ON SCALE-INVARIANT SOLUTIONS OF THE NAVIER-STOKES EQUATIONS

HAO JIA, VLADIMÍR ŠVERÁK

UNIVERSITY OF MINNESOTA

Abstract

We discuss the forward self-similar solutions of the Navier-Stokes equations. It appears these solutions may provide an interesting window into non-perturbative regimes of the solutions of the equations.

1. Introduction

We consider the classical Cauchy problem for the incompressible Navier-Stokes equation

$$\left.\begin{array}{rcl} u_t + u\nabla u + \nabla p - \Delta u &=& 0 \\ \operatorname{div} u &=& 0 \end{array}\right\} \qquad \text{in } \mathbb{R}^3 \times (0,\infty)\,, \tag{1.1}$$

$$u|_{t=0} = u_0 \qquad \text{in } \mathbb{R}^3\,. \tag{1.2}$$

We recall that the problem is invariant under the scaling

$$\begin{array}{rcrcl} u(x,t) &\to& u_\lambda(x,t) &=& \lambda u(\lambda x, \lambda^2 t)\,, \\ p(x,t) &\to& p_\lambda(x,t) &=& \lambda^2 p(\lambda x, \lambda^2 t)\,, \\ u_0(x) &\to& u_{0\lambda}(x) &=& \lambda u_0(\lambda x)\,, \end{array} \tag{1.3}$$

where $\lambda > 0$. We say that a solution u is *scale-invariant* if $u_\lambda = u$ and $p_\lambda = p$ for each $\lambda > 0$. Similarly, we say that an initial condition u_0 is scale-invariant, if $u_{0\lambda} = u_0$ for each $\lambda > 0$. This is of course the same as requiring that u_0 be $(-1)-$ homogeneous.

We will discuss the following result, which we recently proved in [6].

Theorem 1.1. *Assume u_0 is scale-invariant and locally Hölder continuous in $\mathbb{R}^3 \setminus \{0\}$ with $\operatorname{div}\, u_0 = 0$ in $\mathbb{R}^3$. Then the Cauchy problem (1.1), (1.2) has at least one scale-invariant solution u which is smooth in $\mathbb{R}^3 \times (0,\infty)$ and locally Hölder continuous in $\mathbb{R}^3 \times [0,\infty) \setminus \{(0,0)\}$.*

Previously this result has been known only under suitable smallness conditions on u_0, see for example [2, 10]. For small u_0 one can also prove uniqueness (in suitable function classes). It is quite conceivable that uniqueness may fail for large data. We will discuss this point in more detail below.

2. Well-posedness and scale invariant initial data

We recall that a function space X of div-free fields on $\mathbb{R}^3$ is *homogeneous* if $||u_{0\lambda}||_X = \lambda^\alpha ||u_0||_X$ for some $\alpha \in \mathbb{R}$. A homogeneous space X is *scale invariant* (for the Navier-Stokes scaling) if $\alpha = 0$, i.e. $||u_{0\lambda}||_X = ||u_0||_X$. Within the class of the homogeneous function spaces, the borderline spaces for perturbation theory of (1.1), (1.2) should be scale-invariant. Perturbation theory for the well-posedness results for the Navier-Stokes equation with initial data in such spaces was initiated in a well-known paper [8]. Paper [10] can be considered as a culmination of these developments. In [8] the function space X is taken as $X = L^3$ (where we slightly abuse notation by using L^3 for div-free vector fields which belong to L^3). We note that the function $|x|^{-1}$ "just misses" $L^3(\mathbb{R}^3)$. In [10] the space X is taken $X = \mathrm{BMO}^{-1}$ (again restricted to div-free fields). We note that the function $|x|^{-1}$ belongs to BMO^{-1}. The well-posedness result for $X = \mathrm{BMO}^{-1}$ is slightly more subtle here than with $X = L^3$ in that the equations are well-posed in BMO^{-1} only for sufficiently small data, even in the sense of the local-in-time well-posedness. To get a local-in-time well-posedness for large data, one must further restrict the function space. As we shall see, for $X = \mathrm{BMO}^{-1}$ this smallness assumption may in fact be essential. It is conceivable that the equations are not well-posed (even locally in time) for large initial data in BMO^{-1}.

