Math 5588 **Practice Test 2** Spring 2018

Do at least four of the following six problems.[1]

1. *Assume that* $u_0 = u_0(x_1, x_2)$ *is a smooth function on* $\mathbf{R}^2$ *which is* $1-$*periodic in each direction, i. e.* $u(x_1+1, x_2) = u(x_1, x_2) = u(x_1, x_2+1)$. *Let us consider the problem*

$$u_t = -\Delta\Delta u + \Delta u + \frac{\partial u}{\partial x_1}, \quad x \in \mathbf{R}^2, \, t \geq 0 \tag{1}$$

$$u|_{t=0} = u_0 \,. \tag{2}$$

(The double Laplacian $\Delta\Delta$ *could also be written as* Δ^2.*) If we express the solution* u *as*

$$u(x,t) = \sum_{k \in \mathbf{Z}^2} c(k,t) e^{2\pi i (k_1 x_1 + k_2 x_2)} \,, \tag{3}$$

what are the formulae for $c(k,t)$ *(assuming we know the Fourier coefficients of* u_0*)?*

2. *If you have a computer program which can calculate the Fourier coefficient of a functions and also sum a given Fourier series on the square* $[0,1] \times [0,1]$, *how would you use it to solve the problem*

$$-\Delta u = f \quad \text{in } \Omega \tag{4}$$

$$u|_{\partial\Omega} = 0 \quad \text{in } \Omega \tag{5}$$

when Ω *is the parallelogram* $\{(x_1, x_2)\,,\, 0 < \frac{1}{2}x_1 < 1\,,\, 0 < x_1 + x_2 < 1\}$*?*

Hint: Use the change of variables $y_1 = \frac{1}{2}x_1$ and $y_2 = x_1 + x_2$. Note that $x = (x_1, x_2) \in \Omega$ exactly when $y = (y_1, y_2) \in (0,1) \times (0,1)$. Express the problem in the $y-$coordinates and use the Fourier series in y. The equation in the $y-$variables will not be $-\Delta u = f$, but it will be of the form $-(a\frac{\partial^2}{\partial y_1^2} + b\frac{\partial^2}{\partial y_1 \partial y_2} + c\frac{\partial^2}{\partial y_2^2})u = f$. The important thing is that the coefficients a, b, c are constant (independent of y), and therefore the euqation is well-suited for the use of Fourier series. The equation on the Fourier side has to be modified accordingly, of course.

3. *Let* $\Omega \subset \mathbf{R}^3$ *be a bounded smooth domain. Assume that* $\phi_1, \phi_2, \phi_3 \ldots$ *are the eigenfunctions of the Laplacian* $-\Delta$ *in* Ω *with the zero boundary condition and eigenvalues* $\lambda_1 < \lambda_2 \leq \lambda_3 \leq \ldots$. *(In other words,* $-\Delta\phi_j = \lambda_j \phi_j$ *and* $\phi_j|_{\partial\Omega} = 0$.*) Find the solution of the initial-value problem for the Schrödinger equation*

$$iu_t + \Delta u = 0 \quad \text{in } \Omega \times (0, \infty)\,, \tag{6}$$

$$u|_{\partial\Omega} = 0\,, \tag{7}$$

$$u(x,0) = u_0(x) \quad \text{in } \Omega\,, \tag{8}$$

where $i = \sqrt{-1}$ *and* $u_0(x) = \sum_{k=1}^n c_k \phi_k(x)$.

Remark: This is solved in a way very similar to what we did in class for the heat equation.

4. *Let* $\Omega \subset \mathbf{R}^3$ *be a smooth bounded domain and let* g *be a smooth function on* $\partial\Omega$. *What will be the PDE and the boundary conditions corresponding to the minimization of the functional*

$$J(u) = \int_\Omega \frac{1}{2}|\nabla u|^2 \, dx + \frac{1}{\varepsilon} \int_{\partial\Omega} \frac{1}{2}(u-g)^2 \, dx \,, \tag{9}$$

where $\varepsilon > 0$ *is a parameter.*

5. *Let* $b = (b_1, b_2, b_3) \in \mathbf{R}^3, c \in \mathbf{R}$ *and let* $u_0 \colon \mathbf{R}^3 \to \mathbf{R}$ *be a smooth function. Find the solution* $u(x,t)$ *of the equation*

$$u_t + \sum_{j=1}^3 b_j \frac{\partial u}{\partial x_j} a + cu = 0 \qquad \text{in } \mathbf{R}^3 \times (-\infty, \infty) \tag{10}$$

with $u(x,0) = u_0(x)$.

6. *Let* $\Omega \subset \mathbf{R}^3$ *be the complement of the unit ball, i. e.* $\Omega = \{x \in \mathbf{R}^3\,, |x| > 1\}$. *Find the Green's function of the domain* Ω *for the equation* $\Delta u = f$ *with the boundary conditions* $u|\partial\Omega = 0$ *and* $u(x) \to 0$ *as* $x \to \infty$.

[1] For grading purposes, any 4 problems correspond to 100%. You can get extra credit if you do more. You can use the textbook, any notes, and a calculator, as long as it does not have wireless capabilities. Devices with wireless communication capabilities are not allowed. Hints for solutions such as the ones above might not be included on the real exam.