1. Consider a spherical planet of radius R and a uniform density ρ. Assume a straight elevator shaft the radius of which is negligible compared to the radius of the planet goes from the north pole to the south pole of the planet. Assuming there is no frictional forces, if we drop a body into the shaft at the north pole, how long will it take for it to reach the south pole?
Hint: Use the Shell Theorem we discussed in class to calculate the gravitational force inside the planet. In the calculation of the field we neglect the shaft and assume that the planet is just a sphere. Alternatively, use the Poisson equation for radial functions $u''(r) + \frac{2u'(r)}{r} = 4\pi\kappa\rho(r)$ to obtain the gravitational potential. You can save some calculations by observing that $\Delta(x_1^2 + x_2^2 + x_3^2) = 6$.

Solution: Let r be a coordinate along the shaft, with r=0 corresponding to the center of the planet. Let $r(t)$ be the position of the body at time t. By Newton's law, $\ddot{r}$ = force. To calculate the force at position r, one can use the Shell Theorem. The gravitational force due to the shell $\{x\,,\, r < |x| \le R\}$ vanishes, by the Shell Theorem. At the same time, again by the Shell Theorem, the gravitational force due to the ball $\{x\,,\,|x| \le r\}$ is the same as if all the mass of that ball was at the origin, and hence the magnitude of the force is $[\kappa\rho\frac{4\pi}{3}r^3]/r^2 = \gamma r$, where $\gamma = \frac{4\pi}{3}\kappa\rho$. The equation of motion then is $\ddot{r} = -\gamma r$, with a general solution $A\cos\sqrt{\gamma}(t - t_0)$. In our situation we take $A = R$ and $t_0 = 0$, and the body will reach the other end of the shaft at time T with $T\sqrt{\gamma} = \pi$. Hence $T = \pi/\sqrt{\gamma} = \sqrt{\frac{3\pi}{4\kappa\rho}}$.
If we wish to calculate the potential u of the force directly, we note that u should be radial (i. e. depends only on $r = \sqrt{x_1^2 + x_2^2 + x_3^2}$) and satisfy $\Delta u = 4\pi\kappa\rho$ inside the planet. Based on the hint we see that we can seek the solution as $A\frac{r^2}{2}$ + const. A simple calculation gives $3A = 4\pi\kappa\rho$, giving $u = \frac{2\pi}{3}\kappa\rho r^2$ + const. The force is then given by $-u'(r)$, leading to the same result as before.

2. Let Ω be the unit disc in $\mathbf{R}^2$, i. e., $\Omega = \{x \in \mathbf{R}^2\,,\,|x| < 1\}$. For $\varepsilon \in (0,1)$ consider a function $h_\varepsilon\colon [0,\infty) \to \mathbf{R}$ given by

$$h_\varepsilon(s) = \begin{cases} 1\,, & 0 \le s \le 1-\varepsilon\,, \\ \frac{1-s}{\varepsilon}\,, & 1-\varepsilon < s \le 1\,, \\ 0\,, & s > 1\,. \end{cases} \tag{1}$$

Let $g_\varepsilon\colon \mathbf{R}^2 \to \mathbf{R}$ be defined by $g_\varepsilon(x) = h_\varepsilon(\sqrt{x_1^2 + x_2^2})$. Let $f\colon \mathbf{R}^2 \to \mathbf{R}$ be any smooth function.
Explain why

$$\lim_{\varepsilon\to 0_+} \int_{\mathbf{R}^2} \frac{\partial}{\partial x_1}\left(f(x)g_\varepsilon(x)\right)\,dx = \int_\Omega \frac{\partial f}{\partial x_1}\,dx - \int_{\partial\Omega} f(x)n_1(x)\,dx\,, \tag{2}$$

where $n_1(x) = \frac{x_1}{|x|}$ is the first component of the autward unit normal to the boundary of Ω at x (when $x \in \partial\Omega$).
(As we discussed in class, the integral on the left-hand side of (2) vanishes, hence this gives $\int_\Omega \frac{\partial f}{\partial x_1}\,dx = \int_{\partial\Omega} f n_1\,dx$.)

