Comments on cranks and the rank

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thanks to Frank Garvan

Analytic and Combinatorial Number Theory: The Legacy of Ramanujan

in honor of Bruce Berndt

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Ramanujan congruences for $p(n)$

$$p(5n+4) \equiv 0 \mod 5$$

$$p(7n+5) \equiv 0 \mod 7$$

$$p(11n+6) \equiv 0 \mod 11$$

? combinatorial proofs splitting into equal classes ?

Dyson's rank (1944)

$\mathrm{rank}(\lambda) = \lambda_1 - \lambda_1'$

$\mathrm{rank}\Bigg($  $\Bigg) = 4-3=1$

Atkin and Swinnerton-Dyer proved it works for 5, 7 (1954)

Andrews-Garvan crank (June 6, 1987)

$\mathrm{AG\,crank}(\lambda) =$

$$\begin{cases} \lambda_1 & \lambda \text{ has no 1's} \\ \mu(\lambda) - (\#1\text{'s in } \lambda) & \lambda \text{ has} \end{cases}$$

$\mu(\lambda) = \#$ parts of λ which are greater than #(1

works for 5, 7, 11

5-core crank (Garvan-Kim-S. 1990)

Uses the 5-residue diagram of λ

$\lambda = 5421$

$$\begin{array}{|c|c|c|c|c|}
\hline
0 & 1 & 2 & 3 & 4 \\
\hline
4 & 0 & 1 & 2 \\
\cline{1-4}
3 & 4 \\
\cline{1-2}
2 \\
\cline{1-1}
\end{array}$$

$r_0 = 2 \quad r_1 = 2 \quad r_2 = 3 \quad r_3 = 2 \quad r_4 = 3$

$$\text{5-core crank}(\lambda) = r_1 + 2r_2 - 2r_3 - r_4$$

works for 5, 7, 11 (7-core, 11-core)

$$\mathrm{rank}_n(z) = \sum_{\lambda \in p(n)} z^{\mathrm{rank}(\lambda)}$$

$$\mathrm{AGcrank}_n(z) = \sum_{\lambda \in P(n)} z^{\mathrm{AGcrank}(\lambda)}$$

$$5\,\mathrm{core\,crank}_n(z) = \sum_{\lambda \in P(n)} z^{5\text{-}\mathrm{corecrank}(\lambda)}$$

Generating functions

$$\sum_{n=0}^{\infty} \mathrm{rank}_n(z)\, q^n =$$

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(zq, q/z; q)_n}$$

$$\sum_{n=0}^{\infty} \mathrm{AGcrank}_n(z)\, q^n =$$

$$\frac{(q;q)_\infty}{(zq, q/z; q)_\infty}$$

$\left\{\begin{array}{l}\text{Bruce's}\\ \text{George's}\end{array}\right.$ top 10 list for Ramanujan

crank generating function	#3
rank	#1

$$\sum_{n=0}^{\infty} \text{5corecrank}_{5n+4}(z)\, q^{n+1} =$$

$$\frac{1}{(q;q)_\infty^5} \sum_{\substack{\vec{a}\cdot\vec{1}=1 \\ \vec{a}\in\mathbb{Z}^5}} q^{Q(\vec{a})}\, z^{\sum_{i=0}^{4} i a_i}$$

$$Q(\vec{a}) = \sum_{i=0}^{4} a_i^2 - \sum_{i=0}^{4} a_i a_{i+1} \qquad a_5 = a_0.$$

5-fold symmetry $\Rightarrow$ bijection on 5corecrank classes

? Questions? Combinatorial reductions ?

$$\mathrm{rank}_{5n+4}(z) = (1+z+z^2+z^3+z^4) * \left(\begin{array}{c}\text{positive}\\ \text{polynomial}\end{array}\right)$$

$$\mathrm{rank}_4(z) = (1+z+z^2+z^3+z^4)\left(\frac{1-z+z^2}{z^3}\right)$$

NO

DEFN The modified rank, Mrank,

$$\mathrm{Mrank}_n(z) = \mathrm{rank}_n(z) + \left(z^{n-2} - z^{n-1} + z^{2-n} - z^{1-n}\right)$$

$\lambda = n$ $\qquad$ $n-1 \longrightarrow n-2$

$\lambda = 1^n$ $\qquad$ $1-n \longrightarrow 2-n$

<u>CONJ</u> $\dfrac{\mathrm{Mrank}_{5n+4}(z)}{1+z+z^2+z^3+z^4}$ is a positive Laurent polynomial

$\dfrac{\mathrm{Mrank}_{7n+5}(z)}{1+z+z^2+z^3+z^4+z^5+z^6}$ also.

Frank Garvan has verified these for

$5n+4 \le 1000$
$7n+5 \le 1000$.

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Cranks—really, the final problem

**Bruce C. Berndt · Heng Huat Chan ·
Song Heng Chan · Wen-Chin Liaw**

Dedicated to our friend George Andrews on his 70th birthday

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Abstract A survey of Ramanujan's work on cranks in his lost notebook is given. We give evidence that Ramanujan was concentrating on cranks when he died, that is to say, the final problem on which Ramanujan worked was *cranks—not mock theta functions*.

