PDE Aspects of the Navier-Stokes Equations

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Abstract

In this lecture we will discuss mathematical aspects of the Navier-Stokes equations. We will recall some of the important open problems and mention a few recent results.

Consider a ball of radius R moving in an incompressible fluid of constant density ρ at constant velocity U. We expect that a force F is needed to keep the ball in motion (to overcome the "resistance of the medium"). The force is usually called the drag force. What is the formula for the drag force? This classical problem of Fluid Mechanics was considered already by Newton, who derived the formula

$$F = c\,\rho R^2 U^2 \ , \tag{1}$$

where c is a dimensionless constant. The formula (published in 1687) can be found in *Principia*, Corollary 1 of Theorem 30, Book II. From the modern point of view we can see that the formula is dictated by the dimensional analysis: the given expression for F is the only possible expression with the dimension of force which can be formed from the available data ρ, R, and U.

In 1752 d'Alembert published the well-known *Essai d'une nouvelle théorie de la résistance des fluides*, where he reached the surprising conclusion that in ideal fluids the drag force is

$$F = 0 \ . \tag{2}$$

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This is known as d'Alembert's paradox[1].

In 1851 Stokes brought viscosity into the considerations, and derived that for slowly moving objects one should have

$$F = 6\pi\nu\rho RU \ , \tag{3}$$

where ν is the *kinematic viscosity* (which has dimension $[\text{Length}]^2/[\text{Time}]$) of the fluid.

The mathematical description of the fluid motion we use today is the same as the one used by Stokes, the Navier-Stokes equations:

$$\begin{array}{rcl} u_t + u\nabla u + \frac{1}{\rho}\nabla p - \nu\Delta u &=& 0 \\ \operatorname{div} u &=& 0 \ , \end{array} \tag{4}$$

where $u = (u_1(x,t), u_2(x,t), u_3(x,t))$ is the velocity of the fluid particle which is at position x at time t, and $p = p(x,t)$ is the pressure. For $\nu = 0$ the system (4) was derived in by Euler in 1757.

The natural boundary conditions for $\nu > 0$ is that $u = 0$ at the boundaries (in the local coordinates in which the boundary is at rest). For $\nu = 0$ the natural boundary condition is $u(x,t) \cdot n(x) = 0$, where $n(x)$ is the outward unit normal to the boundary.

The formulae of Stokes and d'Alembert are well understood in the context of PDE (4). For the Stokes formula one calculates (following Stokes) an explicit solution of the linearized problem. For d'Alembert's formula one assumes $\nu = 0$ and calculates that a steady-state solution of Euler's equations with the natural boundary conditions indeed leads to $F = 0$. (See, for example, [16].)

The formula of Newton is much more intriguing from the PDE point of view. Before we start its discussion, let us introduce an important dimensionless parameter of the above flow around a ball. We define the *Reynolds number* (introduced by Reynolds in 1880's) by

$$\mathrm{Re} = \frac{RU}{\nu} \ . \tag{5}$$

[1]The paradox has been well understood since the work of Prandtl in the early 1900s. The source of the paradox is the "ideal fluid" assumption, which is not satisfied for the usual fluids. Even very small internal friction in the fluid can have large effects when the fluid interacts with rigid boundaries. This is not captured by the ideal fluid model. See [23].

Flows with the same Reynolds number are equivalent in the sense that the non-trivial scaling symmetries of the Navier-Stokes equations

$$u(x,t),\, p(x,t) \longrightarrow \lambda u(\lambda x, \lambda^2 t),\, \lambda^2 p(\lambda x, \lambda^2 t) \tag{6}$$

can be used to map the situations with the same Reynolds number onto one another[2].

From experiments we know that Newton's formula (1) is nearly correct once the Reynolds number is large ($\mathrm{Re} \geq 10^6$ should be sufficient). From the point of view of PDEs this is remarkable, since ν plays a prominent role in the equation, and yet the force given by Newton's formula is independent of ν.