Solution: We have

$$\frac{\partial}{\partial x_1}(fg_\varepsilon) = \frac{\partial f}{x_1}g_\varepsilon + f\frac{\partial g_\varepsilon}{\partial x_1}\,. \tag{3}$$

The function $f_1 = \frac{\partial f}{\partial x_1}$ is smooth, and the function g_ε is equal to 1 in the ball $B_{1-\varepsilon} = \{x\,,\,|x| < 1-\varepsilon\}$, with $0 \le g_\varepsilon \le 1$ everywhere in B_1, and $g_\varepsilon = 0$ outside of B_1. From this we see that $\int_{\mathbf{R}^2} f_1 g_\varepsilon\,dx \to \int_\Omega f_1\,dx$ as $\varepsilon \to 0_+$. The derivative $\frac{\partial g_\varepsilon}{\partial x_1}$ vanishes outside the region $\mathcal{O}_\varepsilon = \{x\,,\,1-\varepsilon \le |x| \le 1\}$. Inside $\mathcal{O}_\varepsilon$ we have $\frac{\partial g_\varepsilon}{\partial x_1} = -\frac{1}{\varepsilon}\frac{x_1}{|x|}$. Letting $\varphi_1(x) = f(x)\frac{x_1}{|x|}$, which is a smooth outside of the origin, we need to determine the limit of $-\frac{1}{\varepsilon}\int_{\mathcal{O}_\varepsilon}\varphi_1(x)\,dx$ as $\varepsilon \to 0_+$. This limit is $-\int_{\partial\Omega}\varphi_1(x)\,dx$ (boundary integral), as can be seen from example from the following. Using polar coordinates r, θ, given by $x_1 = r\cos\theta$ and $x_2 = r\sin\theta$, and setting $F(r) = \int_0^{2\pi}\varphi_1(r\cos\theta, r\sin\theta)\,d\theta$,we have $F(1) = \int_{\partial\Omega}\varphi_1(x)\,dx$. We now write $\frac{1}{\varepsilon}\int_{1-\varepsilon}^1\int_0^{2\pi}\varphi_1(r,\theta)r\,d\theta\,dr = \frac{1}{\varepsilon}\int_{1-\varepsilon}^1 F(r)\,dr$. The limit of $\frac{1}{\varepsilon}\int_{1-\varepsilon}^1 F(r)\,dr$ as $\varepsilon \to 0_+$ is $F(1)$, by the fundamental theorem of calculus, which gives our statement, once we recall the definition of F.

3. (i) In the three dimensional space $\mathbf{R}^3$ assume that mass is distributed uniformly along the x_3 axis, with uniform (linear) density ρ per unit length. Calculate the force due to gravity on a particle of a unit mass located at a point x not lying on the x_3 axis. Denoting by κ the gravitational constant, verify that the force is given by a potential $u(x) = a\kappa\rho\log\sqrt{x_1^2 + x_2^2}$ for a suitable constant a. Determine a.
(ii) In the three dimensional space $\mathbf{R}^3$ assume that mass is distributed uniformly along in the (x_2, x_3) coordinate plane, with uniform (surface) density ρ per unit area. Calculate the force due to gravity on a particle of a unit mass located at a point x. Verify that outside the (x_2, x_3)-plane the force is given by a potential $u(x) = b\kappa\rho|x_1|$ for a suitable constant b. Determine b.

Solution: (i) The force $F(x)$ at x not lying at the x_3- axis is given by $F(x) = \int_{-\infty}^\infty \kappa\rho\frac{y-x}{|y-x|^3}\,dy_3$,where $y = (0,0,y_3)$. To evaluate the integral, note that by setting $y_3 - x_3 = s$, we can write $F(x) = \kappa\rho(-x_1, -x_2, 0)I\left(\sqrt{x_1^2 + x_2^2}\right)$, where $I(r) = \int_{-\infty}^\infty \frac{ds}{(r^2+s^2)^{\frac{3}{2}}}$. Making a substitution $s = r\sigma$ in this integral, we obtain $I(r) = \frac{1}{r^2}\int_{-\infty}^\infty \frac{d\sigma}{(1+\sigma^2)^{\frac{3}{2}}} = \frac{2}{r^2}$. Hence $F_i(x) = -2\kappa\rho\frac{x_i}{x_1^2+x_2^2}$ when $i = 1,2$ and $F_3(x) = 0$. This force is given by the potential $2\kappa\rho\log\sqrt{x_1^2 + x_2^2}$, and we see that $a = 2$.
(ii) In a similar way the force is given by $\int_{\mathbf{R}^2}\kappa\rho\frac{y-x}{|y-x|^2}\,dy_2dy_3$, where $y = (y_1, y_2, 0)$. Setting $s_2 = y_2 - x_2$, $s_3 = y_3 - x_3$, we obtain $F(x) = \kappa\rho(-x_1, 0, 0)J(|x_1|)$,with $J(r) = \int_{\mathbf{R}^2}\frac{ds_2\,ds_3}{(r^2+s_2^2+s_3^2)^{\frac{3}{2}}}$. The last (double) integral can be calculated in several ways. A fairly straightforward one is

