1. Let (a,b) be a non-empty bounded open interval of the real line, let α,β be two positive real numbers at least one of which is not zero, and let $f(x)$ be a function which is smooth on the closed interval $[a,b]$. Find the differential equation and the boundary conditions which correspond to the following minimization problem:

Among sufficiently regular functions $u\colon (a,b)\to\mathbf{R}$ minimize the functional

$$J(u)=\int_a^b\left[\frac{1}{2}(u'(x))^2-u(x)f(x)\right]dx+\frac{\alpha}{2}u(a)^2+\frac{\beta}{2}u(b)^2\,. \tag{1}$$

Solution: Assume u is a function at which J attends its minimum. Let us fix a smooth function $\varphi\colon[a,b]\to\mathbf{R}$. The function $t\to J(u+t\varphi)$ obviously attends its minimum at $t=0$, therefore $\frac{d}{dt}|_{t=0}J(u+t\varphi)=0$. This gives $0=\int_a^b[u'(x)\varphi'(x)-f(x)\varphi(x)]\,dx+\alpha u(a)\varphi(a)+\beta u(b)\varphi(b)$. Integrating by parts, we see that $\int_a^b u'\varphi'\,dx=\int_a^b -u''\varphi\,dx+u'(b)\varphi(b)-u'(a)\varphi(a)$. Using this in the previous identity, we obtain $0=\int_a^b(-u''-f)\varphi\,dx+(-u'(a)+\alpha u(a))\varphi(a)+(u'(b)+\beta u(b))\varphi(b)$. This has to be true for any function φ as above. By taking all possible (smooth) functions φ satisfying $\varphi(a)=0$ and $\varphi(b)=0$, we see that $u''+f=0$. Once we know that $u''+f=0$, we take φ with $\varphi(a)=1$, $\varphi(b)=0$ to get $-u'(a)+\alpha u(a)=0$ and another φ with $\varphi(a)=0$, $\varphi(b)=1$ to get $u'(b)+\beta u(b)=0$. Therefore u satisfies the equation $-u''=f$ in the interval (a,b) and the boundary conditions $-u'(a)+\alpha u(a)=0$ and $u'(b)+\beta u(b)$ and the points a and b, respectively.

2. Consider the following variant of Problem 1. Let $a=-1, b=1, \alpha=1, \beta=0$, $f(x)=x^2$ and let X be the space of all quadratic functions of the form $u(x)=px^2+qx+r$. Find the minimizer of J over the space X in this particular case.

Solution: When $u=px^2+qx+r$ and $f(x)=x^2$ we have $\int_{-1}^1\left[\frac{1}{2}(u'(x))^2-u(x)f(x)\right]dx=\int_{-1}^1\left[\frac{1}{2}(2px+q)^2-(px^2+qx+r)x^2\right]dx$. The last integral is evaluated by routine calculation as $\frac{4}{3}p^2+q^2-\frac{2}{5}p-\frac{2}{3}r$. Hence the value of J at $u=px^2+qx+r$, which we will denote by $f(p,q,r)$, is given by $f(p,q,r)=\frac{4}{3}p^2+q^2+\frac{1}{2}(p-q+r)^2-\frac{2}{5}p-\frac{2}{3}r$. It is worth noting that the quadratic part of f, the expression $\frac{4}{3}p^2+q^2+\frac{1}{2}(p-q+r)^2$ is always positive and vanishes only when $(p,q,r)=(0,0,0)$. This means that the function f approaches $+\infty$ when (p,q,r) approaches ∞ in $\mathbf{R}^3$, and hence it attains a minimum. At the minimum the three partial derivatives of f have to vanish. This gives the system of equations

$$\begin{pmatrix}\frac{11}{3} & -1 & 1\\ -1 & 3 & -1\\ 1 & -1 & 1\end{pmatrix}\begin{pmatrix}p\\ q\\ r\end{pmatrix}=\begin{pmatrix}\frac{2}{5}\\ 0\\ \frac{2}{3}\end{pmatrix}. \tag{2}$$

Solving this system (either by hand or with the help of Matlab) gives $p=-\frac{1}{10}$, $q=\frac{1}{3}$, $r=\frac{11}{10}$. Hence the minimum of J on the space X is attained at the function $u(x)=-\frac{1}{10}x^2+\frac{1}{3}x+\frac{11}{10}$.

