Math 5588 **Midterm 1** Spring 2018

Do at least four of the following six problems.[1]

1. *Let $\Omega \subset \mathbf{R}^2$ be the domain consisting of the first three quadrants. In other words, $\Omega = \mathbf{R}^2 \setminus Q_{IV}$, where $Q_{IV} = \{x = (x_1, x_2) \in \mathbf{R}^2\,,\; x_1 \geq 0\,\text{and}\; x_2 \leq 0\}$. Find a function $u\colon \Omega \to \mathbf{R}$ which is strictly positive inside Ω, vanishes at the boundary of Ω, and satisfies the equation $\Delta u = 0$ inside Ω.*

Solution: We seek the function u in the form $u = r^\alpha f(\theta)$, where r, θ are the polar coordinates. Using the (two-dimensional) formula $\Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{\partial u}{r \partial r} + \frac{\partial^2 u}{r^2 \partial^2 \theta}$ we obtain the equation $f'' + \alpha^2 f = 0$, the general solution of which is $f(\theta) = A\sin(\alpha(\theta - \theta_0))$, where A and θ_0 are parameters. (Alternatively, one can write the general solution as $A\sin\alpha\theta + B\cos\alpha\theta$, or as $C_1 e^{i\alpha\theta} + C_2 e^{-i\alpha\theta}$.) From the conditions that u is positive in Ω and vanishes at the boundary, we see that we should take $\alpha = \frac{2}{3}$ and $\theta_0 = 0$. We obtain $u = Ar^{\frac{2}{3}} \sin\frac{2}{3}\theta$, with $A > 0$.

2. *In the three dimensional space $\mathbf{R}^3$ consider the equation*

$$-\Delta u + \beta u = 0\,, \tag{1}$$

where β is a parameter. If we assume that a solution u of (1) in the domain $\mathbf{R}^3 \setminus \{0\}$ is of the form $u(x) = \frac{v(r)}{r}$, where $r = \sqrt{x_1^2 + x_2^2 + x_3^2}$, what equation do we get for v?

Solution: In the three dimensional space $\mathbf{R}^3$, when is r as above and $w = w(r)$, we have $\Delta w = w'' + \frac{2w'}{r}$, see for example formula 1.5.22 on page 27 of the textbook, or formula (12) on page 2 of the Lecture Log. Also, it is not hard to obtain this by direct calculation: $\sum_{j=1}^3 \frac{\partial^2}{\partial x_j^2} w(r) = \sum_{j=1}^3 \frac{\partial}{\partial x_j}\left[w'(r)\frac{\partial r}{\partial x_j}\right] = \sum_{j=1}^3 w''(r)\left(\frac{\partial r}{\partial x_j}\right)^2 + w'(r)\Delta r = w'' + \frac{2w'}{r}$. If we now set $w(r) = \frac{v(r)}{r}$ we obtain, after some calculation, $-v'' + \beta v = 0$.

3. *For $a > 0$ we define*

$$\chi_a(x) = \begin{cases} \frac{1}{2a} & x \in (-a, a)\,, \\ 0 & \textit{elsewhere}\,. \end{cases} \tag{2}$$

Let $f(x) = e^x$.
(i) Show that the function $g = \chi_a * f$ satisfies the equation $g' = g$. (We use the standard notation for convolution: $(\chi_a * f)(x) = \int_{\mathbf{R}} \chi_a(x - y) f(y)\, dy = \int_{\mathbf{R}} f(x - y)\chi_a(y)\, dy$.)
(ii) Calculate the function g.

Solution: (i) $g' = (\chi_a * f)' = \chi_a * f' = \chi_a * f = g\,.$
(ii) $g(x) = \frac{1}{2a}\int_{-a}^a e^{x-y}\, dy = e^x \frac{1}{2a}\int_{-a}^a e^{-y}\, dy = e^x \frac{1}{2a}\left(e^a - e^{-a}\right) = \frac{\sinh a}{a}\, e^x$.

4. *Assume that $u\colon \mathbf{R}^3 \times \mathbf{R} \to \mathbf{R}$ satisfies the equation $\frac{\partial^2 u}{\partial t^2} = \Delta u$. Let $\phi\colon \mathbf{R}^3 \to \mathbf{R}$ be a smooth function which vanishes outside the unit ball $B_1 = \{x \in \mathbf{R}^3\,,\, |x| < 1\}$.*
Show that the function $v(x, t) = \int_{\mathbf{R}^3} u(x - y, t)\phi(y)\, dy$ satisfies $\frac{\partial^2 v}{\partial t^2} = \Delta v$.

