ON LANDAU'S SOLUTIONS OF THE NAVIER-STOKES EQUATIONS

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Dedicated to Professor Nikolai V. Krylov on the occasion of his 70th birthday.

1. Introduction.

In this paper we study a special class of solutions of the $n-$dimensional steady-state Navier-Stokes equations

$$\begin{aligned} -\Delta u + u\nabla u + \nabla p &= 0\,, \\ \operatorname{div} u &= 0\,, \end{aligned} \tag{1}$$

where $u = (u_1, \dots, u_n)$. The equations have a non-trivial scaling symmetry $u(x) \to \lambda u(\lambda x)$ and it is natural to try to find solutions which are invariant under this scaling. The simplest natural domain of definition for such solutions is $R^n \setminus \{0\}$. In this case, assuming that the solutions are smooth in $R^n \setminus \{0\}$, we are able to obtain a good classification of the invariant solutions in all dimensions. There are some interesting conclusions for the regularity theory as well as for the long-distance behavior of solutions in exterior domains which can be drawn from this classification, which will be discussed. We will distinguish three cases, namely $n = 2$, $n = 3$ and $n \geq 4$. (Sometimes it is also useful to distinguish the cases $n = 4$ and $n \geq 5$, as $n = 5$ is the lowest dimension in which the $(-1)-$ homogeneous functions which are smooth in $R^n \setminus \{0\}$ have locally finite energy $\int_{|x|<r} |\nabla u|^2$. This will not be important for our purposes in this paper, however.)

It is useful to note that the effects of a $(-1)-$ homogeneous singularity are more serious in low dimensions. Our $(-1)-$ homogeneous solution u will be locally integrable across the origin for any $n \geq 2$ and hence can always be considered as a distribution in R^n, but the validity of the equations across the origin will depend on n. For example, for $n \geq 3$ the equation $\operatorname{div} u = 0$ will be satisfied in R^n, but this may not be the case for $n = 2$, when case $\operatorname{div} u$ may produce a multiple of a Dirac mass at $x = 0$. Similarly, for $(-1)-$ homogeneous smooth Navier-Stokes solutions in $R^n \setminus \{0\}$, the weak form of the equation $(\int_{R^n} (-u\Delta\varphi - u_i u_j \varphi_{i,j}) = 0$ for smooth compactly supported vector fields φ with $\operatorname{div} \varphi = 0)$ will be satisfied across the origin when $n \geq 4$. For $n = 3$ the expression may produce a non-trivial right-hand side (a multiple of a Dirac mass supported at $\{0\}$), while for $n = 2$ the right-hand side may not be well-defined, in general.

Let us start with the case $n = 3$, which is perhaps the most interesting. Explicit examples of (-1) homogeneous solutions in $R^3 \setminus \{0\}$ were first calculated by L.D.Landau in 1944 ([L]) and can be found in standard textbooks ([LL], p. 82, or [B], p. 206, for example). See also formulae (11) in Section 4. The main idea of Landau's calculation is that if we impose an additional symmetry requirement, namely that the solutions are axi-symmetric, the system (1) reduces to a system of ODEs which, surprisingly, can be solved explicitly in terms of elementary functions. (In fact, as it was kindly pointed out to the author by V. Galaktionov, the ODEs were written down already in 1934 by N.A.Slezkin, see [Sl].) The solutions were also independently found by H.B.Squire in 1951 ([Sq]). More recently, the topic has been re-visited in [TX] and [CK], where issues concerning Landau's solutions are addressed from a slightly different viewpoint.

Here we prove that even if we drop the requirement of axi-symmetry, Landau's solutions are still the only solutions of (1) which are invariant under the natural scaling. More precisely, we will prove the following:

Theorem 1. *Assume that* $u: R^3 \setminus \{0\} \to R^3$ *is a non-trivial smooth solution of (1) satisfying* $\lambda u(\lambda x) = u(x)$ *for each* $\lambda > 0$. *Then* u *is a Landau solution. In other words,* u *is axi-symmetric and, in a suitable coordinate frame, is described by formulae (11) in Section 4.*

The proof of the theorem shows a connection between the scale-invariant solutions of (1) and the conformal geometry of the two-dimensional sphere. In fact, once the connection is understood, the formulae for Landau's solutions can be derived without much calculation, using just the geometrical properties of the two-dimensional sphere.

Some implications of Theorem 1 are considered in Sections 2 and 3.

In the case $n = 2$ our assumptions reduce the problem to an ODE on the circle S^1. The ODE has been studied in a 1917 paper by G. Hamel [Ha] where a reasonably complete description of solutions is obtained in terms of elliptic functions. Here we re-visit some of these calculations and classify the solutions satisfying an additional constraint that $\operatorname{div} u = 0$ across the origin, which means that the origin is neither a source nor a sink for the flow. It turns out that with this assumption there is, modulo rotations, a countable family of $(-1)-$ homogeneous solutions (smooth away from the origin). See Theorem 2 in Section 5.

The case $n \geq 4$ has been previously considered by several authors in connection with potential singularities of Navier-Stokes solutions. In particular, a non-trivial $(-1)-$ homogeneous solution which is smooth in $R^5 \setminus \{0\}$ would represent singular weak solution in $\{|x| < 1\}$ with finite energy $\int_{|x|<1} |\nabla u|^2$. It was proved independently by several authors ([FR], [St2], [T], [Sv2]) that for $n \geq 4$ there are no non-trivial $(-1)-$ homogeneous solutions (smooth away from the origin), thus ruling our this particular scenario for singularities. For completeness we reproduce here the proof given by the author and T.-P. Tsai, which appeared in [T]. See Theorem 3 in Section 6.

The problem of finding $(-1)-$ homogeneous solutions can be considered in any domain invariant under the dilations $x \to \lambda x$, and the next domain (after $R^n \setminus \{0\}$) which one should consider is the half-space R^n_+, with u satisfying the boundary condition $u = 0$ on $\partial R^n_+ \setminus \{0\}$. While the problem should be manageable for $n = 2$, it seems to quite harder when $n \geq 3$, in which case the existence of non-trivial solutions in the half-space is open. The most interesting case seems to be $n = 5$, when a non-trivial solution would give a finite-energy boundary singularity for the steady-state equations. (As pointed out in [RF] and [St1], the 5d steady state problem is a good model for some aspects of the 3d time-dependent problem.)

2. Regularity of very weak solutions.

By a very weak solution of the steady-state Navier-Stokes system (1) in a domain $\Omega \subset R^n$ we mean a divergence-free vector field $u = (u_1, \ldots, u_n) \in L^2_{\text{loc}}(\Omega)$ which satisfies

$$\int_\Omega (u_i \Delta\varphi_i + u_i u_j \frac{\partial \varphi_i}{\partial x_j}) = 0$$

for each smooth, compactly supported, divergence-free vector field $\varphi = (\varphi_1, \ldots, \varphi_n)$ in Ω.

