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A Generic Framework for Scalable and Convergent Multi-Robot Active Simultaneous Localization, Mapping and Target Tracking

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Abstract— **In this paper, a new approach is proposed and analyzed for developing efficient and scalable methodologies for multi-robot active Cooperative Simultaneous Localization And Mapping and Target Tracking (C-SLAMTT). The proposed approach employs an active estimation scheme that switches among linear elements and, as a result, its computational requirements scale linearly with the number of estimated quantities (number of number of robots, landmarks and targets). The parameters of the proposed scheme are calculated off-line using a *convex* optimization algorithm which is based on Semi-Definite Programming (SDP) and approximation using Sum-of-Squares (SoS) polynomials. As shown by rigorous arguments, the estimation accuracy of the proposed scheme is equal to the *optimal* estimation accuracy plus a term that is inversely proportional to the number of estimator's switching elements (or, equivalently, to the memory storage capacity of the robots' equipment). The proposed approach can handle various types of constraints such as "stay-within-an-area", obstacle avoidance and maximum speed constraints. The efficiency of the approach is demonstrated on a 3D active cooperative simultaneous mapping and target tracking application employing flying robots.**

I. INTRODUCTION

The majority of techniques and methods for multi-robot sensing and estimation applications are based on local approximations of the overall system dynamics (team of robots + measurement model + the external environment): for instance, in *Passive Sensing (PS)* applications such as passive target tracking, localization, mapping, Cooperative Simultaneous Localization And Mapping (C-SLAM), etc, the majority of existing approaches employ Extended Kalman Filter (EKF) or similar techniques which are based on linearization of the overall system dynamics, see e.g., [16], [13], [2]. A similar situation is also present in *Active Sensing (AS)* applications such as active target tracking, active C-SLAM, etc, where the objective is to generate the robots' trajectories so that estimation accuracy is optimized. In most of the existing approaches, the trajectory generators are usually based on convex or local approximations (relaxations) of non-convex optimization problems, see e.g., [16], [5], [14]. However, linearization (in case of PS) or relaxation (in case of AS) may have a significant or even devastating effect on the overall system efficiency due to e.g., estimator divergence or convergence to local minima. Attempts that have been made to employ techniques that avoid the usage of linearization or convex/local relaxations face the well-known problem of curse of dimensionality: for instance, the algorithms of [3], [12], [5], [11], [14], require the implementation of algorithms that scale poorly with the number of the robot team members and, as a result, their deployment in large-scale, real-life applications is formidable and/or they cannot guarantee of efficient and convergent performance.

In this paper, we propose and analyze a new approach that overcomes the above mentioned shortcomings. The proposed approach adopts a general framework that can treat in a unified manner a large class of multi-robot AS applications such as active target target tracking, localization, active SLAM or any combinations of them. Contrary to the approaches that are based on linearization or local approximation/relaxation, the proposed approach uses the full nonlinear model of the overall multi-robot system dynamics in order to construct an estimator/trajectory generator that fulfills the computational and other physically-imposed requirements imposed by the particular multi-robot application. Moreover, it guarantees aribtrary-close-to-optimal, convergent and efficient performance of the overall estimation process.

Due to space limitations the proof of our theoretical results are omitted and will be reported elsewhere.

A. Notations & Preliminaries

$\|x\|$ is used to denote the standard Euclidean norm of the vector x. For a symmetric matrix A, the notation $A \succ 0$ ($A \succeq 0$) is used to denote that A is a positive definite (resp. positive semidefinite) matrix. $\dim(x)$ denotes the dimension of the vector x. I_n denotes the $n \times n$ identity matrix. For a smooth function $V(x_1, x_2)$ where x_i are vectors, the following notation is used: $V_{x_i}(x_1, x_2) = \frac{\partial V}{\partial x_i}(x_1, x_2)$, $V_{x_i x_i}(x_1, x_2) = \frac{\partial^2 V}{\partial x_i^2}(x_1, x_2)$. The following definition will be finally needed in the paper.

Definition 1: Fix the positive integer L and let the vector $x \in \Re^n$ satisfy $\|x\| \leq C$ for some positive constant C; the notation $\beta(x) = \mathcal{B}_n^L(x)$ will be used to denote the vector $\beta(x) = [\beta_1(x)x^\tau, \ldots, \beta_L(x)x^\tau]^\tau$ where β_i are L smooth *activation functions* satisfying $\beta_i(x) \in [0, 1]$, $\sum_{i=1}^L \beta_i(x) =$

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1, $\sum_{i=1}^{L} \mathcal{I}(\beta_i(x)) \leq 2$ where $\mathcal{I}(y)$ denotes the indicator function $\mathcal{I}(y) = 1$ if $y > 0$ and $\mathcal{I}(y) = 0$ if $y = 0$.