There is another remarkable classical experimental fact (discovered in the early 1900s by Prandtl and Eiffel): in our experiment described above, there is a certain Reynolds number Re_c, typically in the range $10^5 - 10^6$ (where the c in Newton's formula is not yet constant[3]), such that as we increase the velocity, the drag force will suddenly noticeably *decrease* as the Reynolds number crosses the critical value Re_c. This phenomenon is known as the *drag crisis* (see for example [16]), and (experimentally) it is related to a change of the geometry of the flow.

Are the above examples of fluid behavior described by the Navier-Stokes equations? The general belief is that this is indeed the case. However, strictly speaking, we do not really know, since the behavior is known only from experiments and not from computations or theoretical analyses of the equations. It is perhaps worth remarking that since one cannot really do experiments in two dimensions, we do not know if the two-dimensional Navier-Stokes exhibits similar behavior at large Reynolds numbers.

One should mention the following non-trivial issue which comes up in connection with the above problem of calculating the drag force. In our particular situation we want to solve the Navier-Stokes in the exterior domain $\Omega = \{x \in \mathbf{R}^3,\ |x| > R\}$ with the boundary conditions $u(x,t) = 0$ at the boundary $\partial\Omega = \{|x| = R\}$ and $u(x,t) \to U$ (where U is now considered as a vector) as $x \to \infty$. The Navier-Stokes equation (4) in Ω with these boundary conditions has various solutions. For example, one can consider axi-symmetric steady-state solutions. It can be proved that such solutions

[2]Here we of course have in mind flows defined in the exterior of balls.

[3]The graphs of the dependence of c on Re obtained in experiments can be found in many Fluid Mechanics textbooks, see for example [16].

exist (this is essentially due to Leray). However, these solutions do not give the right drag force for the large Reynolds numbers. For example, it is likely (although it may not be known rigorously) that the drag force for these solutions approaches zero as the viscosity ν approaches zero, in contrast with what is observed in experiments. The reason is that the steady-state symmetric solutions are unstable, and the stable flows are neither symmetric nor time-independent. This has to be taken into account in numerical simulations. In reality the drag force F is a time average of the instantaneous force $F(t) = \int_{\partial\Omega}[p(x,t)n(x) - 2\nu e(x,t)n(x)]\,dx$, where $e(x,t)$ is the symmetric part of $\nabla u(x,t)$ and $n(x)$ is the unit normal.

Can the drag force be calculated numerically? With the best present-day computers, we cannot reliably solve the equations for Reynolds numbers exceeding 10^4. The reason is the appearance of various fine-scale structures, which make it difficult to resolve the details of the solutions. How much resolution is needed? There is a statistical-type theory of turbulence due to Kolmogorov and Onsager (see, for example, [16]) which is not based on the equations of motion, but rather on the assumption that certain statistics of the observed vector fields are invariant under natural scalings[4]. In the situation concerning the drag force, the assumed statistics implies that the drag force is independent of the viscosity (for large Reynolds numbers). The rough prediction from this theory is that the amount of computation needed to calculate the drag force at Reynolds number Re by "brute force" (i. e. by fully resolving the equations) is proportional to $\mathrm{Re}^{11/4}$. This rough prediction is probably too optimistic[5], but it is still worth noticing that if we believe it and if we assume we could perhaps do $\mathrm{Re} = 10^4$ today[6], we need roughly 10^4 – fold increase in the computational power to calculate the air flow around tennis balls at speeds common in the game. To resolve the air flow around a car at realistic speeds (it can still be considered incompressible to a very good approximation), we would need roughly a 10^8 – fold increase in the computational power. Engineers of course need to calculate flows around cars (and airplanes) today, and to achieve this it is necessary to give up the hope of finding the full solution of the equations, and try instead to find some

[4]This is somewhat analogous to the situation in Statistical Mechanics where the laws are not really justified from the equations of motion, but are taken as new postulates.

[5]Among other things, it does not take into account the subtleties concerning the boundary layer.

[6]This would not be a calculation which could be done on a PC or a work station, we have in mind a really big computer.

approximations. The art of finding such approximations is a large research area by itself, and there have been many partial successes. The main idea is that we do not need to resolve the solution fully, and that it should be possible to replace most of the unknown degrees of freedom by a suitable statistics. The famous open *problem of turbulence* can be thought of as the problem of finding a good algorithm which would do such a reduction to a relatively small number of variables in a reliable way in the general situation.