It is an open problem whether very weak solutions of (1) are regular. Standard regularity theory can be used to show that very weak solutions are regular under the additional requirement that $u \in L^n_{\text{loc}}$ when $n \geq 3$. (In the case $n = 2$ one can obtain regularity for $u \in L^{n+\varepsilon}_{\text{loc}}$, $\varepsilon > 0$, while the case $\varepsilon = 0, n = 2$ appears to be open.) Equations (1) are usually considered with the assumption that $\nabla u \in L^2_{\text{loc}}$, in which case regularity follows for $n \leq 4$ by a standard bootstrapping argument. (The case $n = 4$ is critical for $\nabla u \in L^2_{\text{loc}}$.) The assumption $\nabla u \in L^2_{\text{loc}}$ is of course very natural when considering solutions describing real physical flows. However, one can speculate that very weak solutions might arise from a blow-up procedure of the usual weak solutions of the time-dependent Navier-Stokes equations at a possible singularity (if a singularity exists). The time-dependent 3-dimensional Navier-Stokes equations are supercritical with respect to the natural energy estimates, and in a blow-up procedure the information about energy can be lost.

A natural first step in understanding the regularity of the very weak solutions above is to study the scale-invariant solutions in R^n which are smooth in $R^n \setminus 0$. Theorem 3 in Section 6 settles this problem in $n \geq 4$. For $n = 3$ we can use Theorem 1: a calculation (which can be found in [B], p. 209, and also in [T], and [CK]) shows that, for $n = 3$, Landau's solutions are not very weak solutions of (1) across the origin. Hence we have

Corollary. *Let $n \geq 3$ and let u be a (-1)-homogeneous very weak solution of the Navier-Stokes equations in R^n, which is smooth away from the origin. Then $u \equiv 0$.*

This result rules out only the simplest conceivable singularity of a very weak solution. For example, the question if one can have a non-trivial very weak solution smooth away from the origin and satisfying $|u(x)| \leq C|x|^{-1}$ in R^n is not answered by Theorem 1 and – as far as I know – remains open. (The results in [FR] can be used to obtain some results for $n \geq 5$, under additional assumptions.)

3. Landau solutions and behavior near ∞ of solutions in exterior domains in dimension $n = 3$.

Theorem 1 has some relevance for the problem of long-range behavior of solutions of the Navier-Stokes equations in three-dimensional exterior domains. (See for example [G] for an overview of this topic.) Let f be a compactly supported vector field in R^n and consider the equations

$$\begin{aligned} -\Delta u + u\nabla u + \nabla p &= f\,, \\ \operatorname{div} u &= 0 \end{aligned} \qquad \text{in} \quad R^n\,, \tag{2}$$

together with a "boundary condition" at ∞, which might take the form $u(x) \to 0$ at ∞ and $\int_{R^n} |\nabla u|^2 < \infty$, when $n \geq 3$. (See below for remarks concerning the case $n = 2$, which is more difficult, essentially due to issues related to the Stokes paradox, see [Am].) The existence of such solutions was proved for $n = 3$ in a classic paper by Leray ([Le]), but there are many open questions about the behavior of these solutions for large x, see [G]. The situation is more favorable in the case when f is small, as in this case one can use perturbation techniques in spaces with prescribed decay for large x to obtain a more detailed control of the solution. This technique was pioneered by R. Finn, see [F, G]. However, the control is only in terms of the decay, it does not give the leading-order term, as the error term is of the same order of magnitude as the main term.

Theorem 1 implies, roughly speaking, the following:

Corollary. *In dimension $n = 3$, if a solution of the above exterior problem is asymptotically (-1)-homogeneous, then the terms of order $|x|^{-1}$ must be given by a Landau solution.*

To give this a more precise meaning, let us consider the scaled functions u_λ and f_λ defined by $u_\lambda(x) = \lambda u(\lambda x)$ and $f_\lambda(x) = \lambda^3 f(\lambda x)$. The functions u_λ and f_λ satisfy the same equations as u and f. Moreover the functions f_λ converge to a distribution $\bar{f}$, given by $\bar{f}(x) = b\delta(x)$, where $b = \int_{R^3} f$ and δ is the Dirac function. *Assume* now that u_λ converges to a limit $\bar{u}$ in, say, $L^3_{\rm loc}(R^3 \setminus \{0\})$ as $\lambda \to \infty$. Our notion of "asymptotically (-1)- homogeneous" used above can be *defined* by requiring that this is really the case. It is known that in the case of small data this is true, see [NP]. The case of general large data remains open. The limit functions $\bar{u}$ and $\bar{f}$ will again satisfy the same equations (in the sense of distributions). Under our assumptions the function $\bar{u}$ is smooth away from the origin, satisfies $\lambda\bar{u}(\lambda x) = \bar{u}(x)$ for each $\lambda > 0$, and, by Theorem 1, must therefore be a Landau solution or vanish identically. (The direction of the vector b will be the axis of symmetry of the solution.) For $b = 0$ we will have $\bar{u} = 0$, which means that, under the above assumptions, the solution u decays faster than $|x|^{-1}$. After the first draft [Sv1] of this paper was written, a small data result similar to the above conclusions (for small data) was proved by a perturbation analysis by A. Korolev and the author in [KS], without the use of Theorem 1 and [NP].

The situation in dimension $n = 2$ is quite different, and when $\int_{R^2} f \neq 0$, we do note expect existence of solutions to (2) with $u(x) \to 0$ as $x \to \infty$, see for example [Am]. When $\int_{R^2} f = 0$ we can write (under our assumptions) $f = \operatorname{div} F$ for a compactly supported smooth F. We will see in Section 5 that the symmetry $u(x) \to \lambda u(\lambda x), F(x) \to \lambda^2 F(\lambda x)$ leads to a formal possibility of a limit given by $(-1)-$ homogeneous solution. However, this can happen only in exceptional cases, see the discussion in Section 5.

The situation in dimensions $n \geq 4$ is simpler, at least at the formal level. In this case the leading term of the solution at $x \to \infty$ should be given by the linearized equation. For small data this can be indeed established by suitable perturbation arguments, whereas the large data situation has not been much studied, it seems.

