Math 5525 Homework Assignment 1 Spring 2013

due Feb 11, 2013, at the beginning of class, 2:30pm

1. Exercise 5, page 12 [1]

2. Exercise 6, page 12

3. Exercise 9, page 12

4. Exercise 13, page 13

5. Exercise (4), page 25

6. Exercise 1, page 39

As an optional part, you can replace equation (7.2) used in this problem by equation

$$\frac{dv}{dt} = g - \sigma v^2 \tag{1}$$

and answer the same questions as in the original problem.

7*. (Optional)

Let a, x_0 be strictly positive real numbers and let $\varepsilon \in (0,1)$. Consider the equation

$$\frac{dx}{dt} = -ax^{1-\varepsilon} \tag{2}$$

with the initial condition

$$x(0) = x_0\,. \tag{3}$$

Find the formula for $x(t)$ and show that the solution will "touch" the trivial solution $\overline{x}(t) \equiv 0$ of (2) in finite time. In other words, there exists $T > 0$ such that $x(t) > 0$ on $[0,T)$ and $x(t) \to 0$ as $t \to T_+$. Another way to put it is that the trajectories $x(t)$ and $\overline{x}(t)$ of two different solutions collide at (finite) time T. [2] This is never possible if the function giving the equation is smooth.

Denoting by $x^{(\varepsilon)}(t)$ the solution for a given ε, show that

$$\lim_{\varepsilon \to 0} x^{(\varepsilon)}(t) = x_0 e^{-at}\,. \tag{4}$$

8*. (Optional)

Let $a(t) > 0$ be a continuous function on $[0,T)$ and let $x_0 > 0$. Consider the equation

$$\frac{dx}{dt} = -a(t)x \tag{5}$$

with the initial condition

$$x(0) = x_0\,. \tag{6}$$

Show that $\lim_{t \to T_+} x(t) = 0$ if and only if $\int_0^T a(t)\,dt = +\infty$.

[1] Here and elsewhere, all the references are to the textbook

[2] This means that if we run the equation backwards in time from T from the initial condition 0, we have at least two solutions. The solution is not uniquely determined by the equation and the initial condition! This is this is only possible here because the function $x^{1-\varepsilon}$ defining the equation does not have a bounded derivative at $x = 0$. Equations given by smooth functions do not exhibit such behavior.