II. Multi-Robot Active Sensing Set-Up

We consider a team of N_R robots deployed on a $2D$ or a $3D$ environment containing N_L static landmarks and N_T moving targets. For simplicity and to avoid unnecessary lenghty formulas, we assume that a low-level controller (fast inner control-loop) is implemented to each of the robots, so that the velocity of each of the robots (after the application of the low-level control) can be directly controlled. Moreover we assume that the target's dynamics are reproduced according to a zero-acceleration model, see e.g., [16]. Under these assumptions the dynamics of the overall system can be described according to the following set of differential equations:

$$\begin{array}{rcll} \dot{x}_R &=& B_R u, & x_R(0) \text{ is known} \\ \ddot{x}_T &=& B_T \omega_T, & x_T(0), \dot{x}_T(0) \text{ are unknown} \\ \dot{x}_L &=& 0, & x_L(0) \text{ is unknown} \end{array} \quad (2.1)$$

where x_R denotes the vector of positions/poses of all N_R robots, u denotes the vector of all robots' control inputs, x_T denotes the vector of positions of all N_T targets and x_L is the (static) vector of all N_L landmarks; B_R and B_T are constant diagonal matrices and ω_T denotes a zero-mean unity-variance Gaussian random vector that drives the targets. Note that by defining the augmented state vector as $\bar{x} = [x_R^\tau, x_T^\tau, \dot{x}_T^\tau, x_L^\tau]^\tau$, we can rewrite the overall system dynamics according to

$$\dot{\bar{x}} = \bar{B}u + \bar{B}_\omega \omega_T \quad (2.2)$$

for some appropriately defined constant matrices $\bar{B}$ and $\bar{B}_\omega$.

We will further assume that the robots use proprioceptive measurements (e.g., from odometric or inertial sensors) to propagate their position or pose (position and orientation) estimates, and are equipped with exteroceptive sensors (e.g., laser range finders, cameras, etc) that provide nonlinear measurements (e.g., distance or bearing) of robot-to-robot, robot-to-landmark and/or robot-to-target relative positions. Let y denote the vector of all sensor measurements and assume that these measurements are related to the system (2.2) states according to

$$y = \bar{h}(\bar{x}, \xi) \quad (2.3)$$

where $\bar{h}$ is a nonlinear vector function and ξ denotes the sensor noise vector. We will assume that $\bar{h}$ is a differentiable function (or that it can be approximated with arbitrary accuracy by a differentiable function). A final assumption is that the sensor noise is generated according to a *colored* noise model, i.e.,

$$\dot{\xi} = -a\xi + \omega_S \quad (2.4)$$

where a is a positive scalar and ω_S is a zero-mean unity-variance Gaussian random vector.

By defining $\chi = [\bar{x}^\tau, \xi^\tau]^\tau$ and $\omega = [\omega_T^\tau, \omega_S^\tau]^\tau$, we can rewrite equations (2.2), (2.3), (2.4) into the following compact form

$$\begin{array}{rcl} \dot{\chi} &=& A\chi + Bu + B_\omega \omega \\ y &=& h(\chi) \end{array} \quad (2.5)$$

where $h(\chi) \equiv \bar{h}(\bar{x}, \xi)$ and A, B, B_ω are constant matrices that depend on B_R, B_T and a.

The task the multi-robot team is called to accomplish is that of <u>*active*</u> *Cooperative Simultaneous Localization And Mapping and Target Tracking – (active C-SLAMTT)*, i.e., the problem of

- use the sensor signals $y(t)$ in order to provide estimates of the overall state vector $\chi(t)$,
- while designing on-line the robot control signals $u(t)$ so that the overall estimation accuracy is maximized (i.e., the robots are moving so that they "capture" as much information as possible) and, moreover, the robot trajectories satisfy physically-imposed constraints such as maximum height, obstacle-avoidance, maximum speed, etc, constraints.

We close this section by noticing that there are two types of physically-imposed constraints treated in this paper:

- Constraints that can be written in the form:

$$S(y) \leq 0 \quad (2.6)$$

where S can be any nonlinear vector function of the sensor measurements y. Obstacle avoidance, maximum allowable height constraints as well as constraints restricting the robots to remain within an area can be written in the form (2.6).

- Control-related physical constraints such as that the control inputs or the robot speeds should not exceed certain bounds. Please note that these constraints cannot be written in the form (2.6).

III. AS Estimator Design

Having formulated the AS design problem for the active C-SLAMTT case treated in this paper, we now proceed to the proposed solution to this problem. The proposed solution employs an estimator of the form

$$\begin{array}{rcl} \dot{\hat{\chi}} &=& Bu + u_o \\ \hat{y} &=& h(\hat{\chi}) \end{array} \quad (3.1)$$

where u_o is the *correction vector*. In this way the AS design problem is transformed to the problem of designing on-line the signals u and u_o.

