UNIMODALITY AND YOUNG'S LATTICE

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Abstract. Young's lattice of a partition λ consists of all partitions whose Ferrers diagrams fit inside λ. Several infinite families of partitions are given whose Young's lattice is not rank unimodal. Some related problems are discussed.

1. Introduction.

It is well known that the q-binomial coefficient

$$\begin{bmatrix} n+m \\ m \end{bmatrix}_q \tag{1.1}$$

is a symmetric unimodal polynomial in q (see, e.g. [1, §3.5]). Recall that a sequence of integers a_i is *unimodal* if there exists an integer N such that

$$a_0 \le a_1 \le \cdots \le a_N \ge a_{N+1} \ge a_{N+2} \ge \cdots .$$

A polynomial is called unimodal if its sequence of coefficients is unimodal. For the q-binomial coefficient in (1.1), $N = nm/2$, half of the degree of the polynomial.

Combinatorially, the q-binomial coefficient has the following interpretation. If a_i is the number of partitions of i which lie inside an $n \times m$ rectangle, then a_i is the coefficient of q^i in (1.1). This is another way of saying that the q-binomial coefficient is the generating function for all partitions which lie inside an $n \times m$ rectangle. These partitions are the elements of a lattice called *Young's lattice*, whose order relation is given by containment of the respective Ferrers diagrams.

Instead of a rectangle, we can consider Young's lattice for any partition λ. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$, where $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_k \ge 1$, and call the lattice $\mathcal{Y}_\lambda$. The purpose of this paper is to study the unimodality properties of $\mathcal{Y}_\lambda$.

We let $G(\mathcal{Y}_\lambda)(q)$ be the generating function for all partitions which lie inside λ. If λ' denotes the conjugate of the partition λ, it is clear that

$$G(\mathcal{Y}_\lambda)(q) = G(\mathcal{Y}_{\lambda'})(q). \tag{1.2}$$

We will call a partition λ *unimodal* if $G(\mathcal{Y}_\lambda)(q)$ is a unimodal polynomial. Note that the non-unimodality of λ is equivalent to the following condition on the coefficients b_i of $(1-q)G(\mathcal{Y}_\lambda)(q)$. There is some $i < j$ satisfying $b_i < 0$ and $b_j > 0$.

In §2 we give the data from the programs which were written for this problem. The theorems which are suggested from the data are stated and proved in §3. Some final remarks, including observations and conjectures, are given in §4. We will use the notation $\lfloor x \rfloor$ and $\lceil x \rceil$ for the greatest integer $\le x$ and the least integer $\ge x$ respectively.

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2. Data.

All partitions of $n \leq 50$ were tested. (There are 204226 partitions of 50.) *Not all partitions λ are unimodal.* The first non-unimodal λ is $\lambda = (8, 8, 4, 4)$, with coefficients

$$1\ 1\ 2\ 3\ 5\ 6\ 9\ 11\ 15\ 17\ 21\ 23\ 27\ 28\ 31\ 30\ 31\ 27\ 24\ 18\ 14\ 8\ 5\ 2\ 1.$$

It is true that all partitions of $n \leq 23$, or all partitions which lie inside a 7×7 square, are unimodal. The following table lists the non-unimodal partitions of $n \leq 36$. Because of (1.2), we list only one of λ and λ'. The value of i for which unimodality fails, and the three offending values a_{i-1}, a_i, and a_{i+1} are also given.

Partition	i	Values	Partition	i	Values
8 8 4 4	15	31 30 31	11 11 6 6	21	67 66 67
10 9 4 4	17	46 45 46	14 13 4 4	21	76 75 76
10 10 4 4	17	46 45 46	16 12 4 4	23	91 90 91
12 10 4 4	19	61 60 61	14 14 4 4	21	76 75 76
12 11 4 4	19	61 60 61	12 12 8 4	23	81 80 81
12 12 4 4	19	61 60 61	12 10 8 6	23	82 81 82
14 11 4 4	21	76 75 76	8 8 8 6 4 2	23	141 140 141
11 11 6 5	21	67 66 67	8 8 6 6 4 4	23	144 143 144
14 12 4 4	21	76 75 76			

Table 1

Many of the partitions on the previous list have the form $\lambda = (a, a, b, b)$. The following table lists all non-unimodal partitions of this form with $a \leq 24$.