4. Proof of Theorem 1.

Let u be a (-1)-homogeneous vector field in R^3, smooth away from the origin. Clearly u is determined by its restriction to the unit sphere $S^2 \subset R^3$. For $x \in S^2$ we decompose $u(x)$ as $u(x) = v(x) + f(x)e(x)$, where $e(x) = x$ is the outer unit normal to S^2, and $v(x)$ is tangent to S^2 at x, i. e. $v(x) \cdot n(x) = 0$. We now write down the Navier-Stokes equations for u and as a system of PDEs on S^2. If u satisfies the Navier-Stokes equation in $R^3 \setminus \{0\}$ in the very weak sense defined above, it is easy to see that there exists a suitable pressure function p in $R^3 \setminus \{0\}$ which is (-2)-homogeneous and smooth away from the origin. The function p is also determined by its values on S^2, and the system (1) can be written down as a system of PDEs on S^2 for v, f and p. The differential operators in what follows will all be differential operators on S^2, defined by the usual conventions of Riemannian geometry. The differential forms on S^2 will be identified with vector fields and vice-versa, as is usual on Riemannian manifolds. The Hodge Laplacian $d\,d^* + d^*d$ on 1-forms will be denoted by $-\Delta_{\rm H}$. (The reason for writing it as $-\Delta_{\rm H}$, with the minus sign, is to keep the equations on S^2 in a form which resembles the standard euclidean form of the equations as much as possible.) The Navier-Stokes equations (1) for u written in terms of v, f and p as equations on S^2 are as follows:

$$\begin{aligned} -\Delta_{\rm H} v + v\nabla v + \nabla(p - 2f) &= 0\,, \\ -\Delta f + v\nabla f - f^2 - |v|^2 - 2p &= 0\,, \\ \operatorname{div} v + f &= 0\,. \end{aligned} \tag{3}$$

A straightforward (although perhaps not the most illuminating) way to derive these equations is to write the system (1) in spherical coordinates (see, for example, [B], p. 601) and check that for (-1)-homogeneous vector fields it reduces to the system (3). We remark that the spherical coordinates version of (1) in the second edition of the book [LL] (p. 49) contains a misprint in the right-hand side of the first equation, where an incorrect expression $\sin^2\theta$ appears instead of the correct $\sin\theta$. For the convenience of the reader we give another derivation of the equations (3) in Appendix 1.

We will denote by ω the function on S^2 given by $dv = \omega\Omega_0$, where Ω_0 is the canonical volume form of S^2. This corresponds to the formula $\omega = \operatorname{curl} v$ used in R^2.

By taking d of the first equation of the system (3) we obtain (see Appendix 2)

$$-\Delta\omega + \operatorname{div}(v\omega) = 0. \tag{4}$$

Lemma 1. With the notation introduced above, we have $\omega \equiv 0$.

Proof. Let L be the differential operator defined by $Lw = -\Delta w + \operatorname{div}(vw)$. The adjoint operator L^* is given by $L^*w = -\Delta w - v\nabla w$. The kernel of L^* consists of constant functions, as can be seen from the strong maximum principle. The kernel of L must therefore also be one dimensional. Let us denote by w_0 a non-trivial function in the kernel of L. If w_0 changed sign on S^2, we could find a strictly positive smooth function h on S^2 with $\int_{S^2} w_0 h = 0$. But this would mean that the equation $L^*w_1 = h$ has a solution. However, the last equation cannot be satisfied at points where w_1 attains its minimum. From this we see that the function ω cannot change sign. At the same time, the definition of ω immediately implies that $\int_{S^2}\omega = 0$, and we see that ω must vanish.

Remark. I assume the above argument is known in one form or another, but I was not able to find a good reference for it.

Once we know that $dv = 0$, the first equation of (3) simplifies. Indeed, when $dv = 0$ we have $-\Delta_{\rm H} v = -\nabla \operatorname{div} v = \nabla f$, and we also have $v\nabla v = \nabla |v|^2/2$. Using this, the first equation of (3) implies

$$\frac{1}{2}|v|^2 + p - f = c\,,$$

where c is a constant. The second equation of (3) now gives

$$-\Delta f - 2f + \operatorname{div}(fv) = 2c\,. \tag{5}$$

Integrating (5) over S^2 and using the third equation of (3) we see that $c = 0$. Since $dv = 0$ we can write $v = \nabla\varphi$ for a suitable smooth function φ on S^2. The equation (5), together with the third equation of (3) and the fact that $c = 0$ now gives

$$\Delta^2\varphi + 2\Delta\varphi - \operatorname{div}(\Delta\varphi\nabla\varphi) = 0\,. \tag{6}$$

Letting $w = 2 - \Delta\varphi$, the last equation can be re-written as

$$-\Delta w + \operatorname{div}(\nabla\varphi w) = 0\,.$$

The solutions of this equation are well-known: They are functions of the form $c_1 e^{\varphi}$, where c_1 is a constant. (An easy way to verify this is for example the following: Write w in the form $c_1(x)e^{\varphi(x)}$. We get an equation for c_1 for which the strong maximum principle implies that the solutions are exactly $c_1(x) \equiv$ const.) Integrating w over the sphere we see that $c_1 > 0$. Hence we have

$$-\Delta\varphi + 2 = c_1 e^{\varphi}$$

for a constant $c_1 > 0$. Changing φ by a constant, if necessary, we can assume $c_1 = 2$ without loss of generality, and we end up with

$$-\Delta\varphi + 2 = 2e^{\varphi}\,. \tag{7}$$

The interpretation of equation (E4) is well-known (see, for example, [CY]): Let $\bar{g}$ be the canonical metric on S^2 and let g be the metric on S^2 defined by $g = e^{\varphi}\bar{g}$. Equation (7) says exactly that the Gauss curvature of the metric g is 1, i. e. the metric g is isometric to the metric $\bar{g}$. In other words, we have $g = h^*\bar{g}$ (pullback of $\bar{g}$ by h) for a suitable diffeomorphism h of S^2. From the definitions we also see that h has to be conformal or anti-conformal. Anti-conformal maps can be obtained from conformal maps by a composition with an isometry, and hence we can only consider the case when h is conformal. For a given conformal h, the function φ is given by

$$\varphi(x) = \log|h'(x)|^2\,, \tag{8}$$

where $h'(x)$ denotes the (complex) derivative of h at x. It is well-known (see e. g. [DFN]) that all conformal diffeomorphisms of S^2 can be produced as follows. Let $P\colon S^2 \to \mathbf{C}$ be the standard stereographic projection, and let $M_\lambda\colon \mathbf{C} \to \mathbf{C}$ be defined by $z \to \lambda z$. Let $h_\lambda = P^{-1} \circ M_\lambda \circ P$. Then any conformal diffeomorphism of S^2 can be produced by composing a suitable h_λ (with $\lambda > 0$) with isometries of S^2. If φ is given by (8) and we compose h with and isometry, then the function φ either does not change or only changes by being shifted by the isometry. Therefore in a suitable coordinate frame all solutions φ of (7) look like the solutions generated by the special h_λ above. We now consider the standard spherical coordinates (θ, ψ) on S^2, given by

$$\begin{aligned} x_1 &= \sin\theta\cos\psi\,,\\ x_2 &= \sin\theta\sin\psi\,,\\ x_3 &= \cos\theta\,. \end{aligned} \tag{9}$$

