi)$, $\psi_b(\pi)$ respectively.

We also consider a third statistic on lattice paths, which MacMahon [13] called the greater+lesser index,

$$(maj + comaj)(L) = \sum_{i \geq 1,\ c_i \neq c_{i+1}} i.$$

The $maj + comaj$ statistic is computed below on all 6 lattice paths from $(0,0)$ to $(2,2)$.

0011	2
0101	$1+2+3$
0110	$1+3$
1001	$1+3$
1010	$1+2+3$
1100	2

so the generating function is $2q^2+2q^4+2q^6$.

Let

$$\begin{bmatrix} n \\ i \end{bmatrix}_q^* = \frac{q^i + q^{n-i}}{1+q^n} \begin{bmatrix} n \\ i \end{bmatrix}_{q^2},$$

and note that $\left[\begin{smallmatrix} 4 \\ 2 \end{smallmatrix}\right]_q^* = 2q^2+2q^4+2q^6$.

MacMahon [13] found the following generating function, which we shall need.

Theorem 2.3 *We have*

$$\sum_{L \in L_{(m,n)}} q^{(maj+comaj)(L)} = \begin{bmatrix} m+n \\ m \end{bmatrix}_q^*.$$

A more general result is given in Theorem 1 of [9], which implies

$$\sum_{L \in L_{(m,n)}} x^{maj(L)} y^{comaj(L)} = x^n \begin{bmatrix} m+n-1 \\ n \end{bmatrix}_{xy} + y^m \begin{bmatrix} m+n-1 \\ n-1 \end{bmatrix}_{xy},$$

Moreover (see [9]) we have

$$\sum_{m,n \geq 0} \begin{bmatrix} m+n \\ n \end{bmatrix}_q^* \frac{(qx)^m (qy)^n}{(q)_{m+n}} = \frac{(xyq^2;q^2)_\infty}{(xq,yq;q)_\infty} = F(x,y,q), \tag{1}$$

so that the power series coefficients of $F(x,y,q)$ are non-negative.

3 k-good paths, k-bad paths and k-types

In this section we consider how a lattice path L intersects a certain set of parallel lines. Paths which always move east along these lines are called good paths, otherwise they are called bad. Good paths include paths which lie strictly inside two parallel lines. This is equivalent to considering partitions with prescribed hook difference conditions [3]. The results (5),(7), and (9) below can be considered as generalized versions of special cases of an alternating sum considered by Andrews et al. [3]

First we define the *k-type* of a path $L \in L_{(m,n)}$. Fix a positive integer k, and consider the infinite set of parallel lines $y = x + ak$, for all integers a. L intersects some of the lines of the form $y = x + ak$. We record all the lines that L visits. Let $(a_0, a_1, a_2, \ldots, a_t)$ be the sequence such that $a_0 = 0$ and for each i, $a_i \neq a_{i-1}$ and L visits the line $y = x + a_i k$ directly after it visits the line $y = x + a_{i-1}k$. Note that $|a_i - a_{i-1}| = 1$, for each i. We associate to L the sequence $type_k(L) = (\alpha_1, \alpha_2, \ldots, \alpha_t)$, where

$$\alpha_i = \begin{cases} 0, & \text{if } a_i = a_{i-1} - 1, \\ 1, & \text{if } a_i = a_{i-1} + 1. \end{cases}$$

The basic intuition is that 0's in $type_k(L)$ mean the path is moving east across the diagonal lines, while 1's represent northward movement.

If L is a path from $(-lk, lk)$ to (m, n), then we translate it to a path in $L_{(m+lk,n-lk)}$ and define $type_k(L)$ as above.

Definition 3.1 *A path L is called* k-good *if $type_k(L) = (\alpha_1, \alpha_2, \ldots, \alpha_t)$ with $\alpha_i = 0$ for all i; otherwise it is called* k-bad. *The path L is said to be* of class v, *$class_k(L) = v$, if $type_k(L)$ has v 1's. The path L is said to be of* k-length t, *if $type_k(L)$ is of length t.*