Assuming that the estimator (3.1) is employed, the *optimal AS estimator design* can be formulated as a *stochastic optimal control problem* described according[1] to

$$\min_{(u(s), u_o(s), s \in [0,\infty))} \mathcal{J} \quad (3.2)$$

$$\begin{array}{rcl} \mathcal{J} &=& J(\chi(0), \hat{\chi}(0)) \\ &=& E\left[\int_0^\infty \left(\|\tilde{\chi}(s)\|^2 + \|\pi(y(s))\|^2\right) ds\right] \end{array} \quad (3.3)$$

[1]For the time-being only the case of constraints of the type (2.6) is considered. For the extension to the case of control-related physical constraints see the end of this section. Also, the proposed approach can be easily extended to cases where other optimization criteria than (3.3) are employed.

where $\tilde{\chi}(t)=\chi(t)-\hat{\chi}(t)$ denotes the *state estimation error*, $\pi(y(s))=\mathrm{vec}(\pi_i(y)), \pi_i(y)=\exp(\alpha S_i(y)-\eta)$, α is a large positive constant and η is a positive constant chosen so that if a particular constraint $S_i(y)\leq 0$ is – or is about to be – violated, then the respective $\pi_i(y)$ takes a very large value, while it is negligible when the constraint is satisfied.

Using the concepts of input/output representations and filtering techniques it can be seen [6] that a possible mathematical description for the optimal u^*, u_o^* (minimizing $\mathcal{J}$) is as follows

$$\begin{array}{rcl} u^*(t) & = & k_c(\breve{Y}) \\ u_o^*(t) & = & k_o(\breve{Y}) \end{array}, \quad \breve{Y}=\left[\begin{array}{c} \breve{y} \\ \breve{Y}_f \end{array}\right] \tag{3.4}$$

for some *unknown* functions k_c, k_o, where

$$\breve{y}=\left[\begin{array}{c} \tilde{y} \\ y \end{array}\right] \equiv \left[\begin{array}{c} y-h(\hat{\chi}) \\ y \end{array}\right], \quad \breve{Y}_f=\left[\begin{array}{c} \frac{1}{\Lambda(s)}\breve{y} \\ \vdots \\ \frac{1}{\Lambda^{p-1}(s)}\breve{y} \end{array}\right]$$

and $\Lambda(s)=(s+\rho)$, with ρ being a positive design constant and p being the smaller positive integer satisfying $p\geq \dim(\chi)/\dim(y)$.

By adopting a stochastic dynamic programming framework, we let $\breve{V}$ denote the *optimal-cost-to-go function*, see e.g., [15], defined according to

$$\begin{array}{rcl} \breve{V}(X(t)) & = & \min J(X(t)) \\ J(X(t)) & = & E\left[\int_t^\infty \left(\|\tilde{\chi}(s)\|^2+\|\pi(y(s))\|^2\right) ds\right] \\ X & = & \left[\begin{array}{c} (\hat{\chi}-\chi)^\tau, \chi^\tau, \breve{Y}_f^\tau \end{array}\right] \end{array} \tag{3.5}$$

By direct application of the Hamilton-Jacobi-Bellman equation, see e.g., [15], we obtain that the optimal-cost-to-go function $\breve{V}$ satisfies

$$\mathcal{L}\breve{V}(X)\Big|_{u(t)=u^*(t), u_o(t)=u_o^*(t)} = -\left(\|\tilde{\chi}\|^2+\|\pi(y)\|^2\right) \tag{3.6}$$

where $\mathcal{L}$ stands for the Ito infinitesimal[2] operator (generator) acting on $\breve{V}(\chi,\hat{\chi})$ along the trajectories of (2.5) and (3.1), see e.g., [4], and u^*, u_o^* denotes the optimal values for the signals u and u_o, respectively.

In order to have a well-posed problem, we will assume that the problem at hand admits a solution, or, equivalently that the optimal-cost-to-go function and the associated optimal signals u^*, u_o^* satisfy the following assumption.

(A1) For all admissible initial conditions, the optimization problem (3.5) – or, equivalently, the associated HJB equation – admits a unique viscosity solution $\breve{V}$ satisfying $\breve{V}(X)=V_0$ if $\chi=\hat{\chi}$ and $\breve{V}(X)>0$ if $\chi\neq\hat{\chi}$, where V_0 is a non-negative constant.

Assumption (A1) requires that the problem at hand makes sense, i.e., that for all admissible $\chi(0), \hat{\chi}(0)$, there exists a control strategy $u^*(t)$ that satisfies the constraints (2.6) and renders system (2.5) stochastically observable. To keep our analysis simple and avoid unnecessary technicalities, we will assume hereafter that $V_0=0$. All the results of this paper can be readily extended to the case where $V_0>0$, in which case the term V_0 should be added to the RHS of inequalities (3.8) and (3.22) presented below.

By using the Ito formula [4] and (3.4), the HJB equation (3.6) becomes

$$\begin{aligned} &-\left(\|\tilde{\chi}\|^2+\|\pi(y)\|^2\right)= \\ &\breve{V}_\chi^\tau\left(A\chi+Bk_c(\breve{Y})\right)+\breve{V}_{\hat{\chi}}^\tau\left(Bk_c(\breve{Y})+k_o(\breve{Y})\right) \\ &+\breve{V}_{\breve{Y}_f}^\tau\left(\bar{A}_f\breve{Y}_f+\bar{B}_f\breve{y}\right)+\frac{1}{2}\mathrm{tr}\Big\{B_\omega^\tau\breve{V}_{\chi\chi}B_\omega\Big\} \end{aligned} \tag{3.7}$$

The following lemma will be proven useful for comparing the performance of the proposed AS scheme with that of the optimal one.