We will use the usual notation $e_\theta = \frac{\partial x}{\partial\theta}$ for the tangent vector field on S^2 corresponding to $\frac{\partial}{\partial\theta}$. Letting $\lambda = e^{-\kappa}$, calculating the maps h_λ above in these coordinates, and using the formula (8), we obtain

$$\varphi(x) = -2\log(\cosh\kappa - \sinh\kappa\,\cos\theta)\,. \tag{10}$$

This gives

$$\begin{aligned} v &= \frac{\partial\varphi}{\partial\theta}\,e_\theta = \frac{-2\sin\theta}{\coth\kappa - \cos\theta}\,e_\theta\,,\\ f &= -\,\Delta\varphi = 2e^{\varphi} - 2 = \frac{2}{(\cosh\kappa - \sinh\kappa\,\cos\theta)^2} - 2\,, \end{aligned} \tag{11}$$

which agrees with the formulae in [B], p. 207 if we set $\coth\kappa = 1 + c$ and with the formulae in [LL], p. 82, if we set $\coth\kappa = A$. The proof of Theorem 1 is finished.

Remarks:

1. As we already mentioned in Section 1, the Landau solutions (given by (11)) do not satisfy the Navier-Stokes equations (1) across the origin. A calculation in [B], p. 209, shows that for Landau's solutions we have, in distributions,

$$-\Delta u + \operatorname{div}(u \otimes u) + \nabla p = b\delta\,,$$

where δ is the Dirac function and $b = b(\kappa)$ is a non-zero vector in R^3 depending in a non-trivial way on the parameter κ which parametrizes the solutions in the above coordinate frame. The exact formula for b can be found in [B], p. 209, and was also calculated in [CK].

2. If $h\colon S^2 \to S^2$ is a non-trivial holomorphic map (which is not necessarily a diffeomorphisms) the formula (8) gives a function φ which is regular away from a finite set $a_1, \dots, a_m \in S^2$ where h' vanishes. The function φ will generate a (-1)-homogeneous solution of the Navier-Stokes equations in the region $R^3 \setminus (\cup_{j=1}^{j=m} R_+ \cdot a_j)$, where $R_+ = [0, \infty)$. However, the vector field will not be locally square integrable in R^3, except for the case of Landau's solutions, when h' does not vanish at any point.

5. (-1) - homogeneous solutions in dimension $n = 2$.

In dimension $n = 2$ the equations derived in Appendix 1 reduce to the circle S^1. We denote by θ the the natural angle variable on the circle. The unknown function are $v = v(\theta)$, the component of the velocity tangent to the circle, $f = f(\theta)$, the component of the velocity normal to the circle, and the pressure $p = p(\theta)$ on the circle. We will use the notation $f' = \frac{d}{d\theta} f$. The equations are

$$\begin{aligned} (p - 2f)' &= 0\,, \\ -f'' + vf' - f^2 - |v|^2 - 2p &= 0\,, \\ v' &= 0\,. \end{aligned} \tag{12}$$

This means that v has to be constant and $p = 2f + \text{const}$. Since we are looking for solutions on the whole circle, corresponding to the periodic solutions in θ and the term vf' can be interpreted as "damping", we see that v or f' must vanish identically. The solutions corresponding to a non-zero v are therefore the solutions for which all unknown functions f, v, p are constant and the constants satisfy $f^2 + v^2 + 2p = 0$.
In the case of $v = 0$ we obtain a single equation for f

$$f'' = -4f - f^2 + b\,,$$

where b is any constant. This is the equation of motion of a particle in the potential $V(f) = 1/3 f^3 + 2f^2 - bf$, and we are interested in its 2π-periodic solutions. From this interpretation and the form of the potential V it is clear than one has many of such solutions. (The key point is that for large b the potential has a local minimum, and the solutions of the linearization of our equation around this equilibrium oscillate at high frequency. By changing the amplitude of the oscillations we can change the period and adjust it so that the solution is periodic with the smallest period $2\pi/m$ for a positive integer m. Together with the freedom to change b, this gives countably many 1-parameter families of solutions. We refer the reader to [Ha] for the details.)
We will be interested in the solutions which satisfy the additional requirement that

$$\int_{S^1} f = 0\,. \tag{13}$$

In dimensions $n \geq 3$ this condition is satisfied automatically due to the equation $(n-2)f = -\operatorname{div} v$. (This is also reflected by the fact that in dimensions $n \geq 3$ any vector field u in R^n which is div-free in $R^n\{0\}$ and bounded by $c/|x|$ is also div-free in R^n in the sense of distribution. In dimension $n = 2$ this is no longer the case.) Condition (13) comes up naturally in the context of the long-distance behavior of steady-state solutions in the following way. For a matrix field $F = F_{ij}(x)$ in R^n we denote by $\operatorname{div} F$ the vector field $\frac{\partial}{\partial x_j} F_{ij}$. We consider the steady Navier-Stoked equations with the right-hand side in the divergence form:

$$
\begin{aligned}
-\Delta u + u\nabla u + \nabla p &= \operatorname{div} F\,,\\
\operatorname{div} u &= 0\,.
\end{aligned}
\tag{14}
$$

In dimension $n = 2$ the scaling symmetry works for this equation in a way similar to the 3d case with $\operatorname{div} F$ replaced by f, which we dealt with in Section 3: for $\lambda > 0$ the quantities

$$
\begin{aligned}
u_\lambda(x) &= \lambda u(\lambda x)\,,\\
p_\lambda(x) &= \lambda^2 p(\lambda x)\,, \quad \text{and}\\
F_\lambda(x) &= \lambda^2 F(\lambda x)
\end{aligned}
\tag{15}
$$

satisfy again equation (14).
Assume now that $\lim_{\lambda\to\infty} u_\lambda = \bar u$ exists. Then $\bar u$ is (-1)-homogeneous. It obviously satisfies $\int_{S^1} u\cdot\nu = 0$, where ν is the outer unit normal to S^1. In the variables (f, v) above this means that $\int_{S^1} f = 0$. Clearly $\lim_{\lambda\to\infty} F_\lambda = \bar F = M\delta$ where $M = \int_{R^2} F$ is a 2×2 matrix and δ is the Dirac function, and formally one has

