Math 5587 **Homework Assignment 3** Fall 2017

due November 28

Please submit via Moodle by midnight, Nov 28

Do at least four of the following six problems.[1]

1. Let (a,b) be a non-empty bounded open interval of the real line, let α,β be two positive real numbers at least one of which is not zero, and let $f(x)$ be a function which is smooth on the closed interval $[a,b]$. Find the differential equation and the boundary conditions which correspond to the following minimization problem:

Among sufficiently regular functions $u\colon (a,b)\to \mathbf{R}$ *minimize the functional*

$$J(u)=\int_a^b \left[\frac{1}{2}(u'(x))^2-u(x)f(x)\right]dx+\frac{\alpha}{2}u(a)^2+\frac{\beta}{2}u(b)^2\,. \tag{1}$$

2. Consider the following variant of Problem 1. Let $a=-1, b=1, \alpha=1, \beta=0$, $f(x)=x^2$ and let X be the space of all quadratic functions of the form $u(x)=px^2+qx+r$. Find the minimizer of J over the space X in this particular case.

3. Consider still another variant of Problem 1. This time choose a positive integer n (think of $n=100$, for example), set $h=(b-a)/n$ and $x_0=a, x_1=a+h, x_2=a+2h,\dots,x_n=b$. Let X_n be the space of continuous functions on the closed interval $[a,b]$ which are of the form p_ix+q_i on the intervals $(x_i,x_{i+1})\,,\ \ i=0,1,\dots,n-1$.

(a) Explain why each function u in X_n is uniquely determined by the vector $u_0=u(x_0), u_1=u(x_1),\dots,u_n=u(x_n)$.

(b) For the case $f(x)\equiv 1$ calculate the equation for the vector $u_0,u_1,u_2,\dots,u_n$ one gets from the problem of minimizing $J(u)$ over X_n.

Hint: Take the derivatives of J is the direction of functions φ in X which are non-zero only at one point of the grid.

4. Consider the $n\times n$ matrix

$$S=\begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0\\ 0 & 0 & 1 & 0 & \dots & 0\\ 0 & 0 & 0 & 1 & \dots & 0\\ \dots & \dots & \dots & \dots & \dots & \dots\\ 0 & 0 & 0 & 0 & \dots & 1\\ 1 & 0 & 0 & 0 & \dots & 0 \end{pmatrix}. \tag{2}$$

(a) Show that this matrix is unitary, in the sense that for any two vectors $z,z'\in\mathbf{C}^n$ with (complex) coordinates $z_1,\dots,z_n$ and $z'_1,\dots,z'_n$ we have $\langle\, z\,,\, z'\,\rangle=\langle\, Sz\,,\, Sz'\,\rangle$, where $\langle\, z\,,\, z'\,\rangle=\sum_{j=1}^n z_j\overline{z}'_j$ is the standard Hermitian product in $\mathbf{C}^n$.

(b) Find the adjoint matrix S^* (defined by $\langle\, Sz\,,\, z'\,\rangle=\langle\, z\,,\, S^*z'\,\rangle$ for each $z,z'\in\mathbf{C}^n$).

(c) Verify that $SS^*=S^*S=I$, where I is the identity matrix. In particular, S is normal. (This of course follows from (a) and general principles, but here the task is to verify this directly.)

(d) As S is normal, the general theory implies that it can be diagonalized (together with S^*) in a basis which is orthogonal with respect to the Hermitian product $\mathbf{C}^n$. Show that the columns of the Fourier matrix which appeared in Problem 6 of hw2 provide exactly such a basis, and calculate the eigenvalue corresponding the each eigenvector for both S and S^*.

(e) Check that the matrix $A=S-2I+S^*$ corresponds to a matrix we used for a finite-dimensional approximation of the operator $\frac{\partial^2}{\partial x^2}$. Calculate the eigenvalues of A.

[1] For grading purposes, any 4 problems correspond to 100%. You can get extra credit if you do more.

5. Solve the following problem for the heat equation

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial x^2} + \sin x\,, \qquad x \in (0,\pi)\,,\ t \in (0,\infty)\,, \\ u(0,t) &= 0\,, \\ u(\pi,t) &= 0\,, \\ u(x,0) &= \sin 2x\,, \end{aligned}$$

and determine $u_\infty(x) = \lim_{t\to\infty} u(x,t)$.

Hint: Seek the solution in the form of the sine-Fourier series, and recall how to solve the ODE $\dot{y} = -y + 1$.

6. Show that for a $2L-$periodic solution of the (generalized) wave equation

$$\rho \frac{\partial^2 u}{\partial t^2} = T \frac{\partial^2 u}{\partial x^2} - \gamma u \tag{3}$$

where ρ, T, γ are positive constants, with ρ and T being strictly positive, the following quantities are constant in time:

(a) The energy: $E(t) = \int_{-L}^{L} \left[\frac{\rho}{2} \left(\frac{\partial u(x,t)}{\partial t} \right)^2 + \frac{T}{2} \left(\frac{\partial u(x,t)}{\partial x} \right)^2 + \frac{\gamma}{2} u(x,t)^2 \right] dx\,.$

(b) The momentum: $P(t) = \int_{-L}^{L} \left[\frac{\partial u(x,t)}{\partial t} \frac{\partial u(x,t)}{\partial x} \right] dx\,.$

Hint: Show that the time derivatives of $E(t)$ and $P(t)$ vanish, using integration by parts and the equation.