to integrate over s_2 first, which gives $\frac{2}{r^2+s_3^2}$ by the previous calculation. Now we have to calculate $\int_{-\infty}^{\infty}\frac{2ds_3}{r^2+s_3^2}$. Setting $s_3 = r\sigma$ the last integral becomes $\frac{2}{r}\int_{-\infty}^{\infty}\frac{d\sigma}{1+\sigma^2} = \frac{2\pi}{r}$. We conclude that $F_1(x) = -2\pi\kappa\rho\frac{x_1}{|x_1|}$, and $F_2 = F_3 = 0$. The force is given by the potential $u(x) = 2\pi\kappa\rho|x_1|$. We see that $b = 2\pi$.

4. Let $G_1, G_2\colon \mathbf{R}\to\mathbf{R}$ be defined by

$$G_1(x) = \frac{1}{2}|x|\,, \qquad G_2 = x^+ \quad \text{(the positive part of } x). \tag{4}$$

Assume that $f\colon \mathbf{R}\to\mathbf{R}$ is a smooth function which vanishes outside of some finite interval, and set

$$u_i(x) = \int_{\mathbf{R}} G_i(x-y)f(y)\,dy\,, \qquad i = 1,2\,. \tag{5}$$

(i) Explain why for $i = 1,2$ we have $u_i'' = f$ in $\mathbf{R}$.
(ii) Show that a necessary and sufficient condition for $u_1 = u_2$ in $\mathbf{R}$ is that $\int f(y)\,dy = 0$ and $\int yf(y)\,dy = 0$.

Solution: (i) This can be proved in many ways, it essentially reduces to the Fundamental Theorem of Calculus - below we give a proof relying on that theorem. Before that, it is worth noting that there is a proof quite similar to what we did in class for the 3d Laplace equation. We will briefly illustrate it for G_1, the case of G_2 being essentially the same. Let $K(x)$ be a smooth function which is equal to G_1 outside of the interval $(-1,1)$, and for $\varepsilon > 0$ define $K_\varepsilon(x) = \varepsilon K(\frac{x}{\varepsilon})$. Note that $K_\varepsilon(x)$ approaches $G_1(x)$ as $\varepsilon\to 0_+$. Moreover, $(K_\varepsilon)''(x) = \frac{1}{\varepsilon}K''(\frac{x}{\varepsilon})$ is an approximation of the Dirac function, converging to it as $\varepsilon\to 0_+$. (Note that $\int_{-\infty}^{\infty}K''(x)\,dx = 1$.) In the same way as in our proof in 3d we can write $(G_1 * f)'' = \lim_{\varepsilon\to 0_+}(K_\varepsilon)'' * f = f$.

Let us now sketch a proof relying just on the Fundamental Theorem of Calculus. Let us again work with G_1 (the case of G_2 being now marginally easier). Note that $G_1'(x) = \frac{1}{2}$ for $x > 0$ and $-\frac{1}{2}$ for $x < 0$. Hence

$$u_1'(x) = (G_1' * f)(x) = -\frac{1}{2}\int_x^{\infty} f(y)\,dy + \frac{1}{2}\int_{-\infty}^{x} f(y)\,dy\,. \tag{6}$$

Now the derivative with respect to x of $-\frac{1}{2}\int_x^{\infty} f(y)\,dy$ is $\frac{1}{2}f(x)$ and the derivative of $\int_{-\infty}^{x} f(y)\,dy$ with respect to x is $\frac{1}{2}f(x)$, which shows that $u_1''(x) = f(x)$.

For the purposes of the homework, it is fine to use (6) without a detailed justification. Note, however, that to be completely rigorous, identity (6) needs to be justified, as the function $G_1(x)$ is not differentiable at $x = 0$. This can be done in many ways. For example, as we assume that f is smooth, we can write $u_1'(x) = (G_1 * f)'(x) = G_1 * f' = \int_{-\infty}^{x}\frac{1}{2}(x-y)f'(y)\,dy + \int_x^{\infty} -\frac{1}{2}(x-y)f'(y)\,dy$ and integrate by parts.

(ii) We note that $H(x) = G_2(x) - G_1(x) = \frac{1}{2}x$. Hence a necessary and sufficient condition for $u_1 = u_2$ is $H * f = 0$ which is equivalent to the two the simultaneous fulfilment of the two conditions $\int_{-\infty}^{\infty} f(y)\,dy = 0$ and $\int_{-\infty}^{\infty} yf(y)\,dy = 0$.