$$-\Delta\bar u + \bar u\nabla\bar u + \nabla\bar p = \operatorname{div}\bar F$$

for a suitably defined $\bar p$. For the linear Stokes problem (obtained by dropping the term $u\nabla u$ from the equations) the above procedure works well and the field $\bar u$ gives the leading terms asymptotics at ∞ for the solution.
The solutions of

$$-\Delta\bar U + \nabla\bar P = \operatorname{div}(M\delta)$$

are given by

$$\bar U_i(x) = M_{jk} G_{ijk}(x)\,,$$

where

$$G_{ijk}(x) = \frac{1}{4\pi}\frac{\partial}{\partial x_k}\left(\delta_{ij}\log\frac{1}{|x|} + \frac{x_i x_j}{|x|^2}\right)\,, \tag{16}$$

and these solutions give the leading-order behavior of the solutions of

$$-\Delta U + \nabla P = \operatorname{div} F \tag{17}$$

as $x\to\infty$. One way to calculate the Green function (16) is to solve the linearization of (12), which can be easily done explicitly. For example, the vector field G_{i11} corresponds to $v = 0$, $f(\theta) = \frac{1}{4\pi}\cos(2\theta)$.
In dimension $n = 3$, with $\operatorname{div} F$ replaced by f this procedure works also at the non-linear level, at least for small data, as we have seen in Section 3.
Can this also work for the non-linear problem in dimension $n = 2$? One difficulty is that for (-1)-homogeneous functions the term $\bar u\nabla\bar u$ no longer has an easy distributional interpretation in the open sets containing $x = 0$. Even if we write is as $\operatorname{div} u\otimes u$, it is still not transparently well-defined as a distribution. We can side-step this issue by considering the equations only in $R^2\setminus\{0\}$. The functions $\bar u$ in $R^2\setminus\{0\}$ which are results of the above "blow-up procedure" will still satisfy the equation $\operatorname{div}\bar u = 0$ across the origin, which translates to $\int_{S^1} f = 0$. We see that it important to characterize the solutions of (12) which satisfy the zero flux condition (13). These solutions are characterized in the following theorem.

Theorem 2. *The solutions of (12) satisfying the zero flux conditions (13) are of the following form: Either*

$$f = 0, \quad v = \text{const.}, \quad \textit{and } p = -|v|^2/2,$$

or

$$f(\theta) = \tilde f_k(\theta - \theta_0), \quad v = 0, \quad \textit{and } p = -f/2 + c_k, \quad k = 3, 4, \ldots\,,$$

where for each $k = 3, 4, \ldots$, the function $\tilde f_k$ is a non-trivial periodic function of θ with minimal period $2\pi/k$, the constant c_k is given by $c_k = 1/2\int_{S^1}|\tilde f_k|^2$, and θ_0 can be chosen arbitrarily. The functions $\tilde f_k$ can be expressed in terms of the classical elliptic functions. The amplitude of oscillations of $\tilde f_k$ is of order k^2.

Before going to the proof of the theorem, let us point out an interesting conclusion one can make from it. Let us consider the equation (14) in dimension $n = 2$ with a smooth compactly supported F satisfying $\int_{R^2} F_{11} \neq 0$, and $F_{12} = F_{21} = F_{22} = 0$. (One can say that the force $\operatorname{div} F$ is approximately a dipole in the x_1-direction.) The solution of the linear Stokes system (17) is given by

$$U_i = G_{i11} * F_{11}$$

and its asymptotics as $x \to \infty$ is given by (a multiple of) G_{i11}, modulo terms of order $1/|x|^2$. The field G_{i11} (the solution corresponding to an exact unit "dipole force" in the x_1-direction) corresponds to the solution of the linearization of the system (12) with $f = \cos(2\theta)$.

One can now ask if in the situation when F is small, one has a solutions of the full Navier- Stokes equation with a similar structure. Theorem 2 shows that, somewhat surprisingly, this is not the case: the system (12) does not have any solution which would be close to the solution $f = \cos(2\theta)$, $v = 0$ of the linearized system. Therefore the linear solution $G_{i11} * F_{11}$ cannot be "deformed" into the solution of the full non-linear system which would still be asymptotically (-1)-homogeneous as $x \to \infty$, no matter how small F is, as long as $\int_{R^2} F_{11} \neq 0$. In particular, one cannot obtain solutions of (14) for small F by perturbation techniques in the spaces of functions with decay $O(1/|x|)$ as $x \to \infty$. The failure of the usual perturbation series to converge in the spaces with decay $O(1/|x|)$ can be analyzed in some detail and is interesting by itself. A noteworthy feature of the situation is that the failure does not occur at the level of the "second iterant" (with the first iterant being the linear solution), but only at the level of the third iterant. Some solutions of (14) can be constructed by Leray's method based on solving the problem in large balls B_R by using energy estimates together with some topological arguments (e. g. degree theory), and then letting $R \to \infty$. However, the precise behavior for large x of the solutions obtained in this way seems to be open.

Proof of Theorem 2. As we have already mentioned, the problem without the zero-flux condition (13) has been investigated in some detail in 1917 by G. Hamel, [Ha]. For the proof we will change our notation and instead of $f = f(\theta)$ we will write $u = u(\theta)$ for the radial component. It is clear that the only non-trivial part of the proof is the investigation of the solutions with $v = 0$. This reduces our task to problem of find all non-trivial 2π-periodic solutions of

$$u'' = -4u - u^2 + b \tag{18}$$

with $\int_0^{2\pi} u(\theta)\, d\theta = 0$, where b is an arbitrary real parameter. As above, we interpret the solutions as motions of a particle of unit mass in the potential $V(u) = u^3/3 + 2u - bu$. Therefore we have the usual energy conservation

$$(u')^2 = 2E - 2V(u) \qquad (19)\,.$$

This is a classical equation defining the elliptic functions (see, for example, [Ch]). Following [Ha], we note that the relevant situation for us occures exactly when the polynomial $2E - 2V(u)$ has three real roots $e_1 \geq e_2 \geq e_3$ satisfying

$$e_1 + e_2 + e_3 = -6\,. \tag{20}$$

(Instead of choosing b and E we choose the roots e_i satisfying (20).) One can therefore write

$$u' = \pm\sqrt{\frac{2}{3}(e_1 - u)(u - e_2)(u - e_3)}\,,$$

and our task is to investigate for which choices of the roots we have

$$T = \int_{e_2}^{e_1} \frac{du}{\sqrt{(e_1 - u)(u - e_2)(u - e_3)}} = \sqrt{\frac{2}{3}}\frac{\pi}{k} \quad \text{for some } k = 1, 2, \ldots \tag{21}$$

(which says that u is 2π-periodic) together with

$$I = \int_{e_2}^{e_1} \frac{u\, du}{\sqrt{(e_1 - u)(u - e_2)(u - e_3)}} = 0\,, \tag{22}$$