Lemma 1: Let (A1) hold. Let also the following design assumption hold:

(A2) The function $S(y)$ in (2.6) is designed so that it incorporates constraints that force the robots to remain within a prespecified area $[x_{\min}, x_{\max}, y_{\min}, y_{\max}, z_{\min}, z_{\max}]$.

Then the optimal AS estimator, i.e., the AS estimator (3.1), (3.4) satisfies

$$E\left[|\tilde{\chi}(t)|^2\right] \quad\leq\quad \lambda_1^* e^{-\lambda_2^* t}\left[|\tilde{\chi}(0)|^2\right] \tag{3.8}$$

where $\lambda_i^*,\ \ i=1,2$ are positive constants that depend on (2.5) and (2.6).

Let us now turn our attention to the proposed approach. The first step in our approach is to employ standard approximators for approximating the functions $\breve{V}, k_c$ and k_o in (3.7). More precisely, consider the vectors $\psi=\left[\begin{array}{c}\tilde{y}^\tau\pi(y)^\tau\end{array}\right]^\tau$,

$$\zeta=\left[\begin{array}{c}\sqrt{\beta_1(\breve{Y})}(\chi-\hat{\chi})^\tau,\ldots,\sqrt{\beta_L(\breve{Y})}(\chi-\hat{\chi})^\tau,\chi^\tau,\psi^\tau\breve{Y}^\tau\end{array}\right]^\tau$$

where $\beta_1(\breve{Y}),\ldots,\beta_L)(\breve{Y})$ denote L activation smooth functions satisfying Definition 1, i.e.,

$$\beta(\breve{Y})=\left[\beta_1(\breve{Y}),\ldots,\beta_L(\breve{Y})\right]^\tau=\mathcal{B}^L_{\dim(\breve{Y})}(\breve{Y}) \tag{3.9}$$

Then, $\breve{V}, k_c$ and k_o can be approximated (with an approximation accuracy that is inversely proportional to the number L of activation functions) as follows:

$$\begin{array}{rcl} \breve{V}(X) & \approx & \zeta^\tau\mathbf{P}\zeta \\ k_c(\breve{Y}) & \approx & \kappa_c(\breve{Y})\mathbf{G_c}\psi, \quad k_o(\breve{Y})\approx\kappa_o(\breve{Y})\mathbf{G_o}\psi \end{array} \tag{3.10}$$

where $\kappa_c(\breve{Y}) = \left[\beta_1(\breve{Y})I_{\dim(u)},\ldots,\beta_L(\breve{Y})I_{\dim(u)}\right]$, $\kappa_o(\breve{Y}) = \left[\beta_1(\breve{Y})I_{\dim(\chi)},\ldots,\beta_L(\breve{Y})I_{\dim(\chi)}\right]$ and $\mathbf{P}$, $\mathbf{G_c}$, $\mathbf{G_o}$ are constant *unknown* matrices of appropriate dimensions. Using assumption (A1) and similar arguments as those of [7]-[10], it can be seen that the matrix $\mathbf{P}$ is positive definite and symmetric and assumes the following *block diagonal form*

$$\mathbf{P}=\left[\begin{array}{cccccc} \mathbf{P}_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \ddots & 0 & 0 & 0 & 0 \\ 0 & \ldots & \mathbf{P}_L & 0 & 0 & 0 \\ 0 & \ldots & 0 & \mathbf{p}I_{\dim(\chi)} & 0 & 0 \\ 0 & \ldots & 0 & 0 & \mathbf{p}I_{\dim(\psi)} & 0 \\ 0 & \ldots & 0 & 0 & 0 & \mathbf{p}I_{\dim(\breve{Y})} \end{array}\right]$$

where $\mathbf{P}_i$ are $\dim(\chi)^2$-dimensional symmetric and positive definite matrices and $\mathbf{p}$ is a positive constant. Returning

[2]In simple words, $\mathcal{L}$ denotes the mean of the time-derivative of $\breve{V}(\chi,\hat{\chi})$ along the trajectories of (2.5) and (3.1).

back to the definition of the vector ζ, it is not difficult for someone to see that the terms $\|\tilde{\chi}\|^2 + \|\pi(y)\|^2, \chi$ and ψ can be rewritten as follows:

$$\begin{aligned} \|\tilde{\chi}\|^2 + \|\pi(y)\|^2 &= \zeta^\tau Q \zeta \qquad (3.11)\\ \psi = H\zeta, \quad \chi = \bar{H}\zeta, \quad & \breve{Y} = \bar{\bar{H}}\zeta \end{aligned}$$

