Please note: *The Friday, May 10 office hour is moved to Thursday, May 9, 1pm–4:30pm.*

The final examination will consists of 5 or 6 problems from the following areas:

1. Linear first order equations $\frac{dx}{dt} = a(t)x + b(t)$.
2. Separable equations $\frac{dx}{dt} = a(t)f(x)$; phase portraits of the (scalar) autonomous equations $x' = f(x)$ and the global behavior the solutions.
3. Newton's equations $mx'' = F(x)$ for systems with one degree of freedom.
4. Linear systems $x' = Ax + f(t)$, where A is a (constant) $n \times n$ matrix; calculation of matrix exponential e^{tA}; Jordan forms.
5. Higher-order linear equations with constant coefficients $x^{(m)} + a_1 x^{(m-1)} + \ldots + a_m x = f(t)$.
6. Second order systems $x'' + Ax = f(t)$ with a (constant) symmetric matrix A; diagonalization of symmetric matrices.
7. Stability of equilibria of general autonomous systems $x' = f(x)$; linearization of the system at an equilibrium. (The various notions of stability introduced in lecture 37 and Theorem 7 in the Lecture Log.)

Sample problems:

1. Find a formula for the solution of the problem

$$x' = (a + \varepsilon \cos \omega t)x + b, \qquad x(0) = 0\,, \tag{1}$$

where $a, b, \varepsilon, \omega$ are constants. (The formula may involve integrals which cannot be evaluated in terms of elementary functions.)

2. Consider the equation

$$x' = x(1 - x^2) \tag{2}$$

for a function $x\colon \mathbf{R} \to \mathbf{R}$.
(i) Find all equilibria of the equation.
(ii) Decide which of the equilibria are stable and which are unstable.
(iii) For any $a \in \mathbf{R}$ determine the limit $\lim_{t\to\infty} x(t)$ for the solution of (2) with the initial condition $x(0) = a$.

3. Let $x\colon [0, \infty) \to \mathbf{R}$ satisfy

$$x' = -\frac{x^2}{1+t^2}\,, \qquad x(0) = 1. \tag{3}$$

Calculate $\lim_{t\to\infty} x(t)$.

4. Consider the system of equations

$$\begin{aligned} x_1' &= -2x_1 + x_2 + x_3\,, \\ x_2' &= x_1 - 2x_2 + x_3\,, \\ x_3' &= x_1 + x_2 - 2x_3\,. \end{aligned}$$

Show that for each solution $x(t)$ and each $j = 1, 2, 3$ we have $x_j(t) \to \frac{x_1(0)+x_2(0)+x_3(0)}{3}$ as $t \to \infty$. (Hint: Note that $\det \begin{pmatrix} a & 1 & 1 \\ 1 & a & 1 \\ 1 & 1 & a \end{pmatrix} = 0$ when $a = 1$.)

5. Consider the system

$$\begin{aligned} x_1'' + 2x_1 + x_2 &= -\cos 2\omega t + 3\sin 5\omega t\,, \\ x_2'' + x_1 + 2x_2 &= \cos \omega t - \sin 7\omega t\,. \end{aligned}$$

Find all values ω for which all solutions of the system in the interval $(0, \infty)$ remain bounded.

6. Given $a \in \mathbf{R}$, find all solutions of the equation $\frac{d^6x}{dt^6} + ax = 1$ which are bounded on $(0, \infty)$.

7. Let

$$A = \begin{pmatrix} 2 & 1 & -2 \\ 2 & 2 & -2 \\ 2 & -1 & 2 \end{pmatrix}. \tag{4}$$

Calculate e^{tA}.

8. For $a, \sigma > 0$ consider the system

$$\begin{aligned} x' &= a - x - xy\,, \\ y' &= -\sigma y + xy\,. \end{aligned} \tag{5}$$

Find the equilibria of the system and investigate their linearized stability.

As an optional part of this problem, you can also try to draw the phase portrait of the system and determine the global dynamics, at least in the region $x \geq 0$, $y \geq 0$. This optional part is more difficult than the rest of the examples, and problems with a similar level of difficulty will not appear in the final exam. Nevertheless, the problem is still recommended as a good exercise.