At a heuristic level it is not hard to see that $(-1)-$homogeneous vector fields should play an important role. If $u_0(x)$ is such a vector field which is smooth away from the origin and $a > 0$, then

$$|\Delta(au_0)| \sim a|x|^{-3}, \qquad |au_0\nabla(au_0)| \sim a^2|x|^{-3} \tag{2.1}$$

We see that for $a << 1$ the linear viscous term dominates, whereas for $a >> 1$ the non-linear term dominates. At $a \sim 1$ both terms should be of the same order of magnitude (assuming the quantities u_0, ∇u_0 are of similar magnitude on the unit sphere).

The solutions obtained by the perturbation theory are often called *mild solutions*. These solutions exist on a certain maximal interval of existence $[0, T)$ and are regular in $\mathbb{R}^3 \times (0, T)$, see, for example, [3, 4]. For small initial data we can take $T = \infty$, but for large initial data then we can conceivably have $T < \infty$, although it is not known whether this really happens. We emphasize again that once some div-free vector field with a singularity of the strength $\sim |x|^{-1}$ belongs to X, then one needs a smallness assumption even for the proofs of the local-in-time well-posedness.

In the classic paper [15] many of these ideas are considered in slightly different spaces, which are not homogeneous.

In addition to the class of mild solutions, we have the class of the *weak solutions*. The solutions of this kind were first constructed in [15] and their construction is based on the energy inequality, weak convergence and compactness. It was realized relatively recently, see [14], that this technique is applicable even when the energy of the initial data u_0 is only locally finite ($u_0 \in L^2_{\mathrm{loc}}$), with the additional assumption

$$\lim_{x\to\infty} \int_{B_{x,r}} |u_0|^2 \, dx = 0 \ . \tag{2.2}$$

In particular, there is no problem in constructing weak solutions when the initial datum is a $(-1)-$homogeneous field u_0 which is locally bounded away from 0. Unlike for the mild solutions, in the construction of the weak solutions the function $|x|^{-1}$ does not play any distinguished role. For example, the scale invariant fields continuous away from the origin satisfy all the assumptions needed for the construction with good margins. The function $|x|^{-1}$ "comes back" when we try to investigate uniqueness of the weak solutions. At present the best available results for the uniqueness of the weak solutions are of the same form as already discussed in [15], and later extended in [18], [14] and other works. The result say, roughly speaking, that if we have two weak solutions u, v for the same initial datum u_0 and one of the solutions has similar regularity as the mild solutions, then the two solutions coincide. The initial datum of the "good solution" must essentially have the same regularity as required by the perturbation theory for the mild solutions. Viewed from the perspective of this proof, the function $|x|^{-1}$ makes its return, even when we deal only with the weak solutions.

Is the borderline role of the $(-1)-$homogeneous functions an artefact of our techniques, or is there something deeper behind it? We will argue for the latter.

3. Proof of Theorem (1.1)

To prove Theorem 1.1, we seek the solution $u(x,t)$ in the form

$$u(x,t) = \frac{1}{\sqrt{t}}\, U\left(\frac{x}{\sqrt{t}}\right) . \tag{3.1}$$

The Navier-Stokes equation for u gives

$$-\Delta U - \frac{1}{2}U - \frac{1}{2}x\nabla U + U\nabla U + \nabla P = 0, \qquad \operatorname{div} U = 0\,, \tag{3.2}$$

in $\mathbb{R}^3$. For a scale-invariant u_0 the problem of finding a scale-invariant solution of the Cauchy problem (1.1), (1.2) is equivalent to the problem of finding a solution of (3.2) with the asymptotics

$$|U(x) - u_0(x)| = o\left(\frac{1}{|x|}\right) , \qquad x \to \infty\,. \tag{3.3}$$