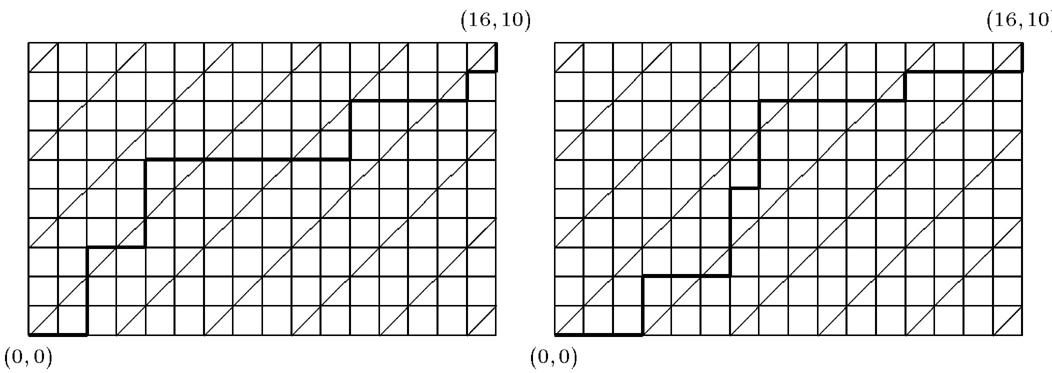

Figure 2. A k-good path L and a k-bad path L' with $k = 3$.
The 3-type of L is $(0, 0)$ and the 3-type of L' is $(0, 1, 0, 0)$.

We let $L^{k,v}_{(m,n)}$ denote the set of paths $L \in L_{(m,n)}$ such that $class_k(L) = v$. Thus any path $L \in L^{k,v}_{(m+lk,m)}$ has k-length $l + 2v$ and is good iff $v = 0$.

For our main conjecture we need a correspondence between good paths and certain bad paths.

Let v be a positive integer. For any v-subset $S \subset \{1, 2, \cdots, l + 2v\}$, let $L^{k,bad(S)}_{(m+lk,m)}$ be those lattice paths in $L_{(m+lk,m)}$ whose k-length is $l+2v$, and whose k-type equals one at exactly at the positions S.

Conjecture 3.2 *Fix $k \geq 2$ and a positive integer v. For any v-subset $S \subset \{1, 2 \cdots, l + 2v\}$, define $L(S) = (c_1, c_2, \ldots, c_{k(l+2v)}) \in L_{(lk+k|S|,k|S|)}$ by $c_{k(j-1)+h} = 1$ for all $h = 1, 2, \cdots, k$ iff $j \in S$. If s is either maj, $001 - 110$, or maj + comaj, we have*

$$\sum_{L \in L^{k,bad(S)}_{(m+lk,m)}} q^{s(L)} = q^{s(L(S))} \sum_{L \in L^{k,0}_{(m+lk+k|S|,m-k|S|)}} q^{s(L)}. \tag{2}$$

We next show that Conjecture 3.2 implies three explicit generating functions for all k-good paths from $(0,0)$ to $(m+lk,m)$. We adopt the notation

$$L^{k,v}_{(m+lk,m)}(s) = \sum_{L \in L^{k,v}_{(m+lk,m)}} q^{s(L)},$$

where s denotes maj, $001-110$, or $maj+comaj$. Using (2) and the fact that the set $L_{(m+lk,m)}$ of all paths from $(0,0)$ to $(m+lk,m)$ is partitioned into

$$\bigcup_v L^{k,v}_{(m+lk,m)} = \bigcup_v \left(\bigcup_{S \subset \{1,\ldots,l+2v\}, |S|=v} L^{k,bad(S)}_{(m+lk,m)} \right),$$

we can consider the paths in $L_{(m+lk,m)}$ weighted by maj, $001-110$, or $maj+comaj$. For example, using maj and Conjecture 3.2 we have

$$\begin{bmatrix} lk+2m \\ m \end{bmatrix}_q = \sum_v L^{k,0}_{(m+(l+v)k,m-vk)}(maj) \begin{bmatrix} l+2v \\ v \end{bmatrix}_{q^k}.$$

Replacing m by $Mk+m$, where $0 \le m < k$, and l by $J-2M$ yields

$$B_{MJ} = \begin{bmatrix} Jk+2m \\ Mk+m \end{bmatrix}_q = \sum_{v=0}^{M} A_{Mv} L^{k,0}_{(m+(J-v)k,m+vk)}(maj), \tag{3}$$

where

$$A_{Mv} = \begin{bmatrix} J-2v \\ M-v \end{bmatrix}_{q^k}.$$

Clearly (3) is a matrix equation, $B = AL$; moreover, since A^{-1} is explicitly known [8],

$$A^{-1}_{vs} = \begin{cases} \begin{bmatrix} J-v-s \\ v-s \end{bmatrix}_{q^k} \frac{1-q^{k(J-2s)}}{1-q^{k(J-v-s)}} (-1)^{v-s} q^{k\binom{v-s}{2}}, & \text{if } v \ge s, \\ 0, \quad \text{if } v < s. \end{cases}$$

we have

$$L^{k,0}_{(m+(J-v)k,m+vk)}(maj) = \sum_{s=0}^{v} \begin{bmatrix} J-v-s \\ v-s \end{bmatrix}_{q^k} \frac{1-q^{k(J-2s)}}{1-q^{k(J-v-s)}} (-1)^{v-s} q^{k\binom{v-s}{2}} \begin{bmatrix} Jk+2m \\ sk+m \end{bmatrix}_q. \tag{4}$$