Solution: $\frac{\partial v}{\partial t}(x, t) = \frac{\partial}{\partial t}\int_{\mathbf{R}^3} u(x - y, t)\phi(y)\, dy = \int_{\mathbf{R}^3} \frac{\partial u}{\partial t}(x - y, t)\phi(y)\, dy = \int_{\mathbf{R}^3} \Delta u(x - y, t)\phi(y)\, dy = \Delta_x \int_{\mathbf{R}^3} u(x - y)\phi(y)\, dy = \Delta v(x, t)\,.$

5. *Let us use the standard coordinates $x = (x_1, x_2, x_3)$ in the three-dimensional space $\mathbf{R}^3$.*
(i) Calculate $\Delta e^{-\frac{|x|^2}{2}}$.
(ii) Evaluate the integral $\int_{\mathbf{R}^3} \Delta e^{-\frac{|x|^2}{2}}\, dx$.
(iii) Use (i) and (ii) to evaluate the integral $\int_{\mathbf{R}^3} |x|^2 e^{-\frac{|x|^2}{2}}\, dx$, given that $\int_{\mathbf{R}^3} e^{-\frac{|x|^2}{2}}\, dx = (2\pi)^{\frac{3}{2}}$.

Solution: (i) By a direct calculation, $\Delta e^{-\frac{|x|^2}{2}} = (|x|^2 - 3)e^{-\frac{|x|^2}{2}}$. (ii) Note that the partial derivatives of $f(x) = e^{-\frac{|x|^2}{2}}$ decay exponentially to zero as $|x| \to \infty$. Letting $B_R = \{x \in \mathbf{R}^3\,,\, |x| < R\}$, we can write for any function f with rapidly decaying derivatives $\int_{\mathbf{R}^3} \Delta f(x) = \lim_{R\to\infty}\int_{B_R} \Delta f(x)\, dx = \lim_{R\to\infty}\int_{\partial B_R} \frac{\partial f}{\partial n}(x)\, dx = 0$. Hence $\int_{\mathbf{R}^3} \Delta e^{-\frac{|x|^2}{2}}\, dx = 0$. (iii) Combining (i) and (ii), we see that $\int_{\mathbf{R}^3} |x|^2 e^{-\frac{|x|^2}{2}} = 3\int_{\mathbf{R}^3} e^{-\frac{|x|^2}{2}} = 3(2\pi)^{\frac{3}{2}}$.

6. *Let Ω be the unit ball in $\mathbf{R}^3$ centered at the origin and let $G(x, y)$ be its Green function. For $y \in \Omega$ evaluate the integral*

$$\int_{\partial\Omega} \sum_{j=1}^3 \frac{\partial G(x, y)}{\partial x_j} x_j x_1 x_2 x_3\, dx. \tag{3}$$

Solution: We note that at the boundary of our particular Ω we have $x_j = n_j(x)$ (the outward unit normal). We recall the identity $\int_\Omega (\Delta u\, v - u\Delta v)\, dx = \int_{\partial\Omega}\left(\frac{\partial u}{\partial n} v - u\frac{\partial v}{\partial n}\right) dx$, and apply it with $u(x) = G(x, y)$ and $v(x) = x_1 x_2 x_3$. Then u vanishes at the boundary $\partial\Omega$ (by the definition of the Green's function) and Δv vanishes in Ω (by a simple calculation). Hence the terms with Δv and u drop out, and we are left with $\int_\Omega \Delta u\, v\, dx = \int_{\partial\Omega} \frac{\partial u}{\partial n} v\, dx$. The integral on the right is our integral (3). For the integral on the left we have $\int_\Omega \Delta u\, v\, dx = \delta(x - y)v(x)\, dx = v(y)$. Hence the integral (3) is equal to $v(y) = v_1 y_2 y_3$.

[1] For grading purposes, any 4 problems correspond to 100%. You can get extra credit if you do more. You can use the textbook, any notes, and a calculator, as long as it does not have wireless capabilities. Devices with wireless communication capabilities are not allowed.