Partition	i	Values	Partition	i	Values
8 8 4 4	15	31 30 31	20 20 4 4	27	121 120 121
10 10 4 4	17	46 45 46	20 20 10 10	37	297 296 298
11 11 6 6	21	67 66 67	20 20 12 12	39	314 313 316
12 12 4 4	19	61 60 61	21 21 8 8	35	278 277 278
14 14 4 4	21	76 75 76	21 21 12 12	41	356 354 355
14 14 8 8	27	123 122 123	22 22 4 4	29	136 135 136
16 16 4 4	23	91 90 91	22 22 11 11	41	384 382 383
16 16 9 9	31	173 172 173	22 22 13 13	43	405 404 406
17 17 8 8	31	188 187 188	23 23 8 8	37	323 322 323
17 17 10 10	33	204 203 204	23 23 14 14	37	458 457 460
18 18 4 4	25	106 105 106	24 24 4 4	31	151 150 151
18 18 10 10	35	235 233 234	24 24 11 11	43	460 459 461
19 19 8 8	33	233 232 233	24 24 14 14	47	512 510 512
19 19 11 11	33	273 272 273			

Table 2

It is also of interest to test $\lambda = (a, a, b, b)$ for particular values of a. Table 3 takes $a = 90$ and $a = 89$.

Partition	i	Values	Partition	i	Values
90 90 58 58	179	21973 21971 21984	89 89 58 58	177	21270 21269 21283
90 90 57 57	179	21964 21959 21968	89 89 57 57	177	21266 21263 21274
90 90 56 56	179	21944 21934 21938	89 89 56 56	177	21254 21247 21254
90 90 54 54	175	21847 21843 21852	89 89 55 55	177	21229 21217 21218
90 90 52 52	175	21682 21681 21693	89 89 53 53	175	21123 21117 21123

90 90 51 51	175	21582 21571 21572	89 89 51 51	173	20948 20944 20951
90 90 50 50	173	21440 21439 21450	89 89 49 49	171	20693 20688 20694
90 90 48 48	171	21111 21107 21114	89 89 47 47	169	20349 20340 20341
90 90 45 45	167	20426 20423 20430	89 89 46 46	167	20131 20130 20139
90 90 42 42	163	19515 19506 19507	89 89 43 43	163	19352 19349 19355
90 90 41 41	161	19153 19149 19154	89 89 38 38	155	17534 17529 17532
90 90 35 35	151	16484 16477 16478	89 89 37 37	153	17091 17089 17095
90 90 34 34	149	15962 15958 15962	89 89 31 31	143	14049 14044 14045
90 90 33 33	147	15423 15422 15428	89 89 30 30	141	13488 13485 13489
90 90 27 27	137	11963 11959 11961	89 89 23 23	129	9397 9394 9396
90 90 26 26	135	11359 11357 11361	89 89 22 22	127	8806 8805 8809
90 90 20 20	125	7761 7757 7758	89 89 16 16	117	5425 5422 5423
90 90 19 19	123	7178 7176 7178	89 89 15 15	115	4901 4900 4902
90 90 18 18	121	6605 6604 6608	89 89 8 8	103	1808 1807 1808
90 90 12 12	111	3487 3485 3486			
90 90 11 11	109	3034 3033 3035			
90 90 4 4	97	646 645 646			

Table 3

Table 4 gives the number of partitions of n $(p(n))$ and the number of non-unimodal partitions of n $(NU(n))$ for $n \leq 50$.

