Math 5525 Homework Assignment 2 Spring 2013

due Feb 25, 2013, at the beginning of class, 2:30pm

1. Exercise 1, page 57 [1]

2. Exercise 2, page 58 (Hint: In lecture 9 we did a similar calculation with the right-hand side $e^{i\omega' t}$. Note that the imaginary part of $e^{i\omega' t}$ is $\sin \omega' t$. By definition, resonance occurs at the frequency at which the amplitude of the oscillation given by the imaginary part of formula (135) in the Lecture Log is maximal, which means that – in the notation of the lecture – we have to find a frequency ω' where $|A|$ attains its maximum. This is where the "resonance curve" is at its peak. This calculation is done on page 57 of the textbook.)

3. Exercise 2, page 66

4. Exercise 1, page 68

5. Exercise 2, page 68

6. Exercise 5, page 40

7*. (Optional)

Let us consider a planet of radius R and mass M. Assume the mass in the planet is distributed symmetrically, so that the density of the planet is only a function of the distance from its center. Under this assumption the gravitational force outside the planet is exactly the same as if all the mass was concentrated at the center: at hight z above the surface of the planet the acceleration due to gravity is $g(z) = \frac{\kappa M}{(R+z)^2}$ where κ is Newton's gravitational constant. Assume the planet has an atmosphere of an ideal gas at a constant temperature with density $\rho = \rho(z)$ and pressure $p = p(z)$. Assume the gravitational effects due to the atmosphere itself can be neglected and that the atmosphere is at rest. Simple physics shows that p and ρ then satisfy $\frac{dp}{dz} = -g(z)\rho(z)$ and $p(z) = C\rho(z)$ for some constant C.[2] If the density at the surface is ρ_0, calculate the density at height $z \geq 0$ above the surface of the planet. Show that an atmosphere of a finite total mass can never give the calculated solutions. Therefore, in the above model a planet cannot keep its atmosphere indefinitely, the atmosphere eventually escapes into space.[3] You can also do the calculation under the assumption $g = \text{const.}$, which is a good approximation when $\frac{z}{R}$ is small.[4] The result of this simpler calculation is sometimes called "exponential atmosphere".

8*. (Optional)

Consider the equation

$$x'' = f(x, x')\,, \tag{1}$$

where $x = x(t)$ is a scalar function of t. Let us set $p = x'$. The quantity p is naturally a function of t. However, on intervals where x is monotone, p can also be considered as a function of x. Show that when viewed in this way, the function $p = p(x)$ satisfies

$$p\frac{dp}{dx} = f(x, p)\,. \tag{2}$$

[1] Here and elsewhere, all the references are to the textbook

[2] If T is the temperature, then $C = \frac{R^* T}{M}$, where R^* is the universal gas constant and M is the molar mass, see `http://en.wikipedia.org/wiki/Barometric_formula` for details and the actual values of these parameters for air and Earth.

[3] If you plug in the actual numbers for Earth, you will see that the potential "escape effect" should be very small, even in our over-simplified model. See `http://en.wikipedia.org/wiki/Atmospheric_escape` for more details and additional references on this phenomena.

[4] The assumption that $g = \text{const.} = \kappa_1 M$ would be true for all $z > 0$ in a one-dimensional world. In a two dimensional world one should take $g = \frac{\kappa_2 M}{(R+z)}$. In this case the planet can keep its atmosphere on our model, provided $\frac{\kappa_2 M}{C} > 2$.