The problem (3.2), (3.3) is reminiscent of the classical Leray's problem of finding steady-state solution of the Navier-Stokes equation in a bounded domain (which is now replaced by the whole space $\mathbb{R}^3$), with a given boundary conditions (which is now replaced by (3.3)). Heuristically it is clear that the main difficulty in pursuing this analogy is the potentially uncontrolled behavior of U for $x \to \infty$. Roughly speaking if we can show that near ∞ the function U and its derivatives have the same decay as $|x|^{-1}$ and the corresponding derivatives, then we can conclude that nothing surprising is happening near ∞, and the situation is indeed analogous to the bounded domain. (One still needs to establish estimates in the finite region, but these are very similar to the classical case of a bounded domain.) The main difficulty is in establishing these estimates. Once such estimates are established, we can essentially follow the classical Leray proof of Leray for the existence of the steady solutions in bounded domains, see [6] for details.

4. Possible Non-uniqueness

As in the case of the bounded domains, the Leray-Schauder approach gives existence of the solutions, but not uniqueness. In the case of bounded domains one does not generically expect uniqueness for large data, and this non-uniqueness is in fact expected to be quite typical in the context of the steady Navier-Stokes, once the data is large. Let us for example consider the problem

$$\begin{array}{rcll} -\Delta u + u\nabla u + \nabla p & = & 0 & \text{in } \Omega \\ \operatorname{div} u & = & 0 & \quad '' \\ u|_{\partial\Omega} & = & \lambda g & \text{at } \partial\Omega \end{array} \tag{4.1}$$

where g is a given smooth vector field at the boundary satisfying the compatibility condition $\int_{\partial\Omega} g = 0$ and $\lambda > 0$ is a parameter. Eventually we aim to take $\lambda = 1$. We know the equations (4.1) have a unique solution for small λ (by perturbation arguments and energy inequality, for example), with $u|_{\lambda=0} = 0$. We can try to continue the solution u into $\lambda > 0$ as $u = u(\lambda)$, but the curve of can "turn back" and will not be a graph of a function of λ. The existence of these turning points signals non-uniqueness. For bounded domain the existence of such turning points is presumably quite typical, and for generic set-ups we do expect non-uniqueness once the function λg is "sufficiently large". (This is true especially in dimension $n = 3$. In dimension $n = 2$ the situation might be in some cases different, see a related result in [16].)

Could this also be the case for the problem (3.2), (3.3)? This would lead to non-uniqueness for the Cauchy problem (1.1), (1.2) with scale-invariant u_0. We believe it is likely that this indeed happens, and that the solution of the Cauchy problem (1.1), (1.2) for the scale-invariant u_0 may not be unique for large data. This would mean, for example, that the initial value problem may not be well-posed in BMO^{-1} if the initial condition is not small.

The possible non-uniqueness might be detected by following the curve of solutions $U = U(\lambda)$ of the problem (3.2) with the "boundary condition" (3.3) replaced by

$$|U(x) - \lambda u_0(x)| = o\left(\frac{1}{|x|}\right), \qquad x \to \infty\,, \tag{4.2}$$

starting at $\lambda = 0$. For λ small we have a unique solution $U(\lambda)$ and we can observe the spectrum of the linearized problem as we increase λ. Let us denote the spectrum by $\Sigma(\lambda)$. One expects that for small λ we will have $\Sigma(\lambda) \subset \Pi = \{z,\ \operatorname{Re} z < 0\}$. As we increase λ, the spectrum may leave Π. If is does so through $z = 0$, we expect a turning point in the curve of the solution and non-uniqueness as discusses above. What happens when the spectrum leaves Π through the imaginary axis? It is natural to expect that (under some natural assumptions) this will correspond to a Hopf bifurcation, with the appearance of periodic solution to the equation

$$U_s - \Delta U - \frac{1}{2}U - \frac{1}{2}x\nabla U + U\nabla U + \nabla P = 0, \qquad \operatorname{div} U = 0\,, \tag{4.3}$$