(which is just another way of stating condition (13).)
Following [Ha], we use the classical change of variables in these elliptic integrals:

$$u = e_2 + (e_1 - e_2)\sin^2\varphi\,,$$

and we also set

$$\kappa = \frac{e_1 - e_2}{e_2 - e_3}, \quad \delta = e_2 - e_3\,.$$

and

$$F(\kappa) = \int_0^{\pi/2} \frac{d\varphi}{\sqrt{1+\kappa\sin^2\varphi}}, \quad E(\kappa) = \int_0^{\pi/2} \sqrt{1+\kappa\sin^2\varphi}\,d\varphi\,.$$

This gives

$$T = \frac{2}{\sqrt{\delta}}F(\kappa) \quad \text{and} \quad \tfrac{\sqrt{\delta}}{2}I = -2 - \tfrac{\delta(2+\kappa)}{3}F(\kappa) + \delta E(\kappa)\,.$$

The functions $F(\kappa)$ and $E(\kappa)$ are variants of the classical complete elliptic integrals of the first and second kind, respectively. (In the classical definition one replaces κ by $-m = -k^2$ and F is denoted by K.) It is easy to check that the triples of roots with $e_1 > e_2 > e_3$ and $e_1 + e_2 + e_3 = -6$ are in one-to-one correspondence with the pairs $\kappa > 0$, $\delta > 0$.
We use the condition $I = 0$ to obtain

$$\frac{2F}{\delta} = E - \frac{2+\kappa}{3}F\,.$$

We we see that $I = 0$ can only be satisfied when $E(\kappa) - \frac{1}{3}(2+\kappa)F(\kappa) \geq 0$ and in that case we have

$$T^2 = \frac{4F^2}{\delta} = 2F(\kappa)\left(E(\kappa) - \frac{2+\kappa}{3}F(\kappa)\right)\,.$$

Let us denote by $H(\kappa)$ the function on the right-hand side. We are interested in the non-negative solutions of the equation

$$H(\kappa) = \frac{2\pi^2}{3k^2}\,, \quad k = 1, 2, \ldots$$

We note that $H(0) = \frac{\pi^2}{6} = \frac{2\pi^2}{3\cdot 2^2}$. The solution $\kappa = 0$ corresponds to the roots $e_1 = e_2 = 0$ and $e_3 = -6$ and the "infinitesimal oscillations" of u around $u = 0$, which is exactly the solution of the linearized equation. Its period is π, as expected. From the definitions of F and E it is also easy to see that $H(\kappa)$ becomes negative for sufficiently large $\kappa > 0$. Hence the proof of the existence of $\tilde{f}_k$ and their uniqueness (modulo the shift by θ_0) will be finished if we show that the derivative $H'(\kappa) = \frac{dH(\kappa)}{d\kappa}$ is strictly negative for $\kappa > 0$.
We will need the classical formulae for the derivatives of E, F

$$\begin{aligned} E' &= \frac{dE}{d\kappa} = \frac{1}{2\kappa}(E - F)\,, \\ F' &= \frac{dF}{d\kappa} = \frac{1}{2\kappa}(\frac{E}{1+\kappa} - F) \end{aligned} \tag{23}$$

together with the inequality

$$1 < \frac{E}{F} < 1 + \frac{\kappa}{2}, \quad \kappa > 0, \qquad (24).$$

see Appendix 3.
We calculate

$$3\kappa(1+\kappa)H' = 3E^2 - 2(2+\kappa)EF + (1+\kappa)F^2 = F^2(3x^2 - 2(2+\kappa)x + (1+\kappa)), \quad x = \frac{E}{F}\,.$$

It is not hard to see that

$$3x^2 - 2(2+\kappa)x + (1+\kappa) < 0, \quad \text{when } \kappa > 0 \text{ and } 1 < x < 1 + \tfrac{\kappa}{2},$$

which shows that $H'(\kappa) < 0$ for $\kappa > 0$.
The amplitude of the oscillation of $\tilde{f}_k$ is

$$e_1 - e_2 = \kappa\delta = \frac{4\kappa F^2}{H} = \frac{6\kappa F^2}{\pi^2} k^2\,,$$

which proves the statement about the amplitude of $\tilde{f}_k$, as for $k \to \infty$ the corresponding values of κ converge to the positive root of the equation $H(\kappa) = 0$. This finishes the proof of Theorem 2.

6. Higher Dimensions.

In this section we show that the system (29) does not have (smooth) solutions in dimensions $n \geq 4$. As we already indicated in Section 2, this result is related to the regularity theory of the steady-state equations. A (-1)-homogeneous solution in dimension $n = 5$ would provide the simplest example of a singular solution with locally finite energy $\int_{B_R} |\nabla u(x)|^2\,dx$. Dimension $n = 5$ is the lowest dimension for which the steady Navier-Stokes is super-critical with respect to the energy $\int_{B_R} |\nabla u(x)|^2\,dx$, in the sense that the classical boot-strapping argument cannot be used to prove regularity. The regularity theory for this case has been studied by Frehse and Růžička, see for example [FR], and by Struwe, see [St1]. The key point of these works is to use the special properties of the quantity $|u|^2/2 + p$. The quantity will also play an important role in the proof Theorem 3 below which is the main result of this section. The theorem follows from the results of Frehse and Růžička, and was also proved by Struwe [St2], and T.-P. Tsai and the author, see [T]. For the convenience of the reader we reproduce below the proof by T.-P. Tsai and the author.

Theorem 3. *When $n \geq 4$, the system (29) has no non-trivial solutions.*

Proof. The key point in the proof is to use a well-known non-trivial identity which is satisfied by the “Bernoulli quantity” $H = |u|^2/2 + p$ for any steady-state Navier-Stokes solutions. Denoting by ω the anti-symmetric part of ∇u, we have

$$-\Delta H + u\nabla H = -2|\omega|^2\,. \tag{25}$$

This identity plays a very important role in the regularity theory for higher-dimensional steady-state Navier-Stokes. For a (-1)-homogeneous solution we will denote, with a slight abuse of notation, by $H, |\omega|^2$, and p also the restriction of these quantities (originally defined in $R^n \setminus \{0\}$) to the sphere S^{n-1}. We recall that we write the restriction of the vector field u to S^{n-1} as $u = v + fe$, where v is tangential to the sphere and e is the normal to the sphere. For the proof of Theorem 3 it is enough to replace the first equation of (29) by the equation (25) expressed in terms of the variables on S^{n-1}. This system is

$$\begin{aligned} -\Delta H + (2n-8)H + v\nabla H - 2fH &= -2|\omega|^2\,, \\ -\Delta f + v\nabla f &= 2H\,, \\ \operatorname{div} v + (n-2)f &= 0\,, \end{aligned} \tag{26}$$

where all the differential operators are now taken on S^{n-1}. When $n = 4$, we can integrate the first equation over the sphere. Integrating by parts and using the third equation we see that the integral of the left-hand side vanishes, and hence ω must vanish identically. When $n > 4$, we let H_+ be the positive part of H and $\alpha = (n-4)/2$. We multiply the first equation by H_+^α and integrate by parts to obtain (with the use of the third equation)

$$\int_{S^{n-1}} \left(\alpha|\nabla H|^2 H_+^{\alpha-1} + (2n-8)H_+^{1+\alpha} + |\omega|^2 H_+^\alpha\right)\,dy = 0\,.$$