The $J=2v$ version of the above equation is

$$L^{(k,0)}_{(m+vk,m+vk)}(maj) = \sum_{s=-v}^{v} (-1)^s q^{k\binom{s}{2}} \begin{bmatrix} 2vk+2m \\ vk-sk+m \end{bmatrix}_q. \tag{5}$$

The steps of the previous paragraph may be repeated using the $001-110$ and $maj+comaj$ statistics. Note that these two statistics coincide on the lattice paths $L(S)$ of Conjecture 3.2. We also have the explicit inverse [8]

$$A^*_{Mv} = \begin{bmatrix} J-2v \\ M-v \end{bmatrix}^*_{q^k},$$

$$((A^*)^{-1})_{vs} = \begin{cases} \begin{bmatrix} J-v-s \\ v-s \end{bmatrix}_{q^{2k}} \frac{1-q^{k(J-2s)}}{1-q^{k(2J-2v-2s)}} (-1)^{v-s} q^{k((v-s)^2-2v)} (q^{kJ}+q^{2vk}), & \text{if } v \ge s, \\ 0, \qquad \text{if } v < s. \end{cases}$$

The results are

$$L^{k,0}_{(m+(J-v)k,m+vk)}(001-110) =$$
$$\sum_{s=0}^{v} \begin{bmatrix} J-v-s \\ v-s \end{bmatrix}_{q^{2k}} \frac{1-q^{k(J-2s)}}{1-q^{k(2J-2v-2s)}} (-1)^{v-s} q^{k((v-s)^2-2v)} (q^{kJ}+q^{2vk}) \begin{bmatrix} Jk+2m \\ sk+m \end{bmatrix}_q, \tag{6}$$

whose $J = 2v$ version is

$$L^{k,0}_{(m+vk,m+vk)}(001-110) = \sum_{s=-v}^{v} (-1)^s q^{ks^2} \begin{bmatrix} 2vk+2m \\ (v-s)k+m \end{bmatrix}_q, \tag{7}$$

and

$$L^{k,0}_{(m+(J-v)k,m+vk)}(maj+comaj) =$$
$$\sum_{s=0}^{v} \begin{bmatrix} J-v-s \\ v-s \end{bmatrix}_{q^{2k}} \frac{1-q^{k(J-2s)}}{1-q^{k(2J-2v-2s)}} (-1)^{v-s} q^{k((v-s)^2-2v)} (q^{kJ}+q^{2vk}) \begin{bmatrix} Jk+2m \\ sk+m \end{bmatrix}^*_q, \tag{8}$$

whose $J = 2v$ version is

$$L^{(k,0)}_{(m+vk,m+vk)}(maj+comaj) = \sum_{s=-v}^{v} (-1)^s q^{ks^2} \begin{bmatrix} 2vk+2m \\ (v-s)k+m \end{bmatrix}^*_q. \tag{9}$$

The equation (7) is a special case of Theorem 1 in [3].

Recall that in (4)-(9) we have $0 \le m < k$.

For $k = 2$, we have explicit evaluations of the sums in (4), (6), (8). We have replaced $m+2v$ by m and $m+2(J-v)$ by $m+2l$ in these three sums for the explicit statements given in Theorem 3.3 below. We shall prove Theorem 3.3 (without assuming Conjecture 3.1) in the next three sections. These three summations are very-well poised evaluations in the theory of basic hypergeometric series [8].

Theorem 3.3 *Let P be the set of all 2-good paths from $(0,0)$ to $(m+2l,m)$. Then we have*

1.
$$\sum_{L\in P} q^{maj(L)} = \begin{bmatrix} m+l \\ l \end{bmatrix}_{q^2} (-q;q)_m\, q^{\binom{m}{2}},$$

2.
$$\sum_{L\in P} q^{001-110(L)} = \begin{bmatrix} m+l \\ l \end{bmatrix}_{q^2} (-q;q^2)_m,$$

3.
$$\sum_{L\in P} q^{maj+comaj(L)} = \begin{bmatrix} m+l \\ l \end{bmatrix}_{q^2} (-q^{2l};q^2)_m\, q^{m^2}.$$

4 Maj statistics

In this section we prove Theorem 3.3 (1), on the *maj* statistic, by a weight preserving bijection.

The right side of Theorem 3.3 (1) can be considered as the generating function for all ordered triples

$$(\lambda, \sigma, \delta),$$

where $\lambda = \lambda_1\lambda_2\cdots\lambda_m$ is a partition with even parts, $\lambda_1 \le 2l$, $\delta = (m-1)(m-2)\cdots 1\,0$ is the staircase partition, and σ is any subset of $\{1, 2, \cdots, m\}$ weighted by the sum of the entries. The main idea is to use $(\lambda, \sigma, \delta)$ to find the Frobenius notation for a uniquely defined partition μ, then let $L = \psi_1(\mu)$.