By using (3.10) and (3.11), we have that the HJB equation (3.7) can be rewritten – after some lengthy but straightforward manipulations – as follows

$$\begin{aligned} \nu + \mathcal{O}(\|\mathbf{P}\|) = \zeta^\tau \{ & Q + \mathbf{P}\zeta_\chi \left(A\bar{H} + B\kappa_c(\breve{Y})\mathbf{G_c}H \right) \\ & + \left(A\bar{H} + B\kappa_c(\breve{Y})\mathbf{G_c}H \right)^\tau \zeta_\chi^\tau \mathbf{P} \\ & + \mathbf{P}\zeta_{\hat{\chi}} \left(B\kappa_c(\breve{Y})\mathbf{G_c}H + \kappa_o(\breve{Y})\mathbf{G_o}H \right) \\ & + \left(B\kappa_c(\breve{Y})\mathbf{G_c}H + \kappa_o(\breve{Y})\mathbf{G_o}H \right)^\tau \zeta_{\hat{\chi}}^\tau \mathbf{P} \\ & + \mathbf{P}\zeta_{\breve{Y}} \left(A_f \bar{\bar{H}} \right) + \left(A_f \bar{\bar{H}} \right)^\tau \zeta_{\breve{Y}}^\tau \mathbf{P} \} \zeta \qquad (3.12) \\ \equiv & \zeta^\tau \mathcal{P}(\mathbf{P}, \mathbf{G}_c, \mathbf{G}_o) \zeta \qquad (3.13) \end{aligned}$$

where A_f is a constant matrix that corresponds on the state-space realization of $\breve{Y}_f$ and ν stands for the *approximation error term* that is present due to the approximations (3.10) and is inversely proportional to the number L of activation functions. Equation (3.13) indicates that the problem of constructing an approximately optimal AS performance can be cast as the following optimization problem:

$$\begin{aligned} \min & \left[\|\zeta^\tau \mathcal{P}(\mathbf{P}, \mathbf{G}_c, \mathbf{G}_o)\zeta\|^2 + \|\mathbf{P}\| \right] \qquad (3.14) \\ \text{s.t.} & \quad \mathbf{P} \succ 0 \end{aligned}$$

Unfortunately, as the term $\mathcal{P}(\cdot)$ is a nonlinear function of $\mathbf{P}, \mathbf{G}_c, \mathbf{G}_o$, attempting to solve (3.14) is a non-convex – and thus difficult to be solved – problem. To circumvent this problem we work similarly to [7]-[10]: by multiplying from the left and the right the term $\mathcal{P}$ by $\mathbf{P}^{-1}$ we obtain that

$$\begin{aligned} \zeta^\tau \mathbf{P}^{-1} & \mathcal{P}(\mathbf{P}, \mathbf{G}_c, \mathbf{G}_o) \mathbf{P}^{-1} \zeta = \\ & \zeta^\tau \{ \bar{\mathbf{Q}} + \zeta_\chi \left(A\bar{H}\bar{\mathbf{P}} + B\kappa_c(\breve{Y})\mathbf{F_c}H \right) \\ & + \left(\bar{\mathbf{P}} A\bar{H} + B\kappa_c(\breve{Y})\mathbf{F}_c H \right)^\tau \zeta_\chi^\tau \\ & + \zeta_{\hat{\chi}} \left(B\kappa_c(\breve{Y})\mathbf{F_c}H + \kappa_o(\breve{Y})\mathbf{F_o}H \right) \\ & + \left(B\kappa_c(\breve{Y})\mathbf{F_c}H + \kappa_o(\breve{Y})\mathbf{F_o}H \right)^\tau \zeta_{\hat{\chi}}^\tau \\ & + \zeta_{\breve{Y}} \left(A_f \bar{\bar{H}} \right) \bar{\mathbf{P}} + \bar{\mathbf{P}} \left(A_f \bar{\bar{H}} \right)^\tau \zeta_{\breve{Y}}^\tau \} \zeta \\ = & \mathcal{F}_{\bar{\mathbf{P}}, \bar{\mathbf{Q}}, \mathbf{F_c}, \mathbf{F_o}}(\chi, \hat{\chi}, \breve{Y}) \qquad (3.15) \end{aligned}$$

where

$$\bar{\mathbf{P}} = \mathbf{P}^{-1}, \quad \bar{\mathbf{Q}} = \mathbf{P}^{-1} Q \mathbf{P}^{-1} \equiv \bar{\mathbf{P}} Q \bar{\mathbf{P}} \qquad (3.16)$$

and $\mathbf{F_c}, \mathbf{F_o}$ denote two constant matrices satisfying

$$\mathbf{F_c}H = \mathbf{G_c}H\bar{\mathbf{P}}, \quad \mathbf{F_o}H = \mathbf{G_o}H\bar{\mathbf{P}} \qquad (3.17)$$

The following lemma establishes that – in the case where ν is negligible – solving the non-convex problem (3.14) is equivalent to a <u>convex</u> problem related to the function $\mathcal{F}_{\bar{\mathbf{P}}, \bar{\mathbf{Q}}, \mathbf{F_c}, \mathbf{F_o}}$ defined above.