3. Consider still another variant of Problem 1. This time choose a positive integer n (think of $n=100$, for example), set $h=(b-a)/n$ and $x_0=a, x_1=a+h, x_2=a+2h,\dots,x_n=b$. Let X_n be the space of continuous functions on the closed interval $[a,b]$ which are of the form p_ix+q_i on the intervals (x_i,x_{i+1}), $i=0,1,\dots,n-1$.

(a) Explain why each function u in X_n is uniquely determined by the vector $u_0=u(x_0),u_1=u(x_1),\dots,u_n=u(x_n)$.

(b) For the case $f(x)\equiv 1$ calculate the equation for the vector $u_0,u_1,u_2,\dots,u_n$ one gets from the problem of minimizing $J(u)$ over X_n.

Hint: Take the derivatives of J is the direction of functions φ in X which are non-zero only at one point of the grid.
Solution: (a) A function of the form p_ix+q_i on an interval (x_i,x_{i+1}) is uniquely determined by its values at x_i and x_{i+1}.
(b) Proceeding in the same way as in Lecture 14 (see page 31 of the Lecture Log), we obtain $-\frac{u_{i+1}-2u_i+u_{i-1}}{h^2}=1$ for $i=1,2,\dots,n-1$. This represents $n-1$ equations. We have $n+1$ unknowns $u_1,u_1,\dots,u_n$. It remains to determine the two remaining equations. By the same reasoning as in Problem 1, we know that $0=\int_a^b[u'(x)\varphi'(x)-f(x)\varphi(x)]\,dx+\alpha u(a)\varphi(a)+\beta u(b)\varphi(b)$ for each $\varphi\in X_n$. Let us choose this identity with a $\varphi\in X_n$ satisfying $\varphi(x_0)=1$ and $\varphi(x_i)=0$ for $i=1,2,\dots,n$. Recalling that $f(x)\equiv 1$, we obtain $-\frac{u_1-u_0}{h}-\frac{h}{2}+\alpha u_0=0$. Similarly, using a $\varphi\in X_n$ given by $\varphi(x_i)=0$ for $i\le n-1$ and $\varphi(x_n)=1$, we obtain $\frac{u_n-u_{n-1}}{h}-\frac{h}{2}+\beta u_n=0$. The resulting system of equations can be written in matrix form as

$$\frac{1}{h^2}\begin{pmatrix}1+\alpha h & -1 & 0 & \dots & 0 & 0\\ -1 & 2 & -1 & 0 & \dots & 0\\ 0 & -1 & 2 & -1 & \dots & 0\\ \dots & \dots & \dots & \dots & \dots & \dots\\ 0 & 0 & \dots & -1 & 2 & -1\\ 0 & 0 & 0 & \dots & -1 & 1+\beta h\end{pmatrix}\begin{pmatrix}u_0\\ u_1\\ \dots\\ \dots\\ u_{n-1}\\ u_n\end{pmatrix}=\begin{pmatrix}\frac{1}{2}\\ 1\\ \dots\\ \dots\\ 1\\ \frac{1}{2}\end{pmatrix} \tag{3}$$

4. Consider the $n \times n$ matrix

$$S = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 1 \\ 1 & 0 & 0 & 0 & \dots & 0 \end{pmatrix}. \tag{4}$$

(a) Show that this matrix is unitary, in the sense that for any two vectors $z, z' \in \mathbf{C}^n$ with (complex) coordinates $z_1, \dots, z_n$ and $z'_1, \dots, z'_n$ we have $\langle z , z' \rangle = \langle Sz , Sz' \rangle$, where $\langle z , z' \rangle = \sum_{j=1}^n z_j \overline{z}'_j$ is the standard Hermitian product in $\mathbf{C}^n$.

(b) Find the adjoint matrix S^* (defined by $\langle Sz , z' \rangle = \langle z , S^* z' \rangle$ for each $z, z' \in \mathbf{C}^n$).