To check that L is a 2-good path, we note that if $\mu = \begin{bmatrix} a_1 & a_2 & \dots & a_l \\ b_1 & b_2 & \dots & b_l \end{bmatrix}$, and $L = \psi_1(\mu)$, then L is a 2-good path in $R(m+2l, m)$ if and only if

1. $a_i \ge b_i$ for all i, $a_1 < m + 2l$ and $b_1 < m$,
2. b_l is 0 or 1, moreover, if $b_l = 1$, then a_l is odd,
3. $b_i - b_{i+1} \le 2$ for all i, moreover, if $b_i - b_{i+1} = 2$, then $a_i - b_i$ is even.

Two examples of 2-good paths in $R(20, 6)$ are given in Figure 3.

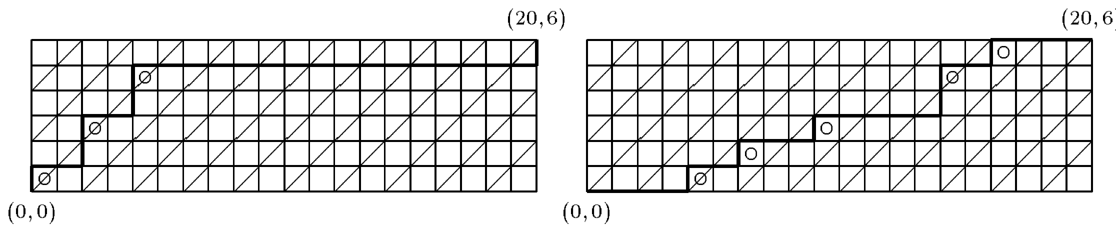

Figure 3. 2-good paths inside 20×6 rectangle with $(m, l) = (6, 7)$.

Given $(\lambda, \sigma, \delta)$, let

$$\pi = \lambda + \delta = \begin{bmatrix} a_1 & a_2 & \dots & a_l \\ b_1 & b_2 & \dots & b_l \end{bmatrix}.$$

Note that π satisfies conditions (1)-(3) above so that $\psi_1(\pi)$ is a 2-good path. We just insert the elements of the subset σ into π one at a time to obtain μ. At each stage the conditions (1)-(3) are preserved.

Suppose we want to insert $1 \in \sigma$ into π. If $b_l = 1$, then enlarge σ by setting $a_{l+1} = b_{l+1} = 0$; if $b_l = 0$, then increase a_l by 1. Suppose we want to insert 2. If $b_l = 1$, then enlarge σ by setting $a_{l+1} = b_{l+1} = 0$ and increase a_l by 1; if $b_l = 0$ and $b_{l-1} = 2$, then increase a_l and b_l by 1; if $b_l = 0$ and $b_{l-1} = 1$, then increase a_l and a_{l-1} by 1. Suppose we want to insert k. If $b_l = 1$, then enlarge σ by setting $a_{l+1} = b_{l+1} = 0$ and increase $k-1$ entries by 1 among $a_l, b_l, a_{l-1}, b_{l-1}, \ldots$; if $b_l = 0$, then increase k entries by 1 among $a_l, b_l, a_{l-1}, b_{l-1}, \ldots$. If some b_i cannot be increased, then we skip it. Note that b_i cannot be increased if $b_i = b_{i-1} - 1$, but a_i can always be increased, since $a_i < a_{i-1} - 1$. Any subset σ can be inserted into π; the largest entry is inserted first and the second largest and so on. Let μ denote the resulting partition.

An example is given in Figure 4. Inserting 5, then 4, and finally 1 into

$$\pi = \begin{bmatrix} 16 & 14 & 10 & 6 & 4 \\ 5 & 4 & 2 & 1 & 0 \end{bmatrix}$$

yields

$$\begin{bmatrix} 16 & 15 & 11 & 7 & 5 \\ 5 & 4 & 3 & 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 16 & 15 & 12 & 8 & 6 \\ 5 & 4 & 3 & 2 & 0 \end{bmatrix}, \quad \mu = \begin{bmatrix} 16 & 15 & 12 & 8 & 7 \\ 5 & 4 & 3 & 2 & 0 \end{bmatrix}.$$

The details that the insertion procedure gives the bijection are left to the reader.

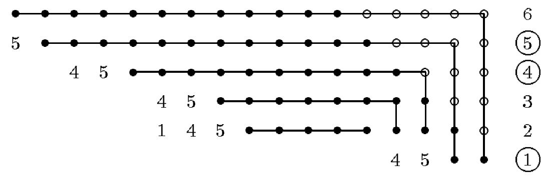

Figure 4. $(m,l)=(6,7)$, $\lambda = 12\,12\,10\,8\,8\,2$, $\delta = 543210$, $\sigma=\{1,4,5\}$
$\pi = \lambda+\delta = \begin{bmatrix} 16 & 14 & 10 & 6 & 4 \\ 5 & 4 & 2 & 1 & 0 \end{bmatrix}$, $\mu = \pi+\sigma = \begin{bmatrix} 16 & 15 & 12 & 8 & 7 \\ 5 & 4 & 3 & 2 & 0 \end{bmatrix}$.

