1. (a) We have $c_k = \frac{1}{2}\int_{-1}^{1} f(x)e^{-\pi ikx}\,dx$. We can do the integration "by hand" or using Wolfram Alpha. The expression we get from the computer in the latter case is $c_k = \frac{-2\pi k\cos(\pi k)+2\sin(\pi k)}{\pi^3 k^3}$. For an integer $k \neq 0$ this gives $c_k = \frac{2(-1)^{k+1}}{\pi^2 k^2}$. For $k = 0$ we can either calculate directly $c_0 = \frac{1}{2}\int_{-1}^{1}(1-x^2)\,dx = \frac{2}{3}$, or obtain the same result by taking the limit $k \to 0$ in the above expression we got from Wolfram Alpha. The formula $\int_{-1}^{1}(1-x^2)^2\,dx = 2\sum_k |c_k|^2$ gives $\sum_{k=1}^{\infty}\frac{1}{k^4} = \frac{\pi^4}{90}$.
(b) One can simply calculate the Fourier series of $f'(x) = -2x$ on the interval $(-1,1)$, and check that its coefficients are πikc_k.
One can also see it without calculation: the Fourier series computed in (a) defines a $2-$periodic function on $f_{\rm per}$ on $(-\infty,\infty)$ which is equal to $1-x^2$ for $x \in (-1,1)$. The function $f_{\rm per}$ is clearly continuous, smooth away from the points $1+2k$ where $k \in \mathbf{Z}$ (the set of integers), and its derivative away from the points of non-differentiability is a $2-$periodic extension of the function $f'(x) = -2x$ from interval $(-1,1)$ to $(-\infty,\infty)\setminus\{1+2k,\ k\in\mathbf{Z}\}$. In particular $f'_{\rm per}$ is piece-wise smooth, and therefore its Fourier series can be differentiated term by term, see for example p. 114 of the textbook.
(c) The extended periodic function $f_{\rm per}$ is given by the expression $1-(x-2)^2$ when $x \in (1,3)$. The derivatives from the left (resp. right) of the function $f_{\rm per}$ at $x = 1$ are easily calculated to be -2 and 2, respectively. Since they are different, the periodically extended function cannot be differentiable at $x = 1$. The partial sums $\sum_{k=-n}^{k=n}$ of Fourier series of the function $f'(x)$ at $x = 1$ are easily seen to vanish (note that in this particular example $c_k e^{\pi ik} + c_{-k}e^{-\pi ik} = 0$ for each k), and hence the Fourier series for $f'(x)$ gives 0 when evaluated at $x = 1$. (Note that 0 is the average of the left and right derivative at $x = 1$.)

2. We have $\cos^2 x = \frac{1}{2} + \frac{1}{2}\cos 2x$ and this is the cosine series of $\cos^2 x$. For $\sin^2 x$ we can similarly write $\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos 2x$, but this clearly *is not* the sine-Fourier series of $\sin^2 x$. If we write $\sin^2 x = \sum_{n=1}^{\infty} B_n \sin nx$, the sum on the right-hand side will be a $2\pi-$ periodic odd function, let us call it $f_{\rm per}$. We have $f_{\rm per}(x) = -\sin^2 x$ for $x \in (-\pi,0)$ and $f_{\rm per}(x) = \sin^2 x$ for $x \in (0,\pi)$. The second derivative $f''_{\rm per}(x)$ is easily seen to have the limit 2 as $x \to 0$ from the right and -2 as $x \to 0$ from the left. Hence $f''_{\rm per}$ cannot be continuous at 0 and the function $f_{\rm per}$ cannot be a finite sum of functions of the form $B_k \sin kx$. For the coefficients B_n we have $B_n = \frac{2}{\pi}\int_0^{\pi}\sin^2 \sin nx\,dx = \frac{-8}{\pi n(n-2)(n+2)}$ when n is odd, and $B_n = 0$ when n is even. As we have seen, the second derivative of $f_{\rm per}$ is discontinuous at $k\pi$ for integer k, and smooth away from those points. Hence the Fourier series of $f''_{\rm per}$ still converges point-wise. On the other hand the Foureir series of $f'''_{\rm per}(x)$ cannot converge (point-wise), as its $n-$th term does not approach zero: differentiation gives (formally) $f'''_{\rm per}(x) = \sum_{n=1}^{\infty} -n^3 B_n \cos nx$, and $n^3 B_n$ does not approach 0 for $n \to \infty$.