n	p(n)	NU(n)	n	p(n)	NU(n)	n	p(n)	NU(n)
24	1575	2	33	10143	4	42	53174	16
25	1958	0	34	12310	4	43	63261	14
26	2436	0	35	14883	2	44	75175	14
27	3010	2	36	17977	12	45	89134	18
28	3718	2	37	21637	14	46	105558	24
29	4565	0	38	26015	20	47	124754	26
30	5604	2	39	31185	16	48	147273	32
31	6842	2	40	37338	16	49	173525	40
32	8349	2	41	44583	6	50	204226	40

Table 4

3. Theorems.

Unfortunately it is not possible to completely classify the non-unimodal partitions λ. In this section we will give several infinite families of partitions which are not unimodal in Theorems 3, 4, 5, 6, 8, and 9. We also give in Theorems 7 and Theorem 11 two infinite families of unimodal partitions.

From Table 2 it appears that the following theorem holds.

Theorem 1. *The partition* $\lambda = (2k, 2k, 4, 4)$ *is non-unimodal for* $k \geq 4$ *at* $i = 2k + 7$, *with consecutive differences of -1 and 1.*

We do not prove Theorem 1 here, because Theorem 3 generalizes Theorem 1. Table 2 also indicates that a similar theorem should hold for $(2k, 2k, 11, 11)$. Note that both 4 and 11 occur on Table 3 for $a = 90$. Then Table 3 might indicate that there is a similar theorem for 4, 11, 12, 18, 19, 20, 26, 27, 33, 34, 35, 41, 42, and 45.

For $\lambda = (2k+1, 2k+1, 8, 8)$ we have the next theorem.

Theorem 2. *The partition* $\lambda = (2k+1, 2k+1, 8, 8)$ *is non-unimodal for* $k \geq 8$ *at* $i = 2k + 15$, *with consecutive differences of -1 and 1.*

Again Table 3 indicates that a similar theorem may hold for 8, 15, 16, 22, 23, 30, 31, 37, 38, and 43.

We now come to the theorems for partitions $\lambda = (a, a, b, b)$ which give the above two sequences of $b's$, and generalize Theorems 1 and 2.

Theorem 3. *Let* a *be an even integer satisfying* $a \geq (4 - \sqrt{3})b + (5 - \sqrt{3})$. *If* b *satisfies*

(1) $\lfloor \sqrt{3}(b+1) \rfloor$ *is even, and*
(2) $\sqrt{3(b+1)^2 + 6} \leq \lfloor \sqrt{3}(b+1) \rfloor + 1 \leq \sqrt{3(b+2)^2 - 8} - 1$,

then $\lambda = (a, a, b, b)$ *is non-unimodal at* $i = a + \lfloor \sqrt{3}(b+1) \rfloor - 1$. *The consecutive differences are*

$$\lceil (3b^2 + 6b - (\lfloor \sqrt{3}(b+1) \rfloor - 2)^2 - 6(\lfloor \sqrt{3}(b+1) \rfloor - 2) - 12)/12 \rceil$$

and

$$\lceil (3b^2 + 12b - (\lfloor \sqrt{3}(b+1) \rfloor - 2)^2 - 6(\lfloor \sqrt{3}(b+1) \rfloor - 2))/12 \rceil.$$

Theorem 4. *Let* a *be an odd integer satisfying* $a \geq (4 - \sqrt{3})b + (5 - \sqrt{3})$. *If* b *satisfies*

(1) $\lfloor \sqrt{3}(b+1) \rfloor$ *is odd, and*
(2) $\sqrt{3(b+1)^2 + 9} \leq \lfloor \sqrt{3}(b+1) \rfloor + 1 \leq \sqrt{3(b+2)^2 - 11} - 1$,

then $\lambda = (a, a, b, b)$ *is non-unimodal at* $i = a + \lfloor \sqrt{3}(b+1) \rfloor - 1$. *The consecutive differences are*

$$\lceil (3b^2 + 6b - (\lfloor \sqrt{3}(b+1) \rfloor - 2)^2 - 6(\lfloor \sqrt{3}(b+1) \rfloor - 2) - 9)/12 \rceil$$

and

$$\lceil (3b^2 + 12b - (\lfloor \sqrt{3}(b+1) \rfloor - 2)^2 - 6(\lfloor \sqrt{3}(b+1) \rfloor - 2) - 3)/12 \rceil.$$

