1. Let us consider for example $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. Then $e^{tA} = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$, $e^{tB} = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}$ and $e^{tA}e^{tB} = \begin{pmatrix} 1+t^2 & t \\ t & 1 \end{pmatrix}$. Let $C = A+B$. Then $C^2 = I$ and therefore $e^{t(A+B)} = e^{tC} = I + tC + \frac{t^2}{2!}I + \frac{t^3}{3!}C + \ldots = \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}$. We see that $e^{tA}e^{tB} \neq e^{t(A+B)}$ for each $t \neq 0$.

In the context of this problem one should mention the following classical calculation. Let A, B be any two $n \times n$ matrices. Expanding the exponentials, we obtain $e^{tA}e^{sB} - e^{tA+sB} = \frac{st}{2}(AB - BA) + O(t^2 + s^2)^{\frac{3}{2}}$, $s, t \to 0$. We see that for small s, t the left-hand side can vanish only when $AB - BA = 0$, i. e. the matrices A, B commute. Therefore for any two non-commuting matrices A, B and sufficiently small $s, t \neq 0$ the matrices tA, sB give an example with the desired property.

2.

Matrix A_1

$\det(A_1 - \lambda I) = (2-\lambda)\lambda^2$. Hence the eigenvalues are $\lambda_1 = 2$ and $\lambda_2 = \lambda_3 = 0$. The equation $(A-2I)x = 0$ is easily seen to be satisfied by $x^{(1)} = e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$.

From general theory we know that in the situation above the eigenspace of λ_1 must be one-dimensional, and hence, up to a multiplicative factor, e_2 is the only eigenvector corresponding to $\lambda_1 = 1$. This is of course seen in many other ways. The matrix $A_1 - \lambda_2 I = A_1 - 0I = A_1$ has rank two, and hence there is only one-dimensional eigenspace associated with the double eigenvalue 0. The eigenvector can be obtained by solving $A_1 x = 0$ and is given (up to a multiplicative factor) by $x^{(2)} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$.

Matrix A_2

$\det(A_2 - \lambda I) = (1-\lambda)^2(2-\lambda)$. The eigenvalues therefore are $\lambda_1 = 2$ and $\lambda_2 = \lambda_3 = 1$. The eigenvector corresponding to λ_1 is easily seen to be $x^{(1)} = e_2$ and the eigenvector corresponding the $\lambda_1 = 1$ is easily seen to be $x^{(2)} = e_1$.

Matrix A_3

$\det(A_3 - \lambda I) = (1-\lambda)^3$. Hence we have $\lambda_1 = \lambda_2 = \lambda_3 = 1$. The eigenspace is easily seen to be one-dimensional, spanned by $x^{(1)} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$. The dimension of the eigenspace corresponding to an eigenvalue λ is called the *geometric multiplicity* of λ.

We recall that the multiplicity of the eigenvalue taken as the multiplicity of the root of the characteristic polynomial is called the *algebraic multiplicity*.

3. For each of the matrices above and each of the eigenvalues λ the dimension of the kernel of $A - \lambda I$ is one. In other words, all eigenspaces of all the matrices are one-dimensional, or, equivalently, the geometric multiplicity of each of the eigenvalues is 1. Therefore in the Jordan canonical form of each of the matrices each Jordan cell is "full", of the form[1] $J_k(\lambda)$, where k is the algebraic multiplicity (=the multiplicity of the eigenvalue taken as the multiplicity of the root of the characteristic polynomial). This means that the minimal polynomials of A_1, A_2, A_3 coincide with their characteristic polynomials.

We now calculate the generalized eigenspaces.

Matrix A_1

We solve $(A_1 - 0I)x^{(3)} = x^{(2)}$ (where $x^{(2)}$ was determined above). The generalized eigenspace of the double eigenvalue 0 will then be given by the linear span of $x^{(2)}$ and $x^{(3)}$. (Note that $x^{(3)}$ is determined only up to $tx^{(2)}$, $t \in \mathbf{C}$.) One easily sees that one can take for example $x^{(3)} = \begin{pmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \end{pmatrix}$.

Matrix A_2

The generalized eigenspace of the double eigenvalue 1 will be spanned by $x^{(2)}$ and a vector $x^{(3)}$ with $(A_2 - I)x^{(3)} = x^{(2)}$. One can take for example $x^{(3)} = e_3$.

Matrix A_3**.**

The generalized eigenspace of the triple eigenvalue 1 will be all $\mathbf{C}^3$. For the later use we calculate vectors $x^{(2)}, x^{(3)}$ with $(A_3 - I)x^{(2)} = x^{(1)}$ and $(A_3 - I)x^{(3)} = x^{(2)}$. It is easy to check that one can take for example $x^{(2)} = \begin{pmatrix} 0 \\ \frac{1}{2} \\ 0 \end{pmatrix}$ and $x^{(3)} = \begin{pmatrix} \frac{1}{12} \\ 0 \\ \frac{1}{12} \end{pmatrix}$.

4. The Jordan forms[2] are $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ for A_1, $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ for A_2, and $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ for A_3, each taken with respect to the basis of the generalized

[1] See p. 63 of the Lecture Log, formula (334).