5 $Maj+comaj$ statistics

In this section we prove Theorem 3.3 (3), on the $maj{+}comaj$ statistic, by a weight preserving bijection. The method of proof is somewhat different from §4. We find a two weight preserving bijections: one for each side of Theorem 3.3 (2), to a third set.

First we find another way to compute the $maj+comaj$ weight of the 2-good paths. This is our first bijection.

Let $L=(s_1,s_2,\cdots,s_{2m+2l})$ be a 2-good path inside $R(m+2l,m)$. Since L is 2-good, 1 occurs consecutively at most twice. Let s_{j_i} be the i-th 1. Then $s_{j_i-1}s_{j_i}s_{j_i+1}$ is either 010, 110, or 011. The second case occurs only if j_i is odd, and the third case occurs only if j_i is even. (We assume that $s_0=s_{2m+2l+1}=1$.)

For $i\in\{1,\cdots,m\}$, we define an integer p_i as follows:

$$p_i = \begin{cases} 2j_i-2i, & \text{if } s_{j_i-1}s_{j_i}s_{j_i+1}=010,\\ j_i-2i+1, & \text{if } j_i \text{ is odd and } s_{j_i-1}s_{j_i}s_{j_i+1}=110,\\ j_i-2i, & \text{if } j_i \text{ is even and } j_i<2m+2l \text{ and } s_{j_i-1}s_{j_i}s_{j_i+1}=011,\\ \underline{2l}, & \text{if } j_i=2m+2l \text{ and } s_{j_i-1}s_{j_i}s_{j_i+1}=011. \end{cases}$$

(We think of $\underline{2l}$ as colored version of $2l$.) Clearly, p_i is even for all i and

$$maj+comaj(L)=\sum_{i=1}^{m}(2i-1+p_i)=m^2+\sum_{i=1}^{m}p_i.$$

The p_i's just described measure how far the path is away from the 'base' path (for instance, the first path in Figure 5). For example, if $(m,l)=(6,7)$, $L=00001000000101000110010000$, (the second path in Figure 5) then $p=(8,20,22,10,10,32)$,

The path L can be recovered from the sequence p. The leader of any string in p of odd length must correspond to a 010 in L. Any consecutive pair $p_i=p_{i+1}$ in a string must correspond to an occurrence of 11 in L.

We can also classify the sequences p by inequalities. For example, suppose $t+1$ consecutive pairs $p_i=p_{i+1},\cdots,p_{i+2t}=p_{i+2t+1}$ lie between two leaders p_{i-1} and p_{i+2t+2}. Since

$$j_{i+2t+1}+2\le j_{i+2t+2},\quad j_{i+2s-1}+3\le j_{i+2s}\quad \text{for } 1\le s\le t,$$

and

$$j_{i-1}\le\begin{cases} j_i-3, & \text{for } j_{i-1} \text{ even},\\ j_i-2, & \text{for } j_{i-1} \text{ odd},\end{cases}$$

we must have

$$p_i \le p_{i+2} \le \cdots \le p_{i+2t}, \quad 2p_{i+2t+1} + 2i + 4t + 4 \le p_{i+2t+2}$$

and

$$p_{i-1} \le \begin{cases} 2p_i + 2i - 4, & \text{for } j_{i-1} \text{ even,} \\ 2p_i + 2i - 2, & \text{for } j_{i-1} \text{ odd.} \end{cases}$$

In our example $p = (8, 20, 22, 10, 10, 32)$, we see that the above inequalities imply that only $9, 9$ or $10, 10$ could be inserted between 22 and 32.

Sort $p = p_1 p_2 \cdots p_m$ in descending order, assuming that $\underline{2l}$ is larger than anything else, to obtain $U = u_1 u_2 \cdots u_m$. The set of sequences U is our third set. In our example, $U = (32, 22, 20, 10, 10, 8)$. Again p can be recovered from U using the above inequalities, but we do not give these details.