Lemma 2: Consider the following optimization problem:

$$\min \left[\|\mathcal{F}_{\bar{\mathbf{P}}, \bar{\mathbf{Q}}, \mathbf{F}_c, \mathbf{F}_o}(\chi, \hat{\chi}, \breve{Y})\|^2 + \operatorname{tr}(X) \right] \qquad (3.18)$$

s.t.

$$\epsilon_1 I \preceq \bar{\mathbf{P}} \preceq \epsilon_2 I, \quad \epsilon_3 Q \preceq \bar{\mathbf{Q}}, \quad 0 \preceq Z, \quad Z = \begin{bmatrix} \bar{\mathbf{P}} & I \\ I & X \end{bmatrix}$$

where $\epsilon_i, i = 1, 2, 3$ are some positive design constants (with $\epsilon_2 > \epsilon_1$) and X is a diagonal matrix. Let also $\mathbf{G}_c^*, \mathbf{G}_o^*, \mathbf{P}^*$ denote the optimal solutions to the optimization problem (3.14), $\bar{\mathbf{P}}^*, \bar{\mathbf{Q}}^*, \mathbf{F}_c^*, \mathbf{F}_o^*$ denote the optimal solutions to the optimization problem (3.18) and

$$\mathbf{P}^{\star\star} = (\bar{\mathbf{P}}^*)^{-1}, \quad \mathbf{G}_c^{\star\star} = \mathbf{F}_c^* H \left(H\bar{\mathbf{P}}^* \right)^H, \quad \mathbf{G}_o^{\star\star} = \mathbf{F}_o^* H \left(H\bar{\mathbf{P}}^* \right)^H$$

where A^H denotes the pseudo-inverse of the matrix A. Then, if the positive constants ϵ_i are chosen so that $\epsilon_1 I \preceq \mathbf{P}^{*-1} \preceq \epsilon_2 I, \quad \epsilon_3 Q \preceq \mathbf{P}^{*-1} Q \mathbf{P}^{*-1}$ we have that

$$\mathbf{P}^{\star\star} = \mathbf{P}^\star + \mathcal{O}(\nu), \quad \mathbf{G}_c^{\star\star} = \mathbf{G}_c^\star + \mathcal{O}(\nu), \quad \mathbf{G}_o^{\star\star} = \mathbf{G}_o^\star + \mathcal{O}(\nu),$$

However the solution of the optimization problem (3.18) requires discretization of the state-space as it is an *infinite-dimensional, state-dependent problem*. Fortunately, due to the particular form of the optimization problem (3.18), the number of discretization points does not have to be as large as it is required in a typical state-dependent optimization problem: as it was seen in [7], [8] the number of discretization points can be as few as the total number of free variables in the matrices $\bar{\mathbf{P}}, \bar{\mathbf{Q}}, \mathbf{F_c}, \mathbf{F_o}$. This is in contrary to alternative approaches that impose approximation schemes for constructing estimation designs: for instance, dynamic programming-based schemes require an extremely larger number of discretization points.

Table I presents the proposed procedure for solving the optimization problem (3.18) and, eventually, constructing the proposed AS scheme. Following the same methodology as in [7], [8], the key idea of the approach in Table I for solving (3.18) is to choose randomly many different[3] triplets $x, \hat{x}, \breve{Y}$; it suffices to choose the number of random triplets to be equal or larger than the number of free variables in the matrices $\mathbf{P}, \mathbf{G}_c, \mathbf{G}_o$. The next theorem establishes the properties of the overall scheme presented in Table I.

Theorem 1: Fix the number L of activation functions and the constants $\epsilon_i, i = 1, 2, 3$ and let $\bar{\mathbf{P}}, \bar{\mathbf{Q}}, \mathbf{F_c}, \mathbf{F_o}$ be constructed according to the design procedure of Table I. Let also (A1), (A2) hold. Then, the following statements hold:
(a) If the following holds for some positive constant C_1:

$$\begin{aligned} & \|\chi^{[i]} - \hat{\chi}^{[i]}\|^2 > C_1 \Rightarrow \qquad (3.21) \\ & \mathcal{F}_{\bar{\mathbf{P}}, \bar{\mathbf{Q}}, \mathbf{F_c}, \mathbf{F_o}}(\chi^{[i]}, \hat{\chi}^{[i]}, \breve{Y}^{[i]}) - \zeta^\tau(\chi^{[i]}, \hat{\chi}^{[i]}) \bar{\mathbf{Q}} \zeta(\chi^{[i]}, \hat{\chi}^{[i]}) < 0 \end{aligned}$$

then

$$E\left[|\tilde{\chi}(t)|^2 \right] \leq \lambda_1 e^{-\lambda_2 t} \left[|\tilde{\chi}(0)|^2 \right] + \lambda_3 \qquad (3.22)$$

where $\lambda_i > 0$ and, moreover, $\lambda_1 = \lambda_1^* + \mathcal{O}\left(\frac{1}{L}\right)$, $\lambda_2 = \lambda_2^* - \mathcal{O}\left(\frac{1}{L}\right)$ and $\lambda_3 \equiv C_1 = \mathcal{O}\left(\frac{1}{L}\right)$.