This shows that H_+ has to vanish, which means that H is non-negative. Now the second equation of (26) together with the strong maximum principle imply f must be constant, and therefore also H vanishes. Going back to the fist equation we see that we obtain again that ω must vanish identically. We now look again at the (-1)-homogeneous field u defined in $R^n \setminus \{0\}$. Since $\omega = 0$ and $\operatorname{div} u = 0$, we see that u is harmonic in $R^n \setminus \{0\}$, and since $n \geq 4$, the (-1)-homogeneous singularity at $x = 0$ is removable. Hence u vanishes identically.

7. Open problems.

An interesting problem is to try to repeat, the above analysis when $R^n \setminus \{0\}$ is replaced by the half-space $R^n_+ = \{x \in R^n,\, x_n > 0\}$ and the boundary condition $u = 0$ is imposed on $\partial R^n_+ \setminus \{0\}$. The case $n = 2$ is amenable to an ODE analysis, along the lines of Section 5, see also [Ha]. When $n \geq 3$, the problem becomes more difficult, and the following question seems to be open.

For $n \geq 3$, does Theorem 3 remain true for in $R^n_+ \setminus \{0\}$, with the boundary condition $u = 0$ at $\partial R^n_+ \setminus \{0\}$?

If an analogue of Theorem 3 would fail in dimension $n \geq 5$ and a non-trivial solution existed, one would have a genuine example of a boundary singularity for steady-state solutions with (locally) finite energy $\int_{B_r \cap R^n_+} |\nabla u|^2$ (in the corresponding dimension).

In dimension $n = 3$, a relatively simple calculation shows that there are no non-trivial axi-symmetric (-1)-homogeneous solutions in that case. However, it is not clear whether this conclusion is still true without assuming the rotational symmetry. We refer the reader to the very interesting paper [Se], where a related situation is studied in a different context.

Another interesting question is the following:

Among smooth vector fields in $R^3 \setminus \{0\}$ satisfying $|u(x)| \leq C|x|^{-1}$ for some $C > 0$, are the Landau solutions the only ones which satisfy the Navier-Stokes equations (1) in $R^3 \setminus \{0\}$?

Such questions are relevant for the problem of asymptotic behavior of steady-state solutions in exterior domains mentioned in Section 3. A first natural step in addressing this question is to look at possible infinitesimal deformations of Landau solutions in the above class. This leads to linear equations which can be reduced to ODEs by classical methods of separation of variables, due to the symmetries of Landau's solutions. Based on numerical experiments with these ODEs, the author conjectures that the Landau solutions are rigid with respect to infinitesimal deformations, i. e. it seems that there are no new solutions bifurcating from Landau's solutions.

Appendix 1.

In this section collect some formulae which can be used for an alternative derivation of equations (3) and (4). As we mentioned in Section 4, (3) and (4) can be checked by straightforward but tedious calculations in polar coordinates. However, it seems to be useful to have a more illuminating derivation.

Let us consider a (-1)-homogeneous vector field u in R^n which is smooth away from the origin. We will write the coordinates in R^n as $x = (x_1, \dots, x_n)$, and denote $r = |x|$ the distance to the origin. We can write

$$x = ry\,,$$

with $y \in S^{n-1}$, where $S^{n-1} \subset R^n$ is the standard unit sphere. For $y \in S^{n-1}$ we let $e(y) = y \in R^n$ be the outward unit normal. The vector field u can be written as

$$u(x) = \frac{1}{r}\left(v(y) + f(y)e(y)\right)\,,$$

where v is a vector field tangent to the sphere and f is a function on the sphere.

We would like to express the Navier-Stokes equations (1) for u in terms of intrinsic equations on S^{n-1} for the field v the function f and the pressure. It is easy to see that the pressure (which is only given up to a constant) can be chosen so that

$$p(x) = \frac{1}{r^2} p(y)\,.$$

The function $p(y)$ can then be considered as function on S^{n-1}.

If we dealt with the Euler equations

$$\begin{aligned} u\nabla u + \nabla p &= 0\,, \\ \operatorname{div} u &= 0\,. \end{aligned}$$

rather than the Navier-Stokes, the derivation would be straightforward: we would get

$$\begin{aligned} v\nabla v + \nabla p &= 0\,, \\ v\nabla f - f^2 - |v|^2 - 2p &= 0\,, \\ \operatorname{div} v + f &= 0\,, \end{aligned}$$

where all the differential operators are the intrinsic operators on S^{n-1}. For example, $v\nabla v$ is the covariant derivative of v in the direction of v. This calculation follows directly from the definition of the covariant derivative in terms of the "usual derivative" and the orthogonal projection on the tangent space, and it is left to the reader as an easy exercise.

For Navier-Stokes we must include the Laplacian Δu and expressing this term in suitable intrinsic operators on S^{n-1} is more subtle, although such calculation are routine in Differential Geometry.

We will consider the following operators:

$\nabla^{\rm S}$ is the standard differentiation (of R^k-valued functions, $k = 1, 2, \ldots$) on S^{n-1},

$\nabla^{\rm C}$ is the covariant differentiation of the vector fields (or one-forms) on S^{n-1},

Δ is the standard Laplacian on R^n, corresponding to the quadratic form $\int_{R^n} \frac{1}{2}|\nabla X|^2$

$\Delta_{\rm S}$ is the standard Laplacian (on R^k-valued functions, $k = 1, 2, \ldots$) on S^{n-1}, corresonding to the quadratic form $\int_{S^{n-1}} \frac{1}{2}|\nabla^{\rm S} X|^2$

$\Delta_{\rm C}$ is the covariant Laplacian (also called "rough Laplacian") on vector fields or one-forms on S^{n-1}, corresponding to the quadratic form $\int_{S^{n-1}} \frac{1}{2}|\nabla^{\rm C} X|^2$

$\Delta_{\rm H}$ is the Hodge Laplacian on vector fields or one-forms on S^{n-1}, corresponding to the quadratic form $\int_{S^{n-1}} \frac{1}{2}(|dX|^2 + |d^*X|^2)$, where d is the exterior differentiation and d^* its adjoint (essentially the operator div).