Keywords Crank · Partitions · Theta functions · Ramanujan's lost notebook · Rank

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At first glance, there does not appear to be any reasoning behind the choice of subscripts; note that there is no subscript for the second value. However, observe that in each case if we set $a = 1$, then the subscript n is equal to the right-hand side. The reason ρ does not have a subscript is that the value of n in this case would be $3 - 2 = 1$, which has been reserved for the first factor. In the table below, we record the content of page 181.

$$
\begin{aligned}
p(1) &= 1, & \lambda_1 &= \rho_1,\\
p(2) &= 2, & \lambda_2 &= \rho_2,\\
p(3) &= 3, & \lambda_3 &= \rho_3,\\
p(4) &= 5, & \lambda_4 &= \rho_5,\\
p(5) &= 7, & \lambda_5 &= \rho_7\rho,\\
p(6) &= 11, & \lambda_6 &= \rho_1\rho_{11},\\
p(7) &= 15, & \lambda_7 &= \rho_3\rho_5,\\
p(8) &= 22, & \lambda_8 &= \rho_1\rho_2\rho_{11},\\
p(9) &= 30, & \lambda_9 &= \rho_2\rho_3\rho_5,\\
p(10) &= 42, & \lambda_{10} &= \rho\rho_2\rho_3\rho_7,\\
p(11) &= 56, & \lambda_{11} &= \rho_4\rho_7(a_5 - a_4 + a_2),\\
p(12) &= 77, & \lambda_{12} &= \rho_7\rho_{11}(a_4 - 2a_3 + 2a_2 - a_1 + 1),\\
p(13) &= 101, & \lambda_{13} &= \rho\rho_1\,(a_{10} + 2a_9 + 2a_8 + 2a_7 + 3a_6\\
& & &\quad + 4a_5 + 6a_4 + 8a_3 + 9a_2 + 9a_1 + 9)\,,\\
p(14) &= 135, & \lambda_{14} &= \rho_5\rho_9(a_5 - a_3 + a_1 + 1),\\
p(15) &= 176, & \lambda_{15} &= \rho_4\rho_{11}(a_7 - a_6 + a_4 + a_1),\\
p(16) &= 231, & \lambda_{16} &= \rho_3\rho_7\rho_{11}(a_5 - 2a_4 + 2a_3 - 2a_2 + 3a_1 - 3),\\
p(17) &= 297, & \lambda_{17} &= \rho_9\rho_{11}(a_7 - a_6 + a_3 + a_1 - 1),\\
p(18) &= 385, & \lambda_{18} &= \rho_5\rho_7\rho_{11}(a_6 - 2a_5 + a_4 + a_3 - a_2 + 1),\\
p(19) &= 490, & \lambda_{19} &= \rho_1\rho_2\rho_5\rho_7(a_9 - a_7 + a_4 + 2a_3 + a_2 - 1),\\
p(20) &= 627, & \lambda_{20} &= \rho\rho_3\rho_{11}(a_{10} + a_6 + a_4 + a_3 + 2a_2 + 2a_1 + 3),\\
p(21) &= 792, & \lambda_{21} &= \rho\rho_3\rho_4\rho_{11}(a_8 - a_6 + a_4 + a_1 + 2).
\end{aligned}
$$

These factors lead to the rapid calculation of values for $p(n)$. For example, since $\lambda_{10} = \rho\rho_2\rho_3\rho_7$, then $p(10) = 1 \cdot 2 \cdot 3 \cdot 7 = 42$.

Ramanujan evidently was searching for some general principles or theorems on the factorization of λ_n so that he could not only compute $p(n)$ but make deductions about the divisibility of $p(n)$. No theorems are stated by Ramanujan. Is it possible to determine that certain factors appear in some precisely described infinite family of

Ramanujan factors for

$$AGcrank_{14}(z) = \left(z^4 + z^2 + 1 + z^{-2} + z^{-4}\right) *$$
$$\rho_9 \left(a_5 - a_3 + a_1 + 1\right)$$

LAST TWO FACTORS =

$$z^{-10} + z^{-7} + z^{-6} + z^{-5} + 2z^{-4} + 2z^{-3} + 2z^{-2} + 2z^{-1}$$
$$+ 3 + 2z + 2z^2 + 2z^3 + 2z^4 + z^5 + z^6 + z^7 + z^{10}$$

<u>CONJ</u> $\dfrac{\mathrm{AGcrank}_{5n+4}(z)}{\left(z^4+z^2+1+z^{-2}+z^{-4}\right)}$ is a positive Laurent polynomial

Frank verified this for

$$5n+4 \leq 1000.$$

DEFN The modified $\mathrm{MAGcrank}_n^{(a)}(z)$

$$= \mathrm{AGcrank}_n(z) + \left(z^{n-a} - z^{n} + z^{a-n} - z^{-n}\right)$$

CONJ The following are non-negative Laurent polys

$$\frac{\mathrm{MAG}^{(5)}_{5n+4}(z)}{1+z+z^2+z^3+z^4}, \quad \frac{\mathrm{MAG}^{(7)}_{7n+5}(z)}{1+z+\dots+z^6}, \quad \frac{\mathrm{MAG}^{(11)}_{11n+6}(z)}{1+z+\dots+z^{10}}$$

Frank checked this for $tn+r \leq 1000$.

CONJ The following are non-negative polynomials

$$\frac{\text{5core crank}_{5n+4}(z)}{1+z+z^2+z^3+z^4},$$

$$\frac{\text{5 core crank}_{5n+4,j}(z)}{1+z+z^2+z^3+z^4}$$

restricted to BG crank $= j$

Alex Berkovich and Frank Garvan 2006, 2008.

Thank you, Bruce!