We obtain the same sequences U from the left side of Theorem 3.3 (2). Again let $\lambda = \lambda_1 \lambda_2 \cdots \lambda_m$ be a partition with even parts, and $\lambda_1 \le 2l$. Let σ be a subset of $\{2l, 2l+2, \cdots, 2l+2m-2\}$. Define a sequence $T = t_1 t_2 \cdots t_m$ by

$$t_i = \begin{cases} \lambda_i + 2m + 2l - 2i, & \text{if } 2m + 2l - 2i \in \sigma, \\ \lambda_i, & \text{else,} \end{cases}$$

for all i. If t_m is $2l$ with $\lambda_m = 0$, then write it as $\underline{2l}$. Sort T in descending order, again assuming that $\underline{2l}$ is larger than anything else, and let the sorted sequence be $U = u_1 u_2 \cdots u_m$. For example, if $(m, l) = (6, 7)$, $\lambda = (10, 10, 10, 8, 6, 6)$, and $\sigma = \{14, 16, 22\}$, we have $T = (10, 32, 10, 8, 22, 20)$, $U = (32, 22, 20, 10, 10, 8)$.

We claim that from U we can recover $T = t_1 t_2 \cdots t_m$, λ and σ. Let $U = u_1 u_2 \cdots u_m$ be the sorted sequence. We determine $\lambda = \lambda_1 \lambda_2 \cdots \lambda_m$ and σ as follows. Assume that $u_1 \ne \underline{2l}$. If all the u_i's are less than or equal to $2l$, then let $u_i = \lambda_i$ and $\sigma = \varnothing$. Otherwise, we declare that each u_i greater than $2l$ is in the wrong position. If the sequence U has any number in a wrong position, then let j be the largest index such that u_j is in a wrong position. If $u_j - (2m + 2l - 2j) \ge u_{j+1}$, then we now declare that u_j is in the right position; otherwise, switch u_j and u_{j+1}. Repeat the same procedure with the resulting sequence until all the u_j's greater than $2l$ are in a right position. The resulting sequence is $T = t_1 t_2 \cdots t_m$. If $t_i \ge 2m + 2l - 2i$, then add $2m + 2l - 2i$ to σ and set $\lambda_i = t_i - (2m + 2l - 2i)$; else set $\lambda_i = t_i$. If $u_1 = \underline{2l}$, then set $\lambda_m = 0$ and declare that $2l$ is an element of σ and proceed as before.

On the example $(m, l) = (6, 7)$, $U = (32, 22, 20, 10, 10, 8)$, we first test 20, which is in a wrong position. It swaps with 10, 10 and 8 to obtain $(32, 22, 10, 10, 8, 20)$. Then the 22 swaps with 10, 10, and 8 to obtain $(32, 10, 10, 8, 22, 20)$, and 32 swaps with 10 to obtain $T = (10, 32, 10, 8, 22, 20)$.

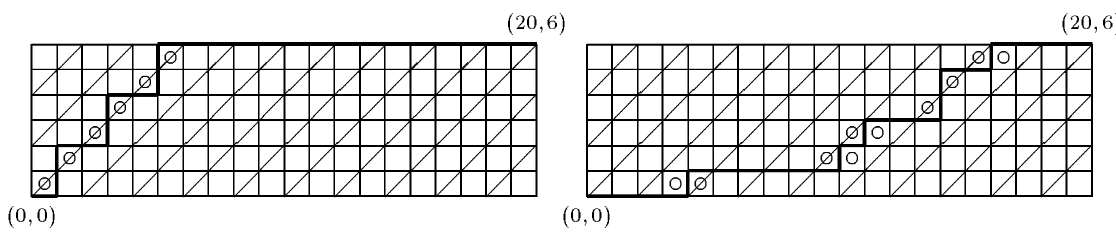

Figure 5. 2-good paths inside 20×6 rectangle with $(m, l) = (6, 7)$.
$\sigma = \{14, 16, 22\}$, $\lambda = 10\,10\,10\,8\,6\,6$.

6 $001 - 110$ statistics

In this section we prove Theorem 3.3 (2), on the $001 - 110$ statistic, by a weight preserving bijection, similar to §4.

Again let $\lambda = \lambda_1\lambda_2\cdots\lambda_m$ be a partition with even parts, and $\lambda_1 \le 2l$, and let σ be a subset of $\{1, 2, \cdots, m\}$. We consider $i \in \sigma$ as the self-conjugate hook of size $(2i-1)$.

Given a pair (λ, σ), we generate a 2-good path L from $(0,0)$ to $(m+2l, m)$. We produce the marked squares which uniquely determine the path, as in ψ_2

First divide each part of λ by 2 to obtain $\tilde{\lambda}$. We compare the largest hook in $\tilde{\lambda}$ (arm x, leg y, hook length $x+y-1$) with the largest hook $(2t-1)$ in σ. If $y > t$, then remove the hook from $\tilde{\lambda}$ and mark the square $(2x+y-2, y-1)$, else remove the largest row from $\tilde{\lambda}$ and the largest element from σ, and mark the square $(2x+t-1, t-1)$. Repeat this process until $\tilde{\lambda}$ becomes empty. Let L be the lattice path whose exposed corner cells, 01- or 10-, are exactly the marked squares.

