1. Let $n \in \{1,2,3\}$ and let $\delta_{\mathbf{R}^n}$ be the Dirac function in $\mathbf{R}^n$. Let A be an $n \times n$ matrix with $\det(A) \neq 0$. Show that

$$\delta_{\mathbf{R}^n}(Ax) = \frac{1}{|\det(A)|}\delta_{\mathbf{R}^n}(x)\,. \tag{1}$$

Solution: Let $\varphi :: \mathbf{R}^n \to \mathbf{R}$ be a smooth function. Consider $\int_{\mathbf{R}^n} \delta(Ax)\varphi(x)\,dx$ and set $Ax = y$. Then $x = A^{-1}y$ and $dx = |\det A^{-1}|dy = \frac{1}{|\det A|}dy$ and the integral becomes $\int_{\mathbf{R}^n} \delta(y)\varphi(A^{-1}y)\frac{1}{|\det A|}\,dy = \varphi(0)\frac{1}{|\det A|} = \int_{\mathbf{R}^n} \varphi(y)\delta(y)\frac{1}{|\det A|}\,dy = \varphi(0)\frac{1}{|\det A|}$.

2. Let a_1, a_2, a_3 be non-zero real numbers. Find the solution of the equation

$$a_1^2\frac{\partial^2 u}{\partial x_1^2} + a_2^2\frac{\partial^2 u}{\partial x_2^2} + a_3^2\frac{\partial^2 u}{\partial x_3^2} = \delta_{\mathbf{R}^3}(x) \tag{2}$$

which satisfies $u(x) \to 0$ as $x \to \infty$.

Solution: Set $\frac{x_j}{a_j} = y_j$, $j = 1,2,3$. Then $a_j^2\frac{\partial^2}{\partial x_j^2} = \frac{\partial^2}{\partial y_j^2}$ and $\delta_{\mathbf{R}^3}(x) = \delta(a_1y_1)\delta(a_2y_2)\delta(a3y_3) = \frac{1}{a_1a_2a_3}\delta_{\mathbf{R}^3}(y)$. The equation becomes $\Delta_y u = \frac{1}{a_1a_2a_3}\delta_{\mathbf{R}^3}(y)$. The solution is $u = -\frac{1}{4\pi a_1a_2a_3|y|} = -\frac{1}{4\pi a_1a_2a_3\sqrt{\frac{x_1^2}{a_1^2}+\frac{x_2^2}{a_2^2}+\frac{x_3^2}{a_3^2}}} = -\frac{1}{4\pi\sqrt{a_2^2a_3^2x_1^2+a_1^2a_3^2x_2^2+a_1^2a_2^2x_3^2}}$

3. Find a solution of

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x_1^2} - \frac{\partial^2 u}{\partial x_2^2} - \frac{\partial^2 u}{\partial x_3^2} = \delta(x_1,x_2,x_3,t) \tag{3}$$

in $\mathbf{R}^3 \times \mathbf{R}$ which vanishes for $t > 0$.

Let $G(x,t)$ be the usual fundamental solution, defined by $G(x,t) = \frac{\delta(t-r)}{4\pi r}$ for $t > 0$ and vanishing for $t < 0$, where $r = |x|$. Then $\tilde{G}(x,t) = G(x,-t)$ will have the desired properties.

4. Calculate the solution of the problem

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} &= \delta(x) \qquad && x \in \mathbf{R}, \qquad t \geq 0 \\ u(x,0) &= 0\,, && x \in \mathbf{R} \\ \frac{\partial u(x,0)}{\partial t} &= 0\,, && x \in \mathbf{R}. \end{aligned} \tag{4}$$

Solution: Let $G(x,t)$ be the fundamental solution of the wave equation in 1 spatial dimension, given by $G(x,t) = \frac{1}{2}$ for $t > |x|$ and 0 otherwise. Then $u(x,t) = \int_0^t \int_{\mathbf{R}} G(x-y,t-s)\delta(y)\,dy\,ds = \int_0^t G(x,t-s)\,ds = \int_0^t G(x,s)\,ds = \frac{1}{2}(t-|x|)_+$, where ξ_+ is the positive part of ξ, i. e. $\xi_+ = \xi$ for $\xi \geq 0$ and $\xi_+ = 0$ otherwise.

