Math 5588 **Practice Test 1** Spring 2018

Do at least four of the following six problems.[1]

1. *Consider the polar coordinates in* $\mathbf{R}^2$*, given by* $x_1 = r\cos\theta\,,\ x_2 = r\sin\theta$*. Let* $\alpha > 0$*, and consider functions* u *in the upper half-plane* $\Omega = \{x = (x_1, x_2) \in \mathbf{R}^2\,,\ x_2 > 0\}$ *which can be expressed in polar coordinates as* $u = r^\alpha f(\theta)$*, where* f *is a smooth function. If we wish that* u *satisfy* $\Delta u = 0$*, what is the equation which should be satisfied by* f*?*

Hint: One way to do this problem is just directly calculate the derivatives using the chain rule. For that you will need to calculate the derivatives $\frac{\partial r}{\partial x_i}$ and $\frac{\partial^2 r}{\partial x_i \partial x_j}$ and also $\frac{\partial \theta}{\partial x_i}$ and $\frac{\partial^2 \theta}{\partial x_i \partial x_j}$, where you take r and θ as functions of x_1, x_2. The expression for r is clear: $r = \sqrt{x_1^2 + x_2^2}$. There are several expressions one can use for θ, such as $\theta = \arcsin\left(\frac{x_2}{r}\right)$, or $\theta = \operatorname{arccot}\frac{x_1}{x_2}$. Another way to calculate the derivatives is to differentiate the expressions $x_1 = r\cos\theta, x_2 = r\sin\theta$ with respect to x_1, x_2, considering r, θ as functions of x_1, x_2, and then express the derivatives we seek from the resulting equations. The most efficient way to do the calculation is to use the expression for Δ in the polar coordinates: $\Delta = \frac{\partial^2}{\partial r^2} + \frac{\partial}{r\partial r} + \frac{\partial^2}{r^2\partial\theta^2}$. It is a good exercise to derive this formula, but for the purpose of the exam you can use it without deriving it.

2. *Let* $\Omega \subset \mathbf{R}^3$ *the domain obtained from* $\mathbf{R}^3$ *by removing the* x_3-*axis. Let* $v\colon (0,\infty) \to R$ *be a smooth function and let* $u\colon \Omega \to \mathbf{R}$ *be defined by* $u(x_1, x_2, x_3) = e^{-x_3}\, v\left(\sqrt{x_1^2 + x_2^2}\right)$*. What equation does* v *have to satisfy if we wish* u *to satisfy* $\Delta u = 0$*?*

3. *For* $a > 0$ *we define*

$$\chi_a(x) = \begin{cases} \frac{1}{2a} & x \in (-a, a)\,, \\ 0 & \textit{elsewhere}\,. \end{cases} \tag{1}$$

(i) Calculate the function $\chi_a * \chi_a$*.*
(ii) If $f\colon \mathbf{R} \to \mathbf{R}$ *is continuous and* $x \in \mathbf{R}$*, calculate* $\lim_{a\to 0_+} (f * \chi_a * \chi_a)(x)$*.*

Hint: One can do this calculation directly from the definition $\chi_a * \chi_a(x) = \int_{-\infty}^{\infty} \chi_a(x-y)\chi_a(y)\,dy$, the resulting integral is easily evaluated. Alternatively, if you are comfortable working with the Dirac function, you can use the fact that $2a\chi_a'(x) = \delta(x+a) - \delta(x-a)$ and the formula $(\chi_a * \chi_a)' = \chi_a' * \chi_a$ to obtain the derivative of $\chi_a * \chi_a$ without much calculation.

4. *Assume that* $u\colon \mathbf{R}^3 \to \mathbf{R}$ *satisfies the equation* $\Delta u + \frac{\partial u}{\partial x_1} = 0$*. Let* $\phi\colon \mathbf{R}^3 \to \mathbf{R}$ *be a smooth function which vanishes outside the unit ball* $B_1 = \{x \in \mathbf{R}^3\,,\ |x| < 1\}$*. Show that the function* $v = u * \phi$ *satisfies the same equation:* $\Delta v + \frac{\partial v}{\partial x_1} = 0\,.$

5. *Let* $f\colon \mathbf{R}^3 \to \mathbf{R}$ *be defined by* $f(x) = e^{-\frac{x_1^2 + x_2^2 + x_3^2}{2}}$*. Given that* $\int_{\mathbf{R}^3} f(x)\,dx = (2\pi)^{\frac{3}{2}}$*, evaluate the integral*

$$I = \int_{\mathbf{R}^3} (x_1^2 + x_2^2 + x_3^2)^2\, \Delta^2 f(x)\,dx\,, \tag{2}$$

where $\Delta^2 f = \Delta(\Delta f)$*.*

Hint: Use integration by parts.

6. *Let* Ω *be the unit ball in* $\mathbf{R}^3$ *centered at the origin and let* $G(x,y)$ *be its Green function. Show that for* $y \in \Omega$ *we have*

$$\int_{\partial\Omega} \sum_j \frac{\partial G(x,y)}{\partial x_j} x_j x_1\ dx = y_1\,. \tag{3}$$

Hint: Use integration by parts, general properties of Green's functions, and the following facts: (i) $\Delta x_1 = 0$, and (ii) when $x \in \partial\Omega$, then x_j coincides with the $j-$th component of the normal to $\partial\Omega$. Although we know G explicitly, it might not be very efficient to try to use the explicit expression for this calculation - it is easier to do a calculation based on general properties of Green's functions.

[1] For grading purposes, any 4 problems correspond to 100%. You can get extra credit if you do more. You can use the textbook, any notes, and a calculator, as long as it does not have wireless capabilities. Devices with wireless communication capabilities are not allowed. Hints for solutions such as the ones above might not be included on the real exam.