(c) Verify that $SS^* = S^*S = I$, where I is the identity matrix. In particular, S is normal. (This of course follows from (a) and general principles, but here the task is to verify this directly.)

(d) As S is normal, the general theory implies that it can be diagonalized (together with S^*) in a basis which is orthogonal with respect to the Hermitian product $\mathbf{C}^n$. Show that the columns of the Fourier matrix which appeared in Problem 6 of hw2 provide exactly such a basis, and calculate the eigenvalue corresponding the each eigenvector for both S and S^*.

(e) Check that the matrix $A = S - 2I + S^*$ corresponds to a matrix we used for a finite-dimensional approximation of the operator $\frac{\partial^2}{\partial x^2}$. Calculate the eigenvalues of A.

Solution: (a) If a vector z has coordinates $(z_1, z_2, \dots, z_n)$, then Sz has coordinates $(z_2, z_3, \dots, z_n, z_1)$. From this it is clear that $\langle z, z' \rangle = \langle Sz, Sz' \rangle$.
(b)We have $\langle Sz, z' \rangle = z_2 \overline{z}'_1 + z_3 \overline{z}'_2 + \dots z_n \overline{z}'_{n-1} + z_1 \overline{z}'_n$, so the vector $S^* z'$ has coordinates $(z'_n, z'_1, z'_2, \dots, z'_{n-1})$. This means that

$$S^* = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 1 & 0 \end{pmatrix}$$

(c) This can be done in many ways, one of them being a direct computation with the above explicit form of the matrices. A more general reasoning one can possibly use: $\langle S^* Sz, z' \rangle = \langle Sz, Sz' \rangle = \langle z, z' \rangle$ for each z, z', which means $S^* Sz = z$ for each z.
(d) The equation $Sz = \lambda z$ means that $z_2 = \lambda z_1$, $z_3 = \lambda z_2$, $z_n = \lambda z_{n-1}, z_1 = \lambda z_n$. Hence $z_{j+1} = \lambda^j z_1$, $j = 1, 2, \dots, n-1$ and $\lambda^n = 1$. Hence $z_1 \neq 0$, which means that we can choose $z_1 = 1$. So any eigenvector can be taken as a vector with coordinates $1, \lambda, \lambda^2, \dots, \lambda^{n-1}$, with $\lambda^n = 1$. These condition exactly characterize the columns of the Fourier matrix. From the calculation it is also clear that the eigenvalue of S associated with $(1, \lambda, \dots, \lambda^{n-1})$ is λ. The eigenvalue of S^* corresponding to the same vector is easily seen to be $\lambda^{-1} = \overline{\lambda}$.
(d) The matrix $S - 2I + S^*$ is

$$\begin{pmatrix} -2 & 1 & 0 & \dots & 0 & 1 \\ 1 & -2 & 1 & \dots & 0 & 0 \\ 0 & 1 & -2 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & -2 & 1 \\ 1 & 0 & 0 & \dots & 1 & -2 \end{pmatrix},$$

which is indeed the matrix coming up in the discretization of $\frac{\partial^2}{\partial x^2}$ with periodic boundary conditions. Its eigenvalues are of the form $\lambda - 2 + \overline{\lambda} = -2(1 - \operatorname{Re} \lambda)$,where lambda can be any $n-$th root of unity. The this the same as $-2(1 - \cos(2\pi k/n))$, $k = 0, 1, \dots, n-1$. Note that $\cos(2\pi k/n) = \cos(2\pi(n-k)/n)$, so the eigenvalues other than 1 and possibly -1 (for even n), have multiplicity two.

5. Solve the following problem for the heat equation

$$\begin{aligned}\frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial x^2} + \sin x\,, \qquad x \in (0,\pi)\,,\ t \in (0,\infty)\,,\\ u(0,t) &= 0\,,\\ u(\pi,t) &= 0\,,\\ u(x,0) &= \sin 2x\,,\end{aligned}$$

and determine $u_\infty(x) = \lim_{t\to\infty} u(x,t)$.