[3]Please note that all the terms $\zeta_x, \zeta_{\hat{x}}, \kappa_c(\breve{Y}), \kappa_o(\breve{Y})$ in (3.18) are functions of $\chi, \hat{\chi}$ and $\breve{Y}$.

Table I: Proposed AS design approach

Step 1. Calculate the matrices $\bar{\mathbf{P}}, \bar{\mathbf{Q}}, \mathbf{F_c}, \mathbf{F_o}$ as follows: Let N denote the total number of free variables of these matrices. Select randomly $\mathcal{N}$ points $\chi^{[i]}, \hat{\chi}^{[i]}, \breve{Y}^{[i]}$, where $\mathcal{N}$ is any integer satisfying $\mathcal{N} \geq N$ and solve the following convex optimization problem (here ϵ_i are user-defined positive constants):

$$\min \left[\sum_{i=1}^{\mathcal{N}} \|\mathcal{F}_{\bar{\mathbf{P}},\bar{\mathbf{Q}},\mathbf{F_c},\mathbf{F_o}}(\chi^{[i]}, \hat{\chi}^{[i]}, \breve{Y}^{[i]})\|^2 + \mathrm{tr}\,(X) \right] \quad (3.19)$$

s.t.

$$\epsilon_1 I \preceq \bar{\mathbf{P}} \preceq \epsilon_2 I, \quad \epsilon_3 Q \preceq \bar{\mathbf{Q}}, \quad 0 \preceq Z$$
$$Z = \begin{bmatrix} \bar{\mathbf{P}} & I \\ I & X \end{bmatrix}$$

Step 2. By using the solution of the above optimization problem, we can extract the matrices $\mathbf{P}, \mathbf{G}_c, \mathbf{G}_o$ according to

$$\mathbf{P} = \bar{\mathbf{P}}^{-1}, \quad \mathbf{G_c} = \mathbf{F_c} H \left(H\mathbf{P}^{-1}\right)^H, \quad \mathbf{G_o} = \mathbf{F_o} H \left(H\mathbf{P}^{-1}\right)^H \quad (3.20)$$

Step 3. The proposed AS scheme is given by equations (3.1), (3.10).

(b) Assume additionally that the design constants ϵ_i satisfy $\epsilon_1 I \preceq \mathbf{P}^{*-1} \preceq \epsilon_2 I, \quad \epsilon_3 Q \preceq \mathbf{P}^{*-1} Q \mathbf{P}^{*-1}$ where $\mathbf{P}^*$ corresponds to the optimal value for the matrix $\mathbf{P}$ with respect to the optimization problem (3.14). Then, for each positive constant C_1, there exists a lower bound $\bar{L}$ on the number L of activation functions so that (3.21) holds for all choices for L satisfying $L \geq \bar{L}$.

Furthermore to Theorem 1 and by using the same arguments as those of [7],[10] it can be seen that, in case (3.21) holds, then the solutions of the overall system satisfy the following inequality:

$$\begin{aligned} |\zeta(t)| &\leq \alpha_1 \exp^{-\alpha_2 t} |\zeta(0)| + \alpha_3 \\ \alpha_1 &= \sqrt{\frac{\epsilon_2}{\epsilon_1}}, \ \alpha_2 = \left(\frac{\epsilon_3}{2\epsilon_1} - \frac{\epsilon_1 \mathcal{O}(1/L)}{2\epsilon_2^2}\right) \\ \alpha_3 &= \mathcal{O}(1/L) \equiv \mathcal{O}(C_1) \end{aligned} \quad (3.23)$$

What is important about (3.23) is that the design constants ϵ_i in the optimization problem (3.19) can serve as tuning/design parameters in a similar fashion as e.g., the noise covariance matrices in Kalman Filter applications: (3.23) can be used to evaluate the effects and trade-offs of different choices for ϵ_i on the overshoot, convergence and steady-state performance of the AS design and thus it can provide a guide on how to choose ϵ_i so that the desired performance is obtained.

We close this section, by presenting two further remarks.

- Incorporation of maximum speed and control constraints can be accomplished within the proposed approach by introducing the "fictitious" control v calculated as $\dot{u} = v$ and use the proposed approach to design v instead of u. The above transformation – quite typical in control applications – can be then used to augment the system dynamics so that the actual control vector u is treated as a state variable and thus control/speed constraints can be treated in a similar fashion as the constraints (2.6).
- All the arguments of this paper can be straightforwardly extended in case of PS applications.

IV. Simulations

To evaluate the efficiency of the proposed scheme we performed a series of simulation experiments of two flying robots – assumed for simplicity to be perfectly localized – deployed to accomplish/meet the following tasks/constraints:

(PC) The robots should fly at maximum height of 1.1 Distance Units (DU) while their xy coordinates should be constrained in the area $[0, 1]^2$ DUs. The robots should also make sure that they do not "hit" the unknown terrain they are deployed to map [see task (M) below].