[2] We note that, in suitable interpretation, A_2 actually already is in a Jordan form so the manipulations of A_2 below are not really necessary. The plane spanned by the x_1, x_3 axis is invariant under A_2 and the restriction of A_2 to that plane is a Jordan block. In addition e_2 is an eigenvector of A_2.

eigenvector $x^{(1)}, x^{(2)}, x^{(3)}$ calculated above for the corresponding matrix. In other words, we have

$$A_1 = \begin{pmatrix} 0 & 1 & \frac{1}{2} \\ 1 & 0 & 0 \\ 0 & -1 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & \frac{1}{2} \\ 1 & 0 & 0 \\ 0 & -1 & \frac{1}{2} \end{pmatrix}^{-1},$$

$$A_2 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1},$$

$$A_3 = \begin{pmatrix} 1 & 0 & \frac{1}{12} \\ 0 & \frac{1}{2} & 0 \\ -1 & 0 & \frac{1}{12} \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & \frac{1}{12} \\ 0 & \frac{1}{2} & 0 \\ -1 & 0 & \frac{1}{12} \end{pmatrix}^{-1}.$$

5*. Recalling the formula for $e^{tJ_k(\lambda)}$ (see, for example, the lecture log, (362), p. 68), we obtain

$$\begin{aligned} e^{tA_1} &= \begin{pmatrix} 0 & 1 & \frac{1}{2} \\ 1 & 0 & 0 \\ 0 & -1 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} e^{2t} & 0 & 0 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & \frac{1}{2} \\ 1 & 0 & 0 \\ 0 & -1 & \frac{1}{2} \end{pmatrix}^{-1}, \\ e^{tA_2} &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} e^{2t} & 0 & 0 \\ 0 & e^t & te^t \\ 0 & 0 & e^t \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1}, \\ e^{tA_3} &= \begin{pmatrix} 1 & 0 & \frac{1}{12} \\ 0 & \frac{1}{2} & 0 \\ -1 & 0 & \frac{1}{12} \end{pmatrix} e^t \begin{pmatrix} 1 & t & \frac{t^2}{2} \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & \frac{1}{12} \\ 0 & \frac{1}{2} & 0 \\ -1 & 0 & \frac{1}{12} \end{pmatrix}^{-1}. \end{aligned}$$

We calculate

$$\begin{aligned} \begin{pmatrix} 0 & 1 & \frac{1}{2} \\ 1 & 0 & 0 \\ 0 & -1 & \frac{1}{2} \end{pmatrix}^{-1} &= \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & -\frac{1}{2} \\ 1 & 0 & 1 \end{pmatrix}, \\ \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\ \begin{pmatrix} 1 & 0 & \frac{1}{12} \\ 0 & \frac{1}{2} & 0 \\ -1 & 0 & \frac{1}{12} \end{pmatrix}^{-1} &= \begin{pmatrix} \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 2 & 0 \\ 6 & 0 & 6 \end{pmatrix}, \end{aligned}$$

and obtain

$$e^{tA_1} = \begin{pmatrix} 1+t & 0 & t \\ 0 & e^{2t} & 0 \\ -t & 0 & 1-t \end{pmatrix},$$

$$
\begin{aligned}
e^{tA_2} &= \begin{pmatrix} e^t & 0 & te^t \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^t \end{pmatrix}, \\
e^{tA_3} &= e^t \begin{pmatrix} 1+3t^2 & 2t & 3t^2 \\ 3t & 1 & 3t \\ -3t^2 & -2t & 1-3t^2 \end{pmatrix}.
\end{aligned}
$$

6*. There are several ways to prove the identity.

Proof 1:
We have $\frac{d}{dt}(e^{-tA}) = -Ae^{-tA}$ and hence $\int_0^\infty -Ae^{-tA}\,dt = \int_0^\infty \frac{d}{dt}(e^{-tA})\,dt = e^{tA}|_{t=0}^{t=\infty} = -I$, as $\lim_{t\to\infty} e^{-tA} = 0$. This is the same as $\int_0^\infty Ae^{-tA} = I$ and multiplying this identity by A^{-1} we obtain the result.

Proof 2:
Let us consider the equation $x' = -Ax+b$ for a constant vector b. This equation has a steady state solution $\overline{x} = A^{-1}b$. By Theorem 6 in lecture 23 (see the lecture log) and our assumptions we know that every solution approaches $\overline{x}$ as $t \to \infty$. From the Duhamel's formula we have

$$x(t) = e^{-tA}x(0) + \int_0^t e^{-(t-s)A}b\,ds = e^{-tA}x(0) + \int_0^t e^{-sA}b\,ds\,. \tag{1}$$

Taking the limit $t \to \infty$ we see that

$$A^{-1}b = \overline{x} = \int_0^\infty e^{-sA}b\,ds\,. \tag{2}$$

The validity of (2) for each $b \in \mathbf{C}^n$ which we just established is clearly equivalent to the formula in the problem.

Proof 3:
The validity of the formula for A is equivalent to its validity for PAP^{-1} for any non-singular matrix P. Hence we can assume without loss of generality that A is in the Jordan canonical form. We see that it is enough to establish the formula for one Jordan block $J_k(\lambda)$ (with $\lambda > 0$). Writing $J_k(\lambda) = \lambda I + M$ (so that M is the $k \times k$ matrix with 1's just above the diagonal and zeroes everywhere else), we have $e^{-t(\lambda I+M)} = e^{-\lambda t}(I - tM + \ldots + (-1)^{k-1}\frac{t^{k-1}}{(k-1)!}M^{k-1}$. Integrating between 0 and ∞ while using $\int_0^\infty t^l e^{-\lambda t}\,dt = \lambda^{-(l+1)}l!$ we obtain $\int_0^\infty e^{-t(\lambda I+M)} = \lambda^{-1}\left(I - \lambda^{-1}M + \lambda^{-2}M^2 - \ldots + (-1)^{k-1}\lambda^{-(k-1)}M^{k-1}\right) = (\lambda I + M)^{-1}$, confirming the formula.
One can in fact avoid using the Jordan blocks of size > 1 by using genericity: we note that both sides of the fomula are continuous in A in the set of matrices with positive eigenvalues. Therefore it is enough to establish the formula only in the generic case when A is diagonalizable, when the above calculation reduces to the particularly simple case $k = 1$.