with the "boundary condition" at ∞ given by (4.2). By a (standard) change of variables

$$u(x,t) = \frac{1}{\sqrt{t}}U\left(\frac{x}{\sqrt{t}}, \log\frac{t}{t_0}\right), \tag{4.4}$$

we see that this would correspond to a solution u of the initial value problem which would be only "discretely scale-invariant" for the scale invariant initial datum u_0. By this we mean that $\lambda u(\lambda x, \lambda^2 t) = u(x,t)$ not for all $\lambda > 0$, but only for a discrete subgroup $\{\lambda_0^k, k \in \mathbf{Z}\}$ of $\mathbb{R}_+$. The existence of such solutions for discretely scale-invariant u_0 with λ_0 close to 1 is proved in a recent paper [20]. Such solutions would still violate uniqueness for the

Cauchy problem (1.1), (1.2) for the scale-invariant initial data u_0. In this case there would be a scale-invariant solution guaranteed by Theorem (1.1) and another solution which is not scale-invariant, but only discretely scale-invariant. We believe that such scenarios are quite likely.

The above considerations apply to the Cauchy problem with the scale-invariant initial data. Can such consideration be taken even further, to some solutions with finite energy obtained by a suitable "truncation at infinity" of the scale-invariant initial data? If this is the case, then we might not only have non-uniqueness for the scale-invariant initial data, but also non-uniqueness for finite-energy initial data, and – in particular – for the Leray-Hopf weak solutions. Moreover, the non-uniqueness would appear right at the borderline of the classes for which uniqueness can be proved via the weak-strong uniqueness theorems mentioned earlier. It is interesting to note the opinion of some prominent mathematicians on the question of the uniqueness of Leray-Hopf weak solutions. In [7] we can find the following comment (p. 217): "It is hard to believe that the initial value problem for the viscous fluid in dimension $n = 3$ could have more than one solution, and more work should be devoted to the study of the uniqueness question." On the other hand, it is known that O. A. Ladyzhenskaya believed in non-uniqueness of the weak solution. The answer to the uniqueness question is still not known, but our current opinion, based on the discussion above, leans towards the non-uniqueness.

5. Estimates

An important theme in [6] can be perhaps called local-in-space regularity estimates near the initial time $t = 0$. The conection to estimates of solutions of (3.2) near ∞ can be seen from (3.1): if, say, $\nabla^k u$ is bounded in $\{x,\ 1 \le |x| \le 2\}$ for times close to 0, it means $\nabla U(x) = O(|x|^{-1-k})$ as $|x| \to \infty$.

The following statement appears to be quite natural

(S) *Modulo the usual (and quite mild) non-local influences of the pressure, local regularity of the initial data propagates for at least a short time.*

Results in the direction of (S) can be found already in the classical paper [1]. More recently, related questions about vorticity propagation have been studied in [19]. Our main result in this direction, which is behind the necessary a-priori estimates for the solutions of (3.2) is as follows.

Theorem 5.1. (Local Hölder regularity of Leray solutions)
Let $u_0 \in L^2_{loc}(R^3)$ with $\sup_{x_0\in R^3} \int_{B_1(x_0)} |u|^2(x)dx \le \alpha < \infty$. Suppose u_0 is in $C^\gamma(B_2(0))$ with $\|u_0\|_{C^\gamma(B_2(0))} \le M < \infty$. Then there exists a positive $T = T(\alpha,\gamma,M) > 0$, such that any Leray solution u with the initial datum u_0 (which implies u is also a local suitable weak solution in the sense of [1]), satisfies

$$u \in C^\gamma_{\rm par}(\overline{B_{1/4}} \times [0,T]), \ \text{ and } \|u\|_{C^\gamma_{\rm par}(\overline{B_{1/4}}\times[0,T])} \le C(M,\alpha,\gamma). \tag{5.1}$$

We refer the reader to [6] for the precise definition of Leray solution.