There is an additional problem one has to face. Namely, it is not known if (in dimension three) the Navier-Stokes equations (4) admit a smooth solution which would describe the flows encountered in the drag force problem. This is known as the regularity problem. The simplest version of the problem is the following: does the initial-value problem for the system (4) in $\mathbf{R}^3 \times (0, \infty)$ with given initial data $u(x, 0) = u_0(x)$ have a smooth solution? The initial datum u_0 is assumed to be smooth and decay "sufficiently fast" to zero as $x \to \infty$.

This problem should be easier than the problem of turbulence, but it is still universally considered as a hard mathematical problem. In dimension two it has been solved a long time ago (by Leray in domains without boundaries and by Ladyzhenskaya in domains with boundaries), see for example [15]. By contrast, the problem of turbulence (as defined above) is open even in dimension two.

The reason why the regularity problem in dimension three is hard (at least for the present-day PDE techniques) can be understood as follows. There is only a limited number of "general" tools available for the analysis of PDEs and none of these techniques seem to be sufficient. No special mathematical structure has been discovered in the equations, and therefore the present-day theory has to treat them, to a large degree, as "the general case". At the same time, one does not expect regularity for all equations in this general class.

The available techniques include:

- Linear estimates,
- Perturbation Analysis, see for example [17, 11, 13],
- Energy methods, see, for example, [17, 3],

- "Scalar techniques", such as the maximum principle and other comparison principles, Harnack inequalities, De Giorgi, Nash-Moser, and Krylov-Safonov estimates, etc.,
- re-scaling and blow-up techniques, classification of entire solutions, etc., as pioneered by De Giorgi for minimal surfaces.

As in other areas of PDEs, ideas which can be traced back to the work of De Giorgi have played play an important role.

In the case of equation (4), we have a good understanding of the estimates for the linear part of the equation (which is called the Stokes system). We also have the energy identity

$$\int_{\mathbf{R}^3} \frac{1}{2}|u(x,t_2)|^2\,dx + \int_{t_1}^{t_2}\int_{\mathbf{R}^3} \nu|\nabla u(x,t)|^2\,dxdt = \int_{\mathbf{R}^3} \frac{1}{2}|u(x,t_1)|^2\,dx\,, \quad (7)$$

which gives us the control of the quantity on the left-hand side ("parabolic energy"). We can now try to combine the energy estimate with the linear estimates and check if we can handle the non-linear term as a perturbation. This is possible in dimension two, but the energy estimate is too weak in dimension three. Two is the borderline dimension for this argument. (Sometimes the term "critical" is used in such borderline situations.) In dimension two we have in fact another quantity which plays to our advantage, the vorticity $\omega = \operatorname{curl} u$. This is a scalar (we are in dimension two!) which satisfies

$$\omega_t + u\nabla\omega = \nu\Delta\omega\ . \quad (8)$$

For this equation we have the maximum principle, and in the absence of boundaries it is easy to see that ω must be bounded. The boundedness of ω is more than enough for regularity. From this point of view the situation is "subcritical" – we have more than we need. However, the argument does not apply near the boundaries, where no "subcritical" argument is known, and one has to work with the energy estimate at the "critical" level.

In dimension three the situation is "super-critical" and all these arguments break down. This can also be understood heuristically in the following terms. The non-linear term in the equation can generate small length scales from large length scales (or, in Fourier terms, large frequencies from low frequencies). The linear part of the equation (which is dissipative) damps the small length scales. In the subcritical case the damping is stronger than the transfer from the longer length scales, and therefore the solution will stay in the realm of the finite length scale, which translates to regularity. In the supercritical case the damping may be insufficient, unless there is some extra mechanism which would slow down the transfer to the small scales. The critical case is the borderline.