3. Our machine can do the Fourier series only for 2π-periodic functions, so we change of variables as follows: For $x \in (0,L)$ we will write $u(x,t) = v(\frac{\pi x}{L},t)$, where $v = v(y,t)$ is an odd 2π-peridoc function on the real line. The function v is defined in three steps: (i) For $y \in (0,\pi)$ we set $v(y,t) = u(\frac{yL}{\pi},t)$. (ii) For $y \in (-\pi,0)$ we let $v(y,t) = -v(-y,t)$. (iii) we extend v from $(-\pi,\pi)$ to $(-\infty,\infty)$ as a $2\pi-$periodic function. Substituting the expression into the equation for u the function $u(x,t) = v(\frac{\pi x}{L},t)$, we obtain the equation satisfied by $v(y,t)$, namely

$$\frac{\partial^2 v}{\partial t^2} = a^2\frac{\partial^2 v}{\partial y^2} - \gamma v\,, \qquad a = c\frac{\pi}{L}\,. \tag{1}$$

We note that the boundary condition for v is $v(0,t) = v(\pi,t) = 0$, and is satisfied automatically in view of the requirement that v be odd and $2\pi-$periodic. The functions u_0, u_1 ate transformed to v_0, v_1 by $u_i(x,t) = v_i(\frac{\pi x}{L})$, $i = 0,1$. We seek $v(y,t)$ as a Fourier series

$$v(y,t) = \sum_k c_k(t)e^{ikx}\,. \tag{2}$$

Our task is to determine the coefficients $c_k(t)$. Once we have them, the machine can be used to calculate $v(y,t)$ and then $u(x,t) = v(\frac{\pi x}{L},t)$. The equation for $c_k = c_k(t)$ is $\ddot{c}_k = -a^2k^2c_k - \gamma c_k$ and its general solution is

$$c_k(t) = A_k \cos\omega_k t + B_k \sin\omega_k t\,, \qquad \omega_k = \sqrt{a^2k^2+\gamma}\,. \tag{3}$$

We now determine the values of A_k, B_k for our particular solution from the conditions $c_k(0) = A_k$ and $\dot c_k(0) = \omega_k B_k$. The values of $c_k(0)$ and $\dot c_k(0)$ are known from the initial conditions: the Fourier coefficients of v_0 are $c_k(0)$ and the Fourier coefficients of v_1 are $\dot c_k(0)$. Our algorithm can be summarized as follows:

1. Set $v_i(y) = u_i(\frac{Ly}{\pi})\ ,\ i = 0,1\,,$ and extend v_i as an odd function of $(-\pi,\pi)$.
2. Let $c_k(0)$ be the Fourier coefficients of v_0 and $\dot c_k(0)$ the Fourier coefficients of v_1. (Here we use our machine for the first time, to calculate Fourier coefficients.)
3. Determine A_k, B_k by the formulae above.
4. Sum the Fourier series $v(y,t) = \sum_k (A_k \cos\omega_k t + B_k \sin\omega_k t)e^{iky}$. (Here we use our machine for the second time, this time to sum a Fourier series.)
5. $u(x,t) = v(\frac{x\pi}{L}, t)$.

4. The general solution of the wave equation in our situation is a sum of terms of the form $B_k \sin(\frac{k\pi x}{L})\sin(\omega_k(t-t_k))$, where $\omega_k = k\frac{c\pi}{L}$, with $c = \sqrt{\frac{T}{\rho}}$. See, for example, Chapter 4 in the textbook (formula 4.4.11). Here we are only interested in the "base frequency" of the string, corresponding to $k = 1$. Hence we can work with the formula $\omega = \frac{\pi}{L}\sqrt{\frac{T}{\rho}}$. The answers can be now easily obtained from the formula. (a) The ratio $\frac{T}{\rho}$ has to remain the same, so we have to change the density to $\frac{\rho}{2}$. (b) The expression $\frac{\pi}{L}\sqrt{\frac{T}{\rho}}$ has to remain the same, so we have to increase T to $4T$.