Proof. We prove Theorem 3. A straightforward but tedious calculation shows that

(3.1)
$$\begin{aligned}(1-q)G(\mathcal{Y}_\lambda)(q) =& \frac{1}{(1-q^2)(1-q^3)(1-q^4)} - \frac{q^{3b+3}}{(1-q)(1-q^2)(1-q^3)} \\ &+ \frac{q^{4b+5}}{(1-q)(1-q^2)(1-q^4)} - \frac{q^{a+1}}{(1-q)(1-q^2)(1-q^3)} \\ &+ \frac{q^{a+2b+3}}{(1-q)^2(1-q^2)} - \frac{q^{a+3b+5}}{(1-q)^2(1-q^3)} \\ &+ \frac{q^{2a+3}}{(1-q)(1-q^2)^2} - \frac{q^{2a+b+4}}{(1-q)^2(1-q^2)} \\ &+ \frac{q^{2a+2b+6}}{(1-q)(1-q^2)^2}.\end{aligned}$$

Clearly each term in (3.1) can be expanded in a Taylor series in q, with coefficients of q^n which are pseudo polynomials in n [7, §4.4]. Assume for the time being that $a \geq 4b+5$. Then for n in the interval from $a+1$ to $a+2b+2$, only the first four terms of (3.1) contribute. A MACSYMA run using these explicit pseudo polynomials shows that the coefficient of q^{a+j+1} is
(3.2)
$$\begin{aligned}\left\lceil \frac{3b^2+6b-j^2-6j-12}{12} \right\rceil &= \left\lfloor \frac{3b^2+6b-j^2-6j-5}{12} \right\rfloor \text{ for } a \text{ even and } j \text{ even},\\ \left\lceil \frac{3b^2+12b-j^2-6j}{12} \right\rceil &= \left\lfloor \frac{3b^2+12b-j^2-6j+7}{12} \right\rfloor \text{ for } a \text{ even and } j \text{ odd},\\ \left\lceil \frac{3b^2+12b-j^2-6j-3}{12} \right\rceil &= \left\lfloor \frac{3b^2+12b-j^2-6j+4}{12} \right\rfloor \text{ for } a \text{ odd and } j \text{ even},\\ \left\lceil \frac{3b^2+6b-j^2-6j-9}{12} \right\rceil &= \left\lfloor \frac{3b^2+6b-j^2-6j-2}{12} \right\rfloor \text{ for } a \text{ odd and } j \text{ odd}.\end{aligned}$$

Some elementary algebra then implies Theorem 3 for $a \geq 4b+5$. This inequality on a may be relaxed to $4b+5 \leq a + \lfloor\sqrt{3}(b+1)\rfloor$, so that the four terms of (3.1) still contribute to the two offending terms. □

We see that the sequence of $b's$ for Theorem 3 (Theorem 4) does not include 45 (44) as suspected. It does appear that the allowed $b's$ for Theorem 3 lie in residue classes modulo 15. However this is not correct. It can be shown, for example, that $b = 15m + 11$, $0 \leq m \leq 26$ satisfies Theorem 3, but $b = 15 \times 27 + 11$ does not. Strictly speaking, Theorem 3 (Theorem 4) with $b = 4$ ($b = 8$) implies Theorem 1 (Theorem 2) for $k \geq 7$ ($k \geq 11$). Nevertheless, these two theorems can be established independent of Theorems 3 and 4.

Note also that condition (1) in Theorems 3 and 4 implies that a given b may not satisfy both theorems. Because $N\sqrt{3} - \lfloor N\sqrt{3}\rfloor$is equidistributed on [0,1) ([5, Prob. 166]), it can be shown that the density of the $b's$ satisfying Theorem 3 or 4 is $(\sqrt{3}-1)/2$.