Our proof of 5.1 in [6] is based on a combination of techniques from [5, 9, 11, 13, 14]. Heuristically, the main point is that one can obtain a sufficient control of the energy flux into "good regions" from the rest of the space, see Section 3. Once we know that only small amount of energy can move into the "good region" one can use (a slight modification of) partial regularity schemes in [11, 13] to prove regularity.

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References

[1] Caffarelli, L., Kohn, R.-V., Nirenberg, L., *Partial regularity of suitable weak solutions of the Navier-Stokes equations*, Comm. Pure Appl. Math., Vol, XXXV (1982), pp. 771-831.

[2] Cannone, M., Planchon, F., *Self-Similar solutions for Navier-Stokes equations in R^3*, Comm. Partial Differential Equations 21 (1996), no. 1-2, 179-193

[3] Dong, H., Du, D., On the local smoothness of solutions of the Navier-Stokes equations, J. Math. Fluid Mech. 9 (2007), no. 2, 139–152.

[4] Germain, P., Pavlovic, N., Staffilani, N., Regularity of solutions to the Navier-Stokes equations evolving from small data in BMO^{-1}, Int. Math. Res. Not. IMRN 2007, no. 21, 35 pp.

[5] Jia, H., Šverák, V., *Minimal L^3-initial data for potential Navier-Stokes singularities*, arXiv:1201.1592.

[6] Jia, H., Šverák, V., *Local-in-space estimates near initial time for weak solutions of the Navier-Stokes equations and forward self-similar solutions*, arXiv:1204.0529

[7] Hopf, E., *Über die Anfangswertaufgabe für die hydrodynamischen Grundgleichungen.* Math. Nachr. 4, (1951), 213–231.

[8] Kato, T. *Strong L^p-solutions of the Navier-Stokes equation in R^m, with applications to weak solutions*, Math. Z. 187 (1984), no. 4, 471-480.

[9] Kikuchi, N., Seregin, G., *Weak solutions to the Cauchy problem for the Navier-Stokes equations satisfying the local energy inequality*, AMS translations, Series 2, Volume 220, pp. 141-164.

[10] Koch, H., Tataru, D., *Well-posedness for the Navier-Stokes equations*, Adv. Math. 157 (2001), no. 1, 22-35.

[11] Ladyzhenskaya, O. A., Seregin, G. A., *On partial regularity of suitable weak solutions to the three-dimensional Navier-Stokes equations*, J. Math. Fluid Mech. 1 (1999), no. 4, 356-387.

[12] Ladyzhenskaya, O.A, *On Uniqueness and smoothness of generalized solutions to the Navier-Stokes equations*, Zapiski Nauchn, Seminar. POMI, 5 (1967), pp. 169-185

[13] Lin, F-H., *A new proof of the Caffarelli-Korn-Nirenberg theorem*, Com. Pure Appl. Math. 51 (1998), no. 3, 241-257.

[14] Lemarié-Rieusset, P. G., *Recent developments in the Navier-Stokes problem*, Chapman & Hall/CRC research Notes in Mathematics, 431 Chapman & Hall/CRC, Boca Raton, FL, 2002

[15] Leray, J., *Sur le mouvement d'un liquide visqueux emplissant l'espace*, Acta Math. 63 (1934), pp. 193-248.

[16] Marchioro, C., *An example of absence of turbulence for any Reynolds number*, Comm. Math. Phys. 105 (1986), 99–106.

[17] Prodi, G., *Un teorema di unicità per el equazioni di Navier-Stokes*, Ann Mat. Pura Appl. 48 (1959), pp. 173-182

[18] Serrin, J., *The initial value problem for the Navier-Stokes equations.* 1963 Nonlinear Problems (Proc. Sympos., Madison, Wis., 1962) pp. 6998 Univ. of Wisconsin Press, Madison, Wis.

[19] Tao, T., *Localization and compactness properties of the Navier-Stokes global regularity problem*, arXiv:1108.1165v3

[20] Tsai, T.-P., *Forward Discrete Self-Similar Solutions of the Navier-Stokes Equations*, arXiv:1210.2783