5. (a) From the chain rule we have $\frac{\partial u}{\partial t} = \frac{\partial u}{\partial \tilde t}\frac{\partial \tilde t}{\partial t} + \frac{\partial u}{\partial x}\frac{\partial x}{\partial \tilde t} = \frac{\partial u}{\partial \tilde t} - v\frac{\partial u}{\partial \tilde x}$. A similar (but easier) calculation gives $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \tilde x}$. (Here we have a convention which is usual in similar situations: when we take $\frac{\partial}{\partial \tilde t}$ we keep $\tilde x$ constant and when we take $\frac{\partial}{\partial \tilde x}$ we keep $\tilde t$ constant, and similarly with the t, x variables. Hence in the new coordinates the equation becomes $(\frac{\partial}{\partial \tilde t} - v\frac{\partial}{\partial \tilde x})^2 u = c^2\frac{\partial^2 u}{\partial \tilde x^2}$, which is the same as $\frac{\partial^2 u}{\partial \tilde t^2} - 2v\frac{\partial^2 u}{\partial \tilde t \partial \tilde x} = (c^2 - v^2)\frac{\partial^2 u}{\partial \tilde x^2}$. If we know c and can measure u (including its derivatives) in the coordinate frame $(\tilde t, \tilde x)$, we can determine v.

(b) Consider the motion of the point $\tilde x = 0$ watched from the frame (t,x). Setting $\tilde x = 0$ in transformation (6) in the hw2 assignment, we obtain $t = \tilde t\cosh\theta$ and $x = c\tilde t\sinh\theta$, which then gives $\frac{dx}{dt} = c\frac{\sinh\theta}{\cosh\theta} = c\tanh\theta$. This is the velocity v of the origin of the frame $(\tilde t, \tilde x)$ when observed from the frame (t,x).

(c) Using the formulae $\cosh^2\theta - \sinh^2\theta = 1\,,\ \tanh\theta = \frac{\sinh\theta}{\cosh\theta}$ and $\tanh\theta = \frac{v}{c}$, one obtains $\cosh\theta = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$ and $\sinh\theta = \frac{\frac{v}{c}}{\sqrt{1-\frac{v^2}{c^2}}}$. This gives $t = \frac{\tilde t}{\sqrt{1-\frac{v^2}{c^2}}} + \frac{\frac{v}{c^2}\tilde x}{\sqrt{1-\frac{v^2}{c^2}}}$ and $x = \frac{v\tilde t}{\sqrt{1-\frac{v^2}{c^2}}} + \frac{\tilde x}{\sqrt{1-\frac{v^2}{c^2}}}$, which one can find in any textbook of special relativity.

6. (a) Let us first show that $AA^* = nI$, where I is the identity matrix. We have $(AA^*)_{kl} = \sum_{m=1}^n A_{km}(A^*)_{ml} = \sum_m w^{(k-1)(m-1)}w^{-(m-1)(l-1)} = \sum_m w^{(m-1)(k-l)}$. When $k = l$, the last sum is clearly equal to n. For $k \neq l$, let us set $\xi = w^{k-l}$. We note that $\xi \neq 1$ but $\xi^n = 1$. The last sun can then be written as $1 + \xi + \cdots + \xi^{n-1} = \frac{\xi^n - 1}{\xi - 1} = 0$.

(b) One can either say that we have shown in (a) that the matrix $\frac{1}{\sqrt n}A$ is unitary and this implies the identity $\frac{1}{n}\sum_k |f_k|^2 = \sum_k |c_k|^2$ in the hw2 assignment. Alternatively, one can show this identity directly, more or less repeating the calculation in (a): we have $\sum_k f_k\overline{f}_k = \sum_{klm} A_{kl}c_l\overline{A}_{km}\overline{c}_m$. In the tripple sum we first sum over k, using $\sum_k A_{kl}\overline{A}_{km} = n\delta_{ml}$, where $\delta_{ml} = 1$ for $k = l$ and 0 for $m \neq l$, and obtaining $\sum_k f_k\overline{f}_k = \sum_{ml} n\delta_{ml}c_m\overline{c}_l = n\sum_l c_l\overline{c}_l$.