It is also clear that the bound for a in Theorems 3 and 4 is not the best possible, for example one might conjecture that $a \geq 2b$ is sufficient. However, $b = 12$ is allowed by Theorem 3 and $(24, 24, 12, 12)$ is unimodal. (It does not appear on

Table 2.) Moreover $a > 2b$ is not sufficient, for $b = 35$, $a \geq 78$. It is possible to give a general theorem in the range $2b+2 \leq a+1 \leq 3b+3$, but the inequalities are not as nice as condition (2) in Theorem 3. For the range $3b/2+1 \leq a+1 \leq 2b+1$, for example $(11, 11, 6, 6)$, there is another simple sufficient condition, which we state in the next two theorems.

Theorem 5. *If $k \geq 2$ and $2 \leq t \leq (1+\sqrt{1+24k})/4$, then the partition $\lambda = (3k+t, 3k+t, 2k, 2k)$ is non-unimodal at $i = 6k+2t-1$. The consecutive differences are*

$$-\left\lceil \frac{t^2-t}{3} \right\rceil$$

and

$$\left\lfloor \frac{3k-2t^2+t+6}{6} \right\rfloor.$$

Theorem 6. *If $k \geq 2$ and $2 \leq t \leq (1+\sqrt{1+24k})/4$, then the partition $\lambda = (3k+t+2, 3k+t+2, 2k+1, 2k+1)$ is non-unimodal at $i = 6k+2t+3$. The consecutive differences are*

$$-\left\lfloor \frac{t^2-1}{3} \right\rfloor$$

and

$$\left\lfloor \frac{3k-2t^2-t+6}{6} \right\rfloor.$$

Proof. This time three terms of (3.1) contribute to the coefficient of q^{3b+3+j}, for $0 \leq j \leq a-b$. The terms given in Theorems 5 and 6 are the differences given by MACSYMA, and the inequality on t insures that the differences are negative and positive. $\square$

Next we see that Table 1 lists partitions with four or six parts, which suggests that a partition with at most three parts is unimodal. This is true, and we will give a proof similar to the proof of Theorems 5 and 6. However the computations can be simplified by using the following lemma.

Lemma 1. *For any partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$, we have*

$$(1-q)G(\mathcal{Y}_\lambda)(q) = G(\mathcal{Z})(q) - q^{\lambda_1+1}G(\mathcal{Y}_\mu)(q),$$

where $\mathcal{Z}$ is the set of all partitions inside λ whose first two parts are equal, and μ is the partition $(\lambda_2, \lambda_3, \ldots, \lambda_k)$.

Proof. Let $\mathcal{A} = \{\emptyset, 1\}$ and consider the set $\mathcal{Y} \times \mathcal{A}$ whose generating function is $(1-q)G(\mathcal{Y}_\lambda)(q)$ if the sign of $1 \in \mathcal{A}$ is -1. A sign-reversing involution σ on $\mathcal{Y} \times \mathcal{A}$ is given by $\sigma((\gamma, \emptyset)) = (\mu, 1)$, where $\mu = (\gamma_1 - 1, \gamma_2, \ldots, \gamma_k)$ if $\gamma_1 > \gamma_2$; and $\sigma((\gamma, 1)) = (\mu, \emptyset)$, where $\mu = (\gamma_1 + 1, \gamma_2, \ldots, \gamma_k)$ if $\gamma_1 < \lambda_1$. Clearly the fixed points of σ have $\gamma_1 = \gamma_2$ or $\gamma_1 = \lambda_1$, whose generating function is given in Lemma 1. $\square$