We claim that L is a 2-good path. Suppose that (i_1, j_1) and (i_2, j_2), $j_1 < j_2$, are two consecutive exposed corner cells. Recall that between two consecutive exposed cells, the lattice path is a staircase, followed by either a long vertical or horizontal path. We must show that the long vertical path does not occur. We first assume that $i_1 + j_1$ and $i_2 + j_2$ are odd. Then $(i_1, j_1) = (2a_1 + l_1 - 2, l_1 - 1)$ and $(i_2, j_2) = (2a_2 + l_2 - 2, l_2 - 1)$ for some (a_1, l_1) and (a_2, l_2), where $a_1 < a_2$ and $l_1 < l_2$. So we have $j_1 - i_1 \ge j_2 - i_2 + 2$, which implies that there is no vertical path between the exposed corner cells (i_1, j_1) and (i_2, j_2). We now assume that $i_1 + j_1$ and $i_2 + j_2$ are even. Then $(i_1, j_1) = (2a_1 + t_1 - 1, t_1 - 1)$ and $(i_2, j_2) = (2a_2 + t_2 - 1, t_2 - 1)$ for some a_1, t_1, a_2, t_2, where $a_1 \le a_2$ and $t_1 < t_2$. So we have $j_1 - i_1 \ge j_2 - i_2$, which, in this case, implies that there is no violation between the exposed corner cells (i_1, j_1) and (i_2, j_2). The other cases are similar.

The inverse mapping is clear.

In the example of Figure 7, $(m, l) = (8, 10)$, $\lambda = (18, 16, 14, 14, 8, 8, 6, 2)$, $\tilde{\lambda} = (9, 8, 7, 7, 4, 4, 3, 1)$, and $\sigma = \{1, 4, 5, 8\}$ (hooks of size $1, 7, 9, 17$.) Here are the steps of the algorithm:
leg $y = 8 = t$, $(8, 7, 7, 4, 4, 3, 1)$, $\{1, 4, 5\}$ mark cell $(25, 7)$,
leg $y = 7, t = 5$, $(6, 6, 3, 3, 2)$, $\{1, 4, 5\}$ mark cell $(21, 6)$,
leg $y = 5 = t$, $(6, 3, 3, 2)$, $\{1, 4\}$ mark cell $(16, 4)$,
leg $y = 4 = t$, $(3, 3, 2)$, $\{1\}$ mark cell $(15, 3)$,
leg $y = 3, t = 1$, $(2, 1)$, $\{1\}$ mark cell $(7, 2)$,
leg $y = 2, t = 1$, $\varnothing$, $\{1\}$ mark cell $(4, 1)$,
leg $y = 0, t = 1$, $\varnothing$, $\varnothing$ mark cell $(0, 0)$.

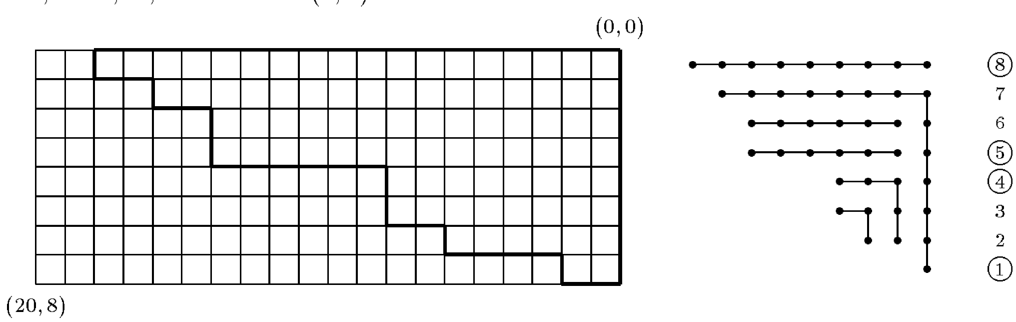

Figure 6. $(m, l) = (8, 10)$, $\tilde{\lambda} = 9\,8\,7\,7\,4\,4\,3\,1$, $\sigma = \{1, 4, 5, 8\}$.

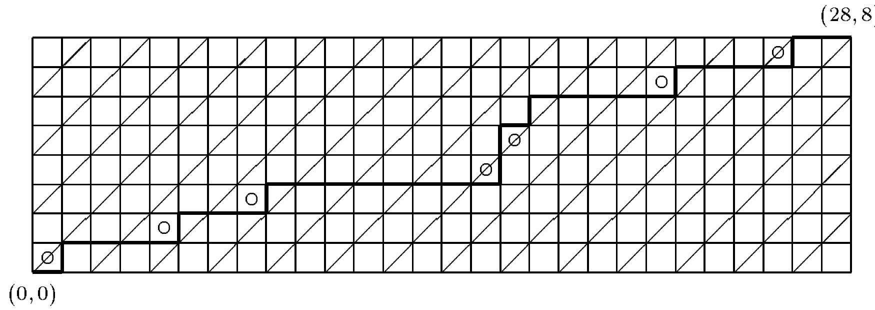

Figure 7. $L \in L^{2,0}_{(28,8)}$ corresponding to $\tilde{\lambda} = 9\,8\,7\,7\,4\,4\,3\,1$, $\sigma = \{1,4,5,8\}$.

