**1.** The general solution of $ku'' + f = 0$ in $(0, L)$ is $u(x) = -\frac{f}{2k}x^2 + ax + b$, where $a, b$ are arbitrary constants. For such a solution we have

$$u(0) = b\,, \quad u'(0) = a\,, \quad u(L) = -\frac{f}{2k}L^2 + aL + b\,, \quad u'(L) = -\frac{f}{k}L + a\,. \tag{1}$$

The boundary conditions are $u'(0) = Hu(0)\,,\, u'(L) = -Hu(L)$, and substituting the expressions (1) into the equations, we get two equations for $a\,,\, b$. Solving the equations, we obtain $a = \frac{fL}{2k}$ and $b = \frac{fL}{2kH}$, and hence

$$u(x) = -\frac{f}{2k}x^2 + \frac{fL}{2k}x + \frac{fL}{2kH}\,. \tag{2}$$

A slightly different way of solving the problem: In lecture 2 we calculated that for the boundary condition $u(0) = u(L) = 0$ the solution is $u(x) = \frac{f}{2k}x(L-x)$. If the boundary condition is changes to the Newton's law of cooling, we can expect that all that happens is that the temperature goes up in the whole rod by a certain constant, so that the solutions will be of the form $u(x) = \frac{f}{2k}x(L-x) + b$. We now use the equation $u'(0) = Hu(0)$ to calculate $b$, obtaining $b = \frac{fL}{2kH}$. (To make sure that our *Ansatz* was correct, we verify that $u'(L) = -Hu(L)$.) Hence

$$u(x) = \frac{f}{2k}x(L-x) + \frac{fL}{2kH} \tag{3}$$

**2.** Let us denote the solution of Problem 1 for a given $H$ by $u_H(x)$. . Formula (1) (or (2)) gives

$$\lim_{H\to 0} u_H(x) = +\infty, \qquad \lim_{H\to\infty} u_H(x) = \frac{f}{2k}x(L-x)\,. \tag{4}$$

**3.** For implicity we can imagine that in the three-dimensional space with coordinates $(x_1, x_2, x_3)$ the outside of the house is given by $x_1 < 0$ and the inside of the house is given by $x_1 > L$, where $L > 0$, so that the wall separating the inside and the outside occupies the region $\{(x_1, x_2, x_3)\,,\ 0 < x_1 < L\}$. If the outside temperature is $T_1$, the inside temperature is $T_2 > T_1$, the temperature in the wall (assuming a steady state) is $u(x_1) = T_1 + (T_2 - T_1)\frac{x_1}{L}\,.$ The energy which is lost per unit of time per the unit area of the wall is given by the heat flux $\phi(x_1)$, defined on page 3 of the textbook. A basic assumption in our analysis of the heat equation is (see formula (1.2.8) on page 7 in the textbook)

$$\phi = -K_0\frac{\partial u}{\partial x_1} = -K_0\frac{T_2 - T_1}{L}\,. \tag{5}$$

To minimize the heat loss (assuming the steady state solution), we should choose the material for which the heat flux will be minimal possible, which means that $K_0$ will be the most important quantity. There are many other ways to arrive at the same conclusion.

**4.** Method (a): Let us denote the values of the relevant quantities in the SI units (meter, second, Kelvin) by $x, t, u$ and the values of these quantities in the new units (mile, month, Fahrenheit) by $X, T, U$. Then

$$x = \lambda X\,, \qquad t = \tau T, \qquad u = \alpha U + \beta\,, \tag{6}$$

where $\lambda, \tau, \alpha, \beta$ are given by the conversion factors between the units. By our assumptions, in terms of $x, t, u$ the equation is

$$\frac{\partial u}{\partial t} = k_{\mathrm{SI}}\frac{\partial^2 u}{\partial x^2}\,. \tag{7}$$

Substituting for $x, t, u$ from (7), we obtain

$$\frac{\alpha\partial U}{\tau\partial T} = k_{\mathrm{SI}}\frac{\alpha\partial^2 U}{\lambda^2\partial X^2}\,, \tag{8}$$

which is the same as

$$\frac{\partial U}{\partial T} = K\frac{\partial^2 U}{\partial X^2}\,, \qquad K = \frac{\tau\, k_{\mathrm{SI}}}{\lambda^2}\,. \tag{9}$$

For our particular choice of units we have $\lambda = 1609.34$ and, assuming 30-day months $\tau = 30 \times 24 \times 60 \times 60$. Hence

$$K = \frac{30 \times 24 \times 60 \times 60}{(1609.34)^2}\,k_{\mathrm{SI}} \sim 1.001\,k_{\mathrm{SI}}. \tag{10}$$

Method (b): The physical dimension of $k$ is $\frac{\text{lenght}^2}{\text{time}}$, so if we increase the unit of length by the factor of $\lambda = 1609.34$ and the unit of time by the factor of $\tau = 30 \times 24 \times 60 \times 60$, $k$ will change by the factor of $\frac{\tau}{\lambda^2} \sim 1.001$, to $k_{\text{new}} \sim 1.001\, k_{\text{SI}}$.