Proposition 1. *If* $\lambda = (a, b, c)$, *then*

(3.3)

$$(1-q)G(\mathcal{Y}_\lambda)(q) = \frac{1}{(1-q^2)(1-q^3)} - \frac{q^{3c+3}}{(1-q^2)(1-q^3)} - \frac{q^{2b+2}}{(1-q)(1-q^2)} + \frac{q^{2b+c+3}}{(1-q)(1-q^2)} - \frac{q^{a+1}}{(1-q)(1-q^2)} + \frac{q^{a+b+2}}{(1-q)^2} - \frac{q^{a+b+c+3}}{(1-q)^2} + \frac{q^{a+2c+3}}{(1-q)(1-q^2)}.$$

Proof. An easy calculation shows that Lemma 1 implies Corollary 1, where the first four terms of (3.3) are $G(\mathcal{Z})(q)$ and the last four terms are $-q^{a+1}G(\mathcal{Y}_\mu)(q)$. □

Theorem 7. *If* λ *has at most three parts, then* λ *is unimodal.*

Proof. We indicate the proof if λ has three parts. From Lemma 1, we see that $(1-q)G(\mathcal{Y}_\lambda)(q)$ is the difference of two terms which are given explicitly in Proposition 1. If each term were unimodal, we could conclude in this case that λ is unimodal. Unfortunately, this is not true, but a careful case-by-case analysis shows that λ is unimodal. □

The next observation is that the non-unimodal λ in Table 1 lie in intervals. For example, (12, 10, 4, 4), (12, 11, 4, 4) and (12, 12, 4, 4) are all non-unimodal at $i = 21$ with the same three values of a_i, and they form the interval [(12, 10, 4, 4), (12, 12, 4, 4)]. The reason is clear: if a cell in position $(j+1, k+1)$ is removed from the Ferrers diagram of λ, the coefficients of q^n in $G(\mathcal{Y}_\lambda)(q)$ do not change for $0 \le n \le jk + j + k$. Thus if j and k are chosen so that $jk + j + k \ge i + 1$, then λ with the cell $(j+1, k+1)$ removed will also be non-unimodal. For example, we see that Theorem 1 implies that $(2k, m, 4, 4)$ is non-unimodal for $m \ge k+4$. It is possible to state a general theorem corresponding to Theorems 3 and 4, instead we give such a theorem for Theorems 5 and 6.

Theorem 8. *Let* $2 \le t \le (1 + \sqrt{1+24k})/4$. *Any partition in the following intervals is non-unimodal:*

1. $[(3k+t, 3k+t, 2k, 2k - \lfloor (2k+3-2t)/4 \rfloor), (3k+t, 3k+t, 2k, 2k)]$ *or*
2. $[(3k+t+2, 3k+t+2, 2k+1, 2k+1 - \lfloor (2k-1-2t)/4 \rfloor), (3k+t+2, 3k+t+2, 2k+1, 2k+1)]$.

By considering the non-unimodal partitions of $n \le 50$, two more infinite families, each singly indexed, can be found: $(k+2, k, k, k)$, for $k = 10$ or $k \ge 12$, non-unimodal at $i = 2k+3$; and $(2k+4, 2k+4, 2k+4, 2k+2)$ for $k \ge 4$, at $i = 4k+7$. In fact, the cases (a, a, a, b) and (a, b, b, b) could be done just as (a, a, b, b) was, but we shall be content to give these two families. In the first case cells from two different rows may be deleted to create non-unimodal intervals.

Theorem 9. *Any partition in the following intervals is non-unimodal:*

1. $[(k+2, k, \lceil (2k+2)/3 \rceil, \lceil (2k+1)/4 \rceil), (k+2, k, k, k)]$ *for* $k = 10$ *or* $k \ge 12$, *or*
2. $[(2k+4, 2k+4, \lceil (4k+5)/3 \rceil, k), (2k+4, 2k+4, 2k+4, k)]$ *for* $k \ge 4$.

