due February 8

Please submit via Moodle by midnight, February 8

Do at least four of the following six problems.[1]

1. Consider a spherical planet of radius R and a uniform density ρ. Assume a straight elevator shaft the radius of which is negligible compared to the radius of the planet goes from the north pole to the south pole of the planet. Assuming there is no frictional forces, if we drop a body into the shaft at the north pole, how long will it take for it to reach the south pole?
Hint: Use the Shell Theorem we discussed in class to calculate the gravitational force inside the planet. In the calculation of the field we neglect the shaft and assume that the planet is just a sphere. Alternatively, use the Poisson equation for radial functions $u''(r) + \frac{2u'(r)}{r} = 4\pi\kappa\rho(r)$ to obtain the gravitational potential. You can save some calculations by observing that $\Delta(x_1^2 + x_2^2 + x_3^2) = 6$.

2. Let Ω be the unit disc in $\mathbf{R}^2$, i. e., $\Omega = \{x \in \mathbf{R}^2\,,\, |x| < 1\}$. For $\varepsilon \in (0,1)$ consider a function $h_\varepsilon\colon [0,\infty) \to \mathbf{R}$ given by

$$h_\varepsilon(s) = \begin{cases} 1\,, & 0 \le s \le 1-\varepsilon\,, \\ \frac{1-s}{\varepsilon}\,, & 1-\varepsilon < s \le 1\,, \\ 0\,, & s > 1\,. \end{cases} \tag{1}$$

Let $g_\varepsilon\colon \mathbf{R}^2 \to \mathbf{R}$ be defined by $g_\varepsilon(x) = h_\varepsilon(\sqrt{x_1^2 + x_2^2})$. Let $f\colon \mathbf{R}^2 \to \mathbf{R}$ be any smooth function.
Explain why

$$\lim_{\varepsilon\to 0_+} \int_{\mathbf{R}^2} \frac{\partial}{\partial x_1}\left(f(x)g_\varepsilon(x)\right)\,dx = \int_\Omega \frac{\partial f}{\partial x_1}\,dx - \int_{\partial\Omega} f(x)n_1(x)\,dx\,, \tag{2}$$

where $n_1(x) = \frac{x_1}{|x|}$ is the first component of the autward unit normal to the boundary of Ω at x (when $x \in \partial\Omega$).
(As we discussed in class, the integral on the left-hand side of (2) vanishes, hence this gives $\int_\Omega \frac{\partial f}{\partial x_1}\,dx = \int_{\partial\Omega} f n_1\,dx$.)

3. (i) In the three dimensional space $\mathbf{R}^3$ assume that mass is distributed uniformly along the x_3 axis, with uniform (linear) density ρ per unit length. Calculate the force due to gravity on a particle of a unit mass located at a point x not lying on the x_3 axis. Denoting by κ the gravitational constant, verify that the force is given by a potential $u(x) = a\kappa\rho\log\sqrt{x_1^2 + x_2^2}$ for a suitable constant a. Determine a.

(ii) In the three dimensional space $\mathbf{R}^3$ assume that mass is distributed uniformly along in the (x_2, x_3) coordinate plane, with uniform (surface) density ρ per unit area. Calculate the force due to gravity on a particle of a unit mass located at a point x. Verify that outside the (x_2, x_3)-plane the force is given by a potential $u(x) = b\kappa\rho|x_1|$ for a suitable constant b. Determine b.

4. Let $G_1, G_2\colon \mathbf{R} \to \mathbf{R}$ be defined by

$$G_1(x) = \frac{1}{2}|x|\,, \qquad G_2 = x^+ \quad \text{(the positive part of } x)\,. \tag{3}$$

Assume that $f\colon \mathbf{R} \to \mathbf{R}$ is a smooth function which vanishes outside of some finite interval, and set

$$u_i(x) = \int_{\mathbf{R}} G_i(x-y)f(y)\,dy\,, \qquad i = 1,2\,. \tag{4}$$

(i) Explain why for $i = 1,2$ we have $u_i'' = f$ in $\mathbf{R}$.
(ii) Show that a necessary and sufficient condition for $u_1 = u_2$ in $\mathbf{R}$ is that $\int f(y)\,dy = 0$ and $\int y f(y)\,dy = 0$.

5. Let us use the standard notation $\mathbf{Z}$ for the integers. For a function $f\colon \mathbf{Z} \to \mathbf{R}$ define $D^+f(x) = f(x+1) - f(x)$ and $D^-f(x) = f(x) - f(x-1)$. Show that if $f, g\colon \mathbf{Z} \to \mathbf{R}$ are two function such that g vanishes outside a finite set, then

$$\sum_{x\in\mathbf{Z}} D^+f(x)g(x) = \sum_{x\in\mathbf{Z}} -f(x)D^-g(x)\,, \tag{5}$$

and

$$\sum_{x\in\mathbf{Z}} D^+f(x)D^+g(x) = \sum_{x\in\mathbf{Z}} -D^-D^+f(x)g(x)\,. \tag{6}$$

6. Let $\mathbf{S}^2 = \{y \in \mathbf{R}^3\,,\, |y|^2 = 1\}$ be the unit sphere in $\mathbf{R}^3$ centered at the origin. Let $f\colon \mathbf{R}^3 \to \mathbf{R}$ be a smooth function and $x \in \mathbf{R}^3$. Show that

$$\Delta f(x) = \lim_{h\to 0} \frac{3}{2\pi h^2} \int_{\mathbf{S}^2} \left(f(x+hy) - f(x)\right)dy\,. \tag{7}$$

Hint: Use the Taylor expansion of f at the point x, and note that the integrals $\int_{\mathbf{S}^2} y_i\,dy$ and $\int_{\mathbf{S}^2} y_i y_j\,dy$ can be evaluated explicitly.

[1] For grading purposes, any 4 problems correspond to 100%. You can get extra credit if you do more.