**5.** Let us use the notation $u_1 = \frac{\partial u}{\partial x_1}, u_{12} = \frac{\partial^2 u}{\partial x_1 \partial x_2}, u_2 = \frac{\partial u}{\partial x_2}$ , etc., and use the convention that $\tilde{u}$ is evaluated at $(x_1, x_2)$ and $u$ is evaluated at $((\cos\theta)x_1 + (\sin\theta)x_2, -(\sin\theta)x_1 + (\cos\theta)x_2)$. Then we can write

$$\frac{\partial \tilde{u}}{\partial x_1} = u_1 \cos\theta - u_2 \sin\theta \tag{11}$$

$$\frac{\partial^2 \tilde{u}}{\partial x_1^2} = u_{11} \cos^2\theta - 2u_{12}\cos\theta\sin\theta + u_{22}\sin^2\theta \tag{12}$$

$$\frac{\partial^2 \tilde{u}}{\partial x_2^2} = u_{11} \sin^2\theta + 2u_{12}\cos\theta\sin\theta + u_{22}\cos^2\theta \tag{13}$$

and hence

$$\frac{\partial^2 \tilde{u}}{\partial x_1^2} + \frac{\partial^2 \tilde{u}}{\partial x_2^2} = u_{11} + u_{22}\,, \tag{14}$$

from which the statement immediately follows.

**6.** First, we note that if we set $kt = s$, the equation becomes $\frac{du}{ds} = Au$, so it is enough to consider the case $k = 1$. If our cooling time is $T$ for $k = 1$, it will be $T/k$ for a general $k > 0$.

(a) Since all the eigenvalues of $A$ are strictly negative (as can be determined by a direct calculation or by using Matlab - see below), any solution converges to zero as $t \to \infty$. This follows for example from the formula for the general solution below.

(b) It is enough to determine $T$ assuming $k = 1$. This can be done is several ways.

(i) Probably the most economical approach (if we do not count the work done by computers) is the "brute force" approach, when we simply calculate the worst-case scenario on a computer. The worst-case scenario for the cooling is when $u_1(0) = u_2(0) = u_3(0) = u_4(0) = 1$ and we wait until the temperature drops to 0.01. Since we are dealing with a $4\times4$ system, we do not have to pay much attention to computational efficiency and can evaluate the solutions in Matlab simply as $u(t) = \exp(tA)u(0)$ (using command `expm(t*A)*u0`). By plotting the solution we see that our cooling time is approximately $T = 12.47$. Remembering that this is for $k = 1$, we see that for general $k$ the cooling time will be $\sim 12.47/k$.

(ii) A simple "back of the envelope" estimate can be obtained as follows. Let $\lambda_1$ be the highest eigenvalue of $A$. Then the long-time decay of a typical solution should be essentially given by $e^{\lambda_1 t}$. Using the command `eig(A)` in Matlab (or doing a calculation by hand - see below), one sees that $\lambda_1 \sim -0.3820$. The function $e^{\lambda_1 t}$ drops by 99% in time $T = \log(0.01)/\lambda_1 \sim 12.06$, and we can take this as a first estimate of our cooling time (for $k = 1$). This estimate already happens to be within the 10% range.[1]

(iii) One can do a calculation without a computer. The characteristic polynomial of the matrix $A$ is $P(\lambda) = \det(A - \lambda I) = (2 + \lambda)^4 - 3(2 + \lambda)^2 + 1$. The roots can be calculated explicitly, and the highest one is[2] $\lambda_1 = -2 + \sqrt{\frac{3+\sqrt{5}}{2}} = \frac{-3+\sqrt{5}}{2} \sim -0.3820$. The second highest is $\lambda_2 = -2 + \sqrt{\frac{3-\sqrt{5}}{2}} = \frac{-5+\sqrt{5}}{2} \sim -1.3820$. The other two eigenvalues will be denoted by $\lambda_3, \lambda_4$. We will denote the eigenvectors of $A$ corresponding to $\lambda_i$ by $b^{(i)}$, normalized to unit length. The general solution of $\dot{u} = Au$ is $u(t) = c_1 b^{(1)} e^{\lambda_1 t} + c_2 b^{(2)} e^{\lambda_2 t} + c_3 b^{(3)} e^{\lambda_3 t} + c_4 b^{(4)} e^{\lambda_4 t}$. It is safe to assume that for times relevant for our purposes the terms $e^{\lambda_j t}$ with $j \geq 2$ are negligible. The coefficient $c_1$ is given by the scalar product $(b^{(1)}, u(0))$, where we again take $u(0) = (1, 1, 1, 1)$. A calculation (which is possible to do by hand, but one can also use Matlab) shows that $\max_j c_1 b_j^{(1)} \sim 1.17$, so the cooling time (for $k = 1$) can be estimated by $(\log(\frac{1}{117}))/\lambda_1 \sim 12.47$.

[1] It is worth remarking that the estimate might not be precise enough if some of the values in the problem were different. For the given values it does get the number within the required range, although it does not provide a complete justification for it. One can consider it as a good "educated guess".

[2] The identify between the expressions with the square roots can be traced back to the special form of the equation: if se set $\xi = 2 + \lambda$ and $\eta = \xi + \frac{1}{\xi}$, the equation $P(\lambda) = 0$ becomes $\eta^2 = 5$. In fact, even when $A$ is an $n \times n$ matrix of the same form, the eigenvalues and eigenfunctions can still be computed explicitly. This is probably best seen using the discrete Fourier transformation.