The respective consecutive differences are

1. *-1 and* $\lfloor k/6 \rfloor - 1$ *for* $k \not\equiv 4 \pmod 6$*; and -1 and* $\lfloor k/6 \rfloor$ *for* $k \equiv 4 \pmod 6$*, and*
2. *-1 and* $\lfloor (k+1)/3 \rfloor - 1$ *for* $k \not\equiv 1 \pmod 3$*; and -1 and* $\lfloor (k+1)/3 \rfloor$ *for* $k \equiv 1 \pmod 3$.

Proof. First we verify the non-unimodality claim for $(k+2,k,k,k)$. This follows from

$$G(\mathcal{Y}_\lambda)(q)=\begin{bmatrix}k+4\\4\end{bmatrix}_q+(q^{k+1}+q^{k+2})\begin{bmatrix}k+3\\3\end{bmatrix}_q$$

and some lengthy calculations involving the appropriate pseudo polynomials. The second part is verified by noting that $(2k+4,2k+4,2k+4,k)$ and $(2k+4,2k+2,2k+2,2k+2)$ contain the same partitions of i for $i\leq 4k+3$. For $i=4k+6,4k+7$, and $4k+8$ respectively, $(2k+4,2k+4,2k+4,k)$ contains 1, 2, and 4 partitions that $(2k+4,2k+2,2k+2,2k+2)$ does not contain. Similarly for $i=4k+6,4k+7$, and $4k+8$, $(2k+4,2k+2,2k+2,2k+2)$ contains 2, 3, and 5 partitions that $(2k+4,2k+4,2k+4,k)$ does not contain. Thus the consecutive differences are the same at $i=4k+7$ and $i=4k+8$, which establishes (2). $\square$

How many non-unimodal partitions of n are there? Table 4 and Theorem 8 imply that these numbers are non-zero for $n\geq 30$. The intervals of Theorem 8 or Theorem 9 imply the following theorem. It is very likely, however, that this number grows much more rapidly than Theorem 10 asserts.

Theorem 10. *As $n\to\infty$, the number of non-unimodal partitions of n is at least cn^2.*

We also see from Table 4 that the number of non-unimodal partitions of n is even for $n\leq 50$. In view of (1.2), this could suggest that self-conjugate partitions are unimodal. In fact, no self-conjugate partition appears on the list of all non-unimodal partitions of $n\leq 50$. Moreover, all self-conjugate partitions of $n\leq 124$ are unimodal. The following theorem is a partial result in this direction.

Theorem 11. *If λ is any self-conjugate partition whose Durfee square has size at most two, then λ is unimodal.*

Proof. We may assume that the Durfee square of λ has size two, $\lambda=(a+2,b+2,2^b,1^{a-b})$, where $b\leq a$. If we apply Lemma 1 to λ we find

(3.4)

$$\begin{aligned}G(\mathcal{Z})(q)=&1+(q^2+\cdots+q^{a+2})+\frac{q^4}{(1-q)(1-q^2)^2}-\frac{q^{a+5}}{(1-q)^2(1-q^2)}+\frac{q^{a+b+6}}{(1-q)^2(1-q^2)}\\&-\frac{2q^{2b+6}}{(1-q)(1-q^2)^2}+\frac{q^{a+2b+7}}{(1-q)^2(1-q^2)}-\frac{q^{a+3b+8}}{(1-q)^2(1-q^2)}+\frac{q^{4b+8}}{(1-q)(1-q^2)^2}\end{aligned}$$

and

(3.5)

$$\begin{aligned}G(\mathcal{Y}_\mu)(q)=&(1+q+\cdots+q^{a+1})+\frac{q^2}{(1-q)^2(1-q^2)}-\frac{q^{a+3}}{(1-q)^3}+\frac{2q^{a+b+4}}{(1-q)^3}\\&-\frac{q^{2b+4}}{(1-q)^2(1-q^2)}-\frac{q^{b+3}}{(1-q)^2(1-q^2)}-\frac{q^{a+2b+5}}{(1-q)^3}+\frac{q^{3b+5}}{(1-q)^2(1-q^2)}.\end{aligned}$$

Again a case-by-case analysis implies Theorem 10. (The case $b\leq a\leq 2b$ is particularly unpleasant.) $\square$

4. Remarks. There are several observations which can be made that have not led to theorems. The purpose of this section is to comment on these possible theorems.