(M) the robots should fly over an unknown terrain that comprises 36 bell-shaped "cliffs" in order to map it as accurately as possible. The height and the width of each of the cliffs is randomly generated according to $z = height \exp\left(-width\left((x - x_c)^2 + (y - y_c)^2\right)\right)$, where $height$ can take any value in the range $[0, 0.9]$ DUs, $widht$ takes any value in $[50, 250]$ and x_c, y_c denote the center of the cliff. The robots are equipped with downlooking cameras that return the following sensor measurement:

$$y_R^M = \begin{cases} 0 & \text{if } |x_R - x_L| \leq 0.2 \\ & \text{or } |x_{R,3} - x_{L,3}| \leq 0.1 \\ x_{L,3} + |x_R - x_L|\xi & \end{cases}$$

where y_R^M denotes the sensor measurement, x_R denotes the location of a robot, x_L denotes the location of the top of the cliff and ξ is the sensor noise. It is worth noticing that the above assumption for the sensor model is quite realistic (although over-simplified) in case the VSLAM algorithm of [1] is employed to map the terrain using camera measurements.

(TT) Concurrently to the task (M) described above, the robots should perform tracking of a target and by employing robot-to-target distance sensor measurements. A multiplicative distance sensor noise model was assumed, described as follows:

$$y_R^T = |x_R - x_T| + \sigma |x_R - x_T| \xi$$

where y_R^M denotes the sensor measurement, x_R denotes the location of a robot, x_T denotes the location of the target, ξ

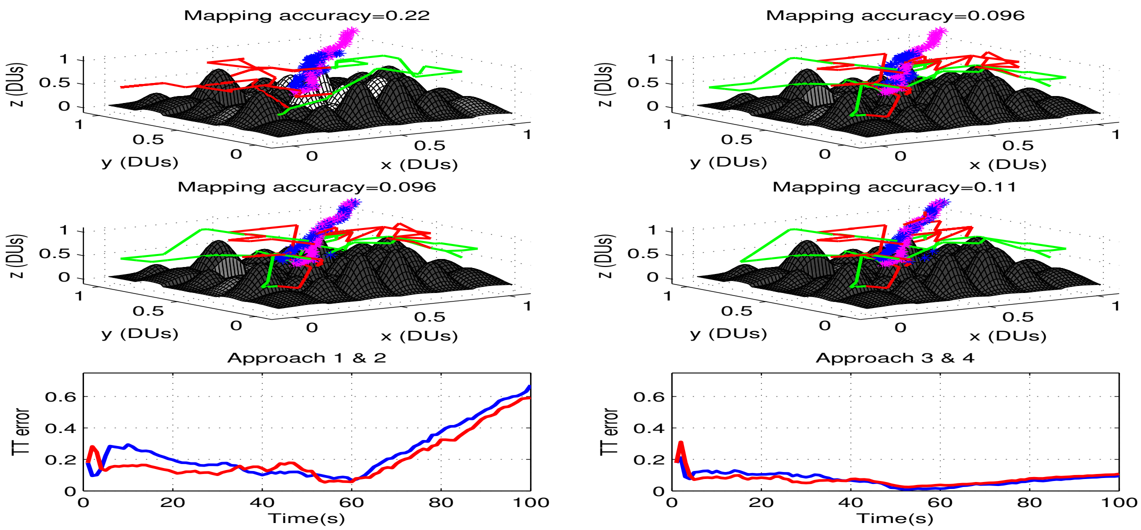

Fig. 1. Application of the proposed methodology: 2 flying robots are deployed to map unknown terrains while concurrently perform target tracking. The green and red curves correspond to the robot trajectories, the magenta curve corresponds to the actual target location and the blue one to the estimated target location. The darker is a cliff, the better is the mapping accuracy. Design 1 (upper left); Design 2 (upper right); Design 3 (middle left); Design 4 (middle right). Lower left: target tracking error for Design 1 (blue curve) and Design 2 (red curve). Lower right: target tracking error for Design 3 (blue curve) and Design 4 (red curve).

is the sensor noise and σ is a positive constant in $[0, 0.5]$. (SC) A maximum robot speed constraint of 0.1 DUs per second, at each of the three xyz-coordinates was also employed. (Localization) In the simulations both robots were assumed to be perfectly localized, i.e., their xyz position was assumed to be known to the AS design.

Four different AS designs were accomplished using the theoretical tools described in the previous section. Designs 1-3 concern an active sensing scheme for the mapping task and a passive scheme for target tracking. Design 4 concerns a combined (simultaneous) active sensing scheme both for mapping and target tracking. The choice for the number L of activation functions was as follows: **Design 1**: For both schemes L was chosen according to $L = 1$; **Design 2**: $L = 50$ for the mapping scheme and $L = 10$ for the target tracking one; **Design 3**: $L = 50$ for the mapping scheme and $L = 50$ for the target tracking one; **Design 4**: $L = 50$ for the mapping scheme and $L = 50$ for the target tracking one. Figure 1 exhibits the results obtained for a particular terrain and target trajectory, for $\sigma = 0.1$ and by using the Designs 1-4 described above. As it is clear from these figures, by increasing the number L of activation functions, the efficiency of the overall scheme is improved.

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