5. Let a, b, c, d be real numbers, with $a, c > 0$. Find the solution of the problem

$$\begin{aligned} a\frac{\partial u}{\partial t} + b\frac{\partial u}{\partial x} - c\frac{\partial^2 u}{\partial^2 x} + d\,u = 0 \qquad x \in \mathbf{R}\,, t > 0\,, \\ u(x,0) = \delta(x)\,. \end{aligned} \tag{5}$$

Solution: We can interpret the solution as a result of three processes: (i) $au_t + bu_x = 0$, with the solution $u_0(x - \frac{b}{a}t)$ (where u_0 is the value of the solution at $t = 0$ for this process); (ii) $au_t - cu_{xx} = 0$, with the solution $\int_{-\infty}^{\infty} u_0(x-y)\Gamma(y, \frac{c}{a}t)$, where $\Gamma(x,t) = \frac{1}{4\pi t}e^{-\frac{x^2}{t}}$ is the fundamental solution of the heat equation and u_0 is again the initial contdition for this process (not necessarily the same the the previous u_0), and (iii) $au_t + du = 0$, with the solution $u_0(x)e^{-\frac{d}{a}t}$, where u_0 is again the initial condition for this process (not necessarily the same as the previous u_0s). The three processes commute, so the solution can be obtain by composing them in an arbitrary order. Applying (ii) first with $u_0 = \delta$ gives $\Gamma(x, \frac{c}{a}t)$. Starting from $\Gamma(x, \frac{c}{a}t)$ and applying (i) gives $\Gamma(x - \frac{b}{a}t, \frac{c}{a}t)$. Finally, starting from $\Gamma(x - \frac{b}{a}t, \frac{c}{a}t)$ and applying (iii) gives $\Gamma(x - \frac{b}{a}t, \frac{c}{a}t)e^{-\frac{d}{a}t}$, which is the solution of our problem.

6. For smooth functions $u\colon \mathbf{R}^3 \times (t_1, t_2) \to \mathbf{R}$ consider the functional

$$J(u) = \int_{t_1}^{t_2}\int_{\mathbf{R}^3}\left(\frac{1}{2}(u_t)^2 - \frac{1}{2}|\nabla u|^2 - \frac{c}{2}u^2 - fu\right)dx\,dt\,, \tag{6}$$

where c is a real number, $f = f(x,t)$ is a given function, and ∇u denotes the spatial gradient of u, i. e. the 3-vector with coordinates $\frac{\partial u}{\partial x_j}$, $j = 1,2,3$. To make sure that the integral is well-defined, we can assume that u and f vanish outside a bounded region. Let X be the class of smooth functions on $\mathbf{R}^3 \times [t_1, t_2]$ which vanish outside a bounded set and also vanish for all x whenever $t = t_1$ or $t = t_2$. Calculate the equation which we obtain from the requirement that for each $\varphi \in X$ the derivative of the function $\varepsilon \to J(u + \varepsilon\varphi)$ vanishes at $\varepsilon = 0$.

Solution: Calculating the derivative $\frac{d}{d\varepsilon}|_{\varepsilon=0}J(u+\varepsilon\varphi)$, which we will denote by $J'(u)\varphi$, we obtain $J'(u)\varphi = \int_{t_1}^{t_2}\int_{\mathbf{R}^3}(u_t\varphi_t - \nabla u\nabla\varphi - cu\varphi - f\varphi)\,dx\,dt$. Integration by parts gives $J'(u)\varphi = \int_{t_1}^{t_2}\int_{\mathbf{R}^3}(-u_{tt} + \Delta u - cu - f)\,\varphi\,dx\,dt$. This can vanish for each φ with the specified properties on if $-u_{tt} + \Delta u - cu - f$ vanishes identically. Hence the equation is $u_{tt} - \Delta u + cu + f = 0$.