7 Remarks and a limiting case

Conjecture 3.2 does not hold for the *inv* statistic. We do have a proof of Conjecture 3.2 for $k=2$ and $s = 001-110$. However it uses special properties of the $k=2$ paths, which are not easy to generalize.

It is possible to explicitly state the conditions on a partition π such that $\psi_2(\pi) = P$, for $P \in L^{k,0}_{(m+lk,m)}$. Here we take $k \geq 3$. (The conditions for $k=2$ are given in §4.) If

$$\pi = \begin{bmatrix} a_1 & a_2 & \cdots & a_t \\ b_1 & b_2 & \cdots & b_t \end{bmatrix},$$

then we have

$a_1 \leq m + lk - 1, \quad t \leq m$
$a_i \leq b_i + (l+1)k - 2$ for all i,
if $a_t < b_t$ then $a_t \geq b_t - (k-2)$,
if $a_t \geq b_t$ then $bound_t := \lfloor (a_t + 1 - b_t)/k \rfloor$,
if $b_{i-1} - b_i > a_{i-1} - a_i$ then $bound_{i-1} := bound_i$, $a_{i-1} - b_{i-1} \geq k * bound_{i-1} + 2$,
if $b_{i-1} - b_i \leq a_{i-1} - a_i$ then $bound_{i-1} := max\{bound_i, \lfloor (a_{i-1} + 1 - b_{i-1})/k \rfloor\}$,

For example, if $l=0$, then $bound_i = -1$ for all i and the above conditions become $|a_i - b_i| \leq k-2$, the hook difference conditions in [3].

We can take the $m \to \infty$ limit in the above conditions and find the generating function of the resulting partitions by using (6),

$$\frac{1}{(q;q)_\infty} \sum_{s=0}^{\infty} \begin{bmatrix} l+s \\ s \end{bmatrix}_{q^{2k}} \frac{(1-q^{(2s+l)k})(1+q^{lk})}{(1-q^{(2s+2l)k})(1-q^{lk})} (-1)^s q^{ks^2} \tag{10}$$

Thus sum is evaluable by a limiting case of the very-well poised ${}_6\phi_5$ evaluation [8], to

$$\frac{(q^{2k}, q^k, q^k; q^{2k})_\infty}{(q;q)_\infty (q^k;q^k)_l}. \tag{11}$$

The case $l=0$ appears in [3], (Theorem 3).

It follows from the q-binomial theorem that the generating function in l of (11) is

$$\frac{(q^{2k}, q^k, q^k; q^{2k})_\infty}{(q;q)_\infty (x;q^k)_\infty} = \frac{(q^{2k}, q^k, q^k; q^{2k})_\infty}{(q;q)_\infty} F(x/q^k, 0, q^k). \tag{12}$$

The limiting partitions can be described by modifying the bijection ψ_2 to be independent of m and n: just truncate P after the last marked cell. Thus (12) can be considered the $001-110$ generating function for all such paths which are j-good, $j \leq k$. So if we multiply (12) by $(1-x)$, we obtain the generating function for paths which are exactly k-good,

$$\frac{(q^{2k}, q^k, q^k; q^{2k})_\infty}{(q;q)_\infty} F(x, 0, q^k).$$

This motivates the next conjecture. For a partition π, let $r(\pi)$ $(u(\pi))$ be the number of 0's (1's) in the k-type of the truncation of $\psi_2(\pi)$. We make the following conjecture:

Conjecture 7.1

$$\sum_{\pi:\ a\ partition} q^{|\pi|} x^{r(\pi)} y^{u(\pi)} = \frac{(q^{2k}, q^k, q^k; q^k)_\infty}{(q;q)_\infty} F(x, y, q^k).$$

The fraction in Conjecture 7.1 is the generating function for partitions with $|a_i - b_i| \leq k-2$, i.e. paths strictly inside the lines $y = x \pm k$. We have already seen in (1) that the coefficients in the expansion of $F(x, y, q^k)$ are positive, and represent certain paths with steps of size k. To verify Conjecture 7.1, it remains to find an appropriate insertion algorithm of the k-good paths of k length 0 into these larger paths.

In the mathematical physics literature [7], [11] the sum (7) is called bosonic, and has an explicit fermionic representation which shows it is positive. (5) also has such a fermionic sum [12]. We do not know the corresponding fermionic sums for (4), (6), (8), although it is likely they exist [12].

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