It is perhaps interesting to note that when the large length scales (or low frequencies) become the focus, the situation is reversed. This happens for example when we study the behavior at infinity of the steady-state solutions of Navier-Stokes in exterior domains. The steady-state equations are subcritical in dimension two and three, and (local) regularity presents no problems. On the other hand, the study the solutions near ∞ is in some sense the study of the behavior at very large length scales (or low frequencies), and the subcritical case becomes more difficult. In fact, in both dimension two and three the exact behavior of the steady state solutions as $x \to \infty$ represents a difficult open problem[7], see for example [1, 10, 14]. The issues become somewhat similar to the issues one has to face in the 3d time-dependent regularity problem, except that the problems are on the other end of the length scale spectrum, and are most likely easier[8]. There seems to be some kind of vague duality here.

In the above approach one uses only the energy estimate, estimates for the linear part, and some simple properties of the non-linear term, which are satisfied for many other equations. It is expected that the class of equations which share with Navier-Stokes the properties which have been used in the regularity theory so far contains some equations which allow the singularity formation in finite time from smooth data, see for example [21] and [22].

There has been some limited success with trying to find hidden scalar quantities in the equations, and to use the rescalings together with suitable blow-up procedures. Such techniques have been used for example to rule out self-similar singularities [20, 25], certain type of axi-symmetric singularities [4, 5, 12], and to prove unexpected regularity results in the (super-critical) model case of the 5-dimensional steady-state Navier-Stokes [9, 28]. However, for the general time-dependent solutions is 3d no such quantities are known.

There has been a lot of research on conditions which are sufficient for regularity of solutions. After the well-known work of Leray, Prodi, Serrin

[7]The 2d problem is more difficult than the 3d problem in this case, as the 2d equation is "more subcritical".

[8]Nevertheless, still sufficiently hard to have remained open since the early papers of Leray on the subject in 1930s. See [10].

and Ladyzhenskaya (see, for example, [17, 24, 15, 27]) this program has been further developed for example in [3, 6]. One interesting recent development in this direction is the result that the boundedness of the spatial L^3 - norm $||u(t)||_3 = \{\int_{\mathbf{R}^3} |u(x,t)|^3 \, dx\}^{1/3}$ of the solution (independently of t) is sufficient for regularity, see [8]. An interesting feature of the proof of the result is a somewhat unexpected connection to the control theory of parabolic equations.

The L^3 norm is special in that it is invariant under the scaling symmetry of the equation. It has the same dimension as the kinematic viscosity, and hence the quantity $||u(t)||_3/\nu$ is dimensionless. In some sense, this quantity can play the role of the Reynolds number in the absence natural length-scales[9].

It should be mentioned that already in 1934 Leray proved that the 3d Navier-Stokes equations always admit global weak solutions. These are solutions which make sense even when singularities appear. It is known that the set of possible singular points must be relatively small: its 1-dimensional parabolic Hausdorff measure has to be zero, see [3].

The main drawback of the theory of the weak solutions is that it is unknown whether they are unique[10]. This implies some difficulties for applications. For example, in the problem of the calculation of the drag force we mentioned in the beginning, it would not be trivial to define what the drag force is if the classes of solutions we deal with are not unique. Presumably one would have to find a suitable invariant measure on the set of all possible weak solutions and use some averaging process to define the drag force. It is not clear to me whether this has been done.

As interesting as the above mentioned results may be, they are inadequate for a real understanding the full 3d regularity problem. The key for the understanding of the problem might be in Euler's equation (the case $\nu = 0$). The regularity problem for Euler's equation is also open in dimension three (and the existence of global regular solutions is known in dimension two). One remarkable feature of Euler's equation is that the equation is completely canonical, there are no free parameters. In some sense it is really a geometric

[9] The fluid occupying the whole space $\mathbf{R}^3$ is a good example.

[10] In fact, some mathematicians consider the problem of the uniqueness of the weak solutions to be more important than the regularity problem.

equation, and there is indeed a lot of beautiful geometry behind it, see for example [2, 7, 19]. So far there has not been much success in combining the PDE tools used for Navier-Stokes with the geometry which is behind Euler. For example, one of the main non-trivial mathematical facts about the solutions of Euler's equation, the Kelvin-Helmholtz law (see e. g. [26]), has not really found too much use in the regularity theory. It seems that Analysis and Geometry need to be brought together in some new way to make progress on these problems.

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