5. Let us use the standard notation $\mathbf{Z}$ for the integers. For a function $f\colon \mathbf{Z}\to\mathbf{R}$ define $D^+f(x) = f(x+1) - f(x)$ and $D^-f(x) = f(x) - f(x-1)$. Show that if $f, g\colon \mathbf{Z}\to\mathbf{R}$ are two function such that g vanishes outside a finite set, then

$$\sum_{x\in\mathbf{Z}} D^+f(x)g(x) = \sum_{x\in\mathbf{Z}} -f(x)D^-g(x)\,, \tag{7}$$

and

$$\sum_{x\in\mathbf{Z}} D^+f(x)D^+g(x) = \sum_{x\in\mathbf{Z}} -D^-D^+f(x)g(x)\,. \tag{8}$$

Solution: To show (7), we can write

$$\sum_x D^+f(x)g(x) = \sum_x [f(x+1)g(x) - f(x)g(x)] = \sum_x [f(x)g(x-1) - f(x)g(x)] = \sum_x -f(x)D^-g(x)\,. \tag{9}$$

To show (8), we apply (7) with g replaced by D^+g.

6. Let $\mathbf{S}^2 = \{y\in\mathbf{R}^3\,,\ |y|^2 = 1\}$ be the unit sphere in $\mathbf{R}^3$ centered at the origin. Let $f\colon \mathbf{R}^3\to\mathbf{R}$ be a smooth function and $x\in\mathbf{R}^3$. Show that

$$\Delta f(x) = \lim_{h\to 0}\frac{3}{2\pi h^2}\int_{\mathbf{S}^2}(f(x+hy) - f(x))\,dy\,. \tag{10}$$

Hint: Use the Taylor expansion of f at the point x, and note that the integrals $\int_{\mathbf{S}^2} y_i\,dy$ and $\int_{\mathbf{S}^2} y_iy_j\,dy$ can be evaluated explicitly.

Solution: Let us write $f(x+hy) - f(x) = \sum_i f_ihy_i + \sum_{ij}\frac{1}{2}f_{ij}h^2y_iy_j + R(x,y,h)$, where $f_i = \frac{\partial f}{\partial x_i}(x)$, $f_{ij} = \frac{\partial^2 f}{\partial x_i\partial x_j}(x)$ and $R(x,y,h)$ is the "remainder". The important thing for us is that the remainder is of order h^3 for small h. In particular,

$$\lim_{h\to 0}\frac{1}{h^2}\int_{\mathbf{S}^2} R(x,y,h)\,dy = 0\,. \tag{11}$$

We note that $\int_{\mathbf{S}^2} y_i = 0$ for each $i = 1,2,3$, due to the symmetries $(y_1,y_2,y_3)\to(-y_1,y_2,y_3)$, $(y_1,y_2,y_3)\to(y_1,-y_2,y_3)$, $(y_1,y_2,y_3)\to(y_1,y_2,-y_3)$. For example, under the first symmetry the function y_1 changes sign, and hence $\int_{\mathbf{S}^2} y_1 = -\int_{\mathbf{S}^2} y_1$, which means that the integral vanishes. The same argument works for the integrals $\int_{\mathbf{S}^2} y_iy_j$ with $i\neq j$. These also have to vanish. For example, when $i = 1, j = 2$ we can use again the symmetry $(y_1,y_2,y_3)\to(-y_1,y_2,y_3)$. Under this symmetry the function y_1y_2 goes to $-y_1y_2$, and hence $\int_{\mathbf{S}^2} y_1y_2 = \int_{\mathbf{S}^2} -y_1y_2$, showing that $\int_{\mathbf{S}^2} y_1y_2 = 0$. Finally, we need to evaluate the integrals $\int_{\mathbf{S}^2} y_i^2$, i=1,2,3. By using the symmetry $(y_1,y_2,y_3)\to(y_2,y_3,y_1)$, it is easy to see that these integrals all have the same value. We can calculate it as follows $\int_{\mathbf{S}^2} y_1^2 = \frac{1}{3}\int_{\mathbf{S}^2}(y_1^2+y_2^2+y_3^2) = \frac{4\pi}{3}$, because $y_1^2+y_2^2+y_3^2 = 1$ on the sphere. We see that $\int_{\mathbf{S}^2}(f(x+hy) - f(x))\,dy = \frac{2\pi}{3}h^2(f_{11}+f_{22}+f_{33}) + \int_{\mathbf{S}^2} R(x,y,h)\,dy$, and (10) follows in view of (11).