Observation 1. *All examples of non-unimodal partitions are bimodal.*

Observation 2. *All examples of non-unimodal partitions are non-unimodal at an odd integer i.*

Observation 3. *All examples of non-unimodal partitions have their absolute peaks at $i-1$ or $i+1$ if they are non-unimodal at i.*

It would appear very unlikely that Observations 1-3 are theorems, rather they are properties of the infinite families that have been found so far.

Observation 4. *There are no examples of non-unimodal partitions with 5, 7, or 9 parts.*

This has been checked for 5 parts with part size ≤ 30, 7 parts with part size ≤ 15, and 9 parts with part size ≤ 10. Again it appears that there is just not enough data in this case.

Observation 5. *All examples of infinite families of non-unimodal partitions have four parts. The only examples of non-unimodal partitions with six parts lie in intervals associated with* $(10,9,9,9,9,9)$*,* $(8,8,8,8,8,2)$*, or* $(8,8,6,6,6,6)$*.*

It is remarkable that $(10,9,9,9,9,9)$ is non-unimodal, being so close to $(9,9,9,9,9,9)$, which is unimodal. These three examples have resisted all attempts to be placed in an infinite family.

Observation 6. *The probability that a partition of n is non-unimodal roughly decreases to* $.00014$ *at* $n=52$*.*

The word "roughly" is used because the probability is not strictly decreasing. For $42 \leq n \leq 52$ the probability lies between .00014 and .00030. (The last integer for which it has been computed is $n = 52$.) One might conjecture that the probability $\rightarrow 0$ as $n \rightarrow \infty$.

Conjecture 1. *All self-conjugate partitions are unimodal.*

Conjecture 1 has been verified for all self-conjugate partitions of $n \leq 124$. (There are 174181 such partitions). It is also supported by Theorem 11.

Conjecture 2. *The staircase partition* $\lambda = (n, n-1, \ldots, 1)$ *is unimodal.*

Conjecture 2 has been verified for $n \leq 22$. The generating function was considered by Carlitz [2]. It is also related to the Rogers-Ramanujan continued fraction [4, §19.15]. If $G_n(\mathcal{Y}_\lambda)(q)$ is the generating function for $\lambda = (n-1, n-2, \ldots, 1)$, and $G_0(\mathcal{Y}_\lambda)(q) = 1$, it is well-known [3] that $G_n(\mathcal{Y}_\lambda)(q)$ is q-analogue of the nth

Catalan number. It is not hard to see that

$$\sum_{n=0}^{\infty} G_n(\mathcal{Y}_\lambda)(1/q)q^{n(n-1)/2}x^n = \cfrac{1}{1-\cfrac{x}{1-\cfrac{xq}{1-\cfrac{xq^2}{\ddots}}}} \tag{4.1}$$

$$= \sum_{n=0}^{\infty} \frac{(-x)^n q^{n^2}}{(q)_n} \Big/ \sum_{n=0}^{\infty} \frac{(-x)^n q^{n^2-n}}{(q)_n},$$

where

$$(q)_n = \prod_{k=1}^{n}(1-q^k).$$

Thus, Conjecture 2 is equivalent to a unimodality property of the continued fraction in (4.1).

Several other questions about Young's lattice remain open. The existence of a symmetric chain decomposition for a $m \times n$ rectangle, $m \geq 5$ is open. Clearly the rectangles are the only partitions which are symmetric. What happens if skew shapes are allowed? It is also known that Young's lattice of a rectangle has the Sperner property [8]. Susanna Fishel and the author have shown that the Young's lattice of any partition of $n \leq 26$ has the Sperner property. Finally, it is clear that one would not have found the infinite families of non-unimodal partitions without aid of a computer. What is missing is an algebraic formulation for a general partition λ (see [6] and [8]).

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