Hint: Seek the solution in the form of the sine-Fourier series, and recall how to solve the ODE $\dot{y} = -y + 1$.

Solution: We write the solution as a sine-Fourier series $u(x,t) = \sum_{n=1}^{\infty} B_n(t)\sin nx$. Let $f_1 = 1$ and $f_n = 0$ for $n \geq 2$, so that $\sin x = \sum_n f_n \sin nx$ is the sine-Fourier series of the forcing term $\sin x$. The equations for $B_n(t)$ are $\dot{B}_n(t) + n^2 B_n(t) = f_n\,, \quad n = 1, 2, \ldots$. The initial conditions $B_n(0)$ are given by $B_2(0) = 1$ and $B_n(0) = 0$ when $n \neq 2$. Based on the simple equations for $B_n(t)$ and their initial conditions, we see that $B_1(t) = 1 - e^{-t}$, $B_2(t) = e^{-4t}$, and $B_n = 0$ for $n \geq 3$. Hence $u(x,t) = (1 - e^{-t})\sin x + e^{-4t}\sin 2x$. The limit $\lim_{t\to\infty} u(x,t)$ is $u_\infty(x) = \sin x$.

6. Show that for a $2L-$periodic solution of the (generalized) wave equation

$$\rho\frac{\partial^2 u}{\partial t^2} = T\frac{\partial^2 u}{\partial x^2} - \gamma u \tag{5}$$

where ρ, T, γ are positive constants, with ρ and T being strictly positive, the following quantities are constant in time:

(a) The energy: $E(t) = \int_{-L}^{L}\left[\frac{\rho}{2}\left(\frac{\partial u(x,t)}{\partial t}\right)^2 + \frac{T}{2}\left(\frac{\partial u(x,t)}{\partial x}\right)^2 + \frac{\gamma}{2}u(x,t)^2\right]dx\,.$

(b) The momentum: $P(t) = \int_{-L}^{L}\left[\frac{\partial u(x,t)}{\partial t}\frac{\partial u(x,t)}{\partial x}\right]dx\,.$

Hint: Show that the time derivatives of $E(t)$ and $P(t)$ vanish, using integration by parts and the equation.

Solution: We will denote derivatives by sub-indices, so u_t means $\frac{\partial u}{\partial t}$, and similarly $u_{tt} = \frac{\partial^2 u}{\partial t^2}$, or $u_{xt} = \frac{\partial^2 u}{\partial x \partial t}$, etc. We also recall the following simples observation: if f is a smooth $2L-$periodic function, then $\int_{-L}^{L} f_x\,dx = 0$. This follows the identity $\int_{L}^{L} f_x\,dx = f(L) - f(-L) = 0$, where the last equality follows from the periodicity of the function.

(a) We calculate $\frac{d}{dt}E(t) = \int_{-L}^{L}[\rho u_t u_{tt} + T u_x u_{xt} + \gamma u u_t]\,dx = \int_{-L}^{L}[u_t(T u_{xx} - \gamma u) + T u_x u_{xt} + \gamma u u_t]\,dx = \int_{-L}^{L}[T u_t u_{xx} + T u_x u_{xt}]\,dx =$ (using integration by parts on the first term) $\int_{-L}^{L}[-T u_{xt} u_x + T u_x u_{xt}]\,dx = 0$.

(b) $\frac{d}{dt}P(t) = \int_{-L}^{L}[u_{tt}u_x + u_t u_{tx}]\,dx = \int_{-L}^{L}[\frac{T}{\rho}u_{xx}u_x - \frac{\gamma}{\rho}u u_x + u_t u_{xt}]\,dx$. Now note that all three terms in the last integral can be written as derivatives: $u_x u_{xx} = (u_x^2/2)_x$, $u u_x = (u^2/2)_x$, and $u_t u_{tx} = (u_t^2/2)_x$. By the remark above about the integral of a derivative of a periodic function, we see that $\int_{-L}^{L}[\frac{T}{\rho}u_{xx}u_x - \frac{\gamma}{\rho}u u_x + u_t u_{xt}]\,dx = 0$.