Ric is the Ricci curvature tensor on S^{n-1}. Recall that $\text{Ric} = \{R_{ij}\}_{i,j=1}^{(n-1)}$, and $R_{ij} = (n-2)g_{ij}$, where g_{ij} denotes the metric.

We also recall the formula

$$\int_{S^{n-1}} \left(|\nabla^{\rm C} u|^2 + \text{Ric}(u,u)\right) = \int_{S^{n-1}} \left(|du|^2 + |d^*u|^2\right)\,, \tag{27}$$

which follows by integration by parts. This formula implies the identity $-\Delta_{\rm C} + \text{Ric} = -\Delta_{\rm H}$. Taking into account that we are on S^{n-1}, we can write $-\Delta_{\rm C} + (n-2) = -\Delta_{\rm H}$.

We also recall that

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{(n-1)\partial}{r\partial r} + \frac{1}{r^2}\Delta_{\rm S}\,. \tag{28}$$

Therefore, returning to our -1 -homogeneous field $u = \frac{1}{r}(v(y) + f(y)e(y))$, we have

$$\Delta u = \frac{1}{r^3}\left((3-n)(v(y) + f(y)e(y)) + \Delta_{\rm S}(v(y) + f(y)e(y))\right)\,.$$

Here the Laplacian on the right-hand side is the usual sphere Laplacian of the R^n-valued function $v(y)$ on S^{n-1}, i. e. we calculate it "component by component". We need to decompose this expression into the tangent part and the normal part, and write each part in terms of intrinsic operators on on vector-fields/one forms and functions on the sphere. An easy way to do this is to use the corresponding quadratic forms. Let $X(y) = v(y) + f(y)e(y)$ and let us consider the quadratic form corresponding the $\Delta_{\rm S}X$, which is

$$\int_{S^{n-1}} \frac{1}{2}|\nabla^{\rm S} X|^2$$

For a fixed vector b tangent to the sphere we have

$$\nabla^{\rm S}_b X = \nabla^{\rm C}_b v - II(b,v)e + (\nabla^{\rm C}_b f)e + fb\,,$$

where $II(b,v)$ denotes the second fundamental form, which in our case is simply the scalar product (v,b). Evaluating $|\nabla^{\rm S}_b X|^2$ and summing over orthonormal vectors b, we obtain

$$|\nabla^{\rm S} X|^2 = |\nabla^{\rm C} v|^2 + 2f\,{\rm div}\,v + (n-1)|f|^2 + |v|^2 - 2v\nabla^{\rm S} f + |\nabla^{\rm S} f|^2\,.$$

Integrating this identity over the sphere and using (27) we see that

$$\int_{S^{n-1}} |\nabla^{\rm S} X|^2 = \int_{S^{n-1}} \left(|dv|^2 + |d^*v|^2 + (3-n)|v|^2 + |\nabla^{\rm S} f|^2 + (n-1)|f|^2 + 4f\,{\rm div}\,v\right)\,.$$

Taking variations of the form $\delta X = \varphi(y) + \eta(y)e(y)$ with $\varphi(y)$ tangent to the sphere, we see that the tangential part of $-\Delta_{\rm S}X$ is

$$[-\Delta_{\rm S}(v+fe)]_{\rm tangential} = -\Delta_{\rm H} v + (n-3)v - 2\nabla^{\rm S} f$$

and the normal part is

$$[-\Delta_{\rm S}(v+fe)]_{\rm normal} = -\Delta_{\rm S} f + (n-1)f + 2{\rm div}\,v\,.$$

We recall that the continuity equation ${\rm div}\,u = 0$ implies ${\rm div}\,v = -(n-2)f$ and hence we can write

$$[-\Delta_{\rm S}(v+fe)]_{\rm normal} = -\Delta_{\rm S} f + (3-n)f.$$

Using (28) together with

$$[\frac{\partial^2}{\partial r^2} + (n-1)\frac{\partial}{r\partial r}]\;\frac{1}{r} = \frac{(3-n)}{r^3}\,,$$

we arrive at

$$[-\Delta u]_{\rm tangential} = \frac{1}{r^3}(-\Delta_{\rm H} v - 2\nabla^{\rm S} f)\,,$$
$$[-\Delta u]_{\rm normal} = \frac{1}{r^3}(-\Delta_{\rm S} f)\,.$$

Putting this together with the Euler part above, and dropping the indices S and C in $\nabla^{\rm S}, \nabla^{\rm C}, \Delta_{\rm S}$ since all equations are now intrinsic on the sphere and there is no danger of confusion, we see that the Navier-Stokes for u becomes the following system on S^{n-1}:

$$\begin{aligned} -\Delta_{\rm H} v + v\nabla v + \nabla(p-2f) &= 0\,,\\ -\Delta f + v\nabla f - f^2 - |v|^2 - 2p &= 0\,,\\ {\rm div}\,v + (n-2)f &= 0\,. \end{aligned} \tag{29}$$

Appendix 2.

We consider equation (4), which was obtained in Section 4 from the first equation of (3) by applying d. We recall that Ω_0 denotes the volume form on S^2, and that $\omega = {\rm curl}\,v$ is defined by $dv = \omega\Omega_0$. Both v and $v\nabla v$ can be considered as vector fields or one-forms, depending on the context. For a vector field X we will denote by L_X the Lie derivative along X.

The formula which we would like to prove is

$$d(v\nabla v) = \Omega_0 \ \mathrm{div}\,(v\omega)$$

or, equivalently,

$$\mathrm{curl}\,(v\nabla v) = \mathrm{div}\,(v\omega).$$

This equation explains the special behavior of vorticity in two dimension. It can verified by mechanical calculation. However, we prefer a more geometric derivation, which avoids the calculations and gives a better explanation of this identity, even in the flat case. We will use the traditional notation $v^i_{,j}$ and $v_{i,j}$ for covariant differentiation. We have

$$(v\nabla v)_i = v^j v_{i,j} = v^j v_{i,j} + v^j v_{j,i} - v^j v_{j,i} = (L_v v)_i - \frac{1}{2}(v^j v_{j,i} + v_j v^j_{,i}) = (L_v v)_i - \frac{1}{2}(v_j v^j)_{,i}\,.$$

As it is hopefully clear from the context, in the expression $L_v v$ the first v is considered as a vector field, whereas the second v is considered as a one-form. We see that

$$d(v\nabla v) = d(L_v v) = L_v(dv) = L_v(\omega\Omega_0) = (v\nabla\omega + \omega\mathrm{div}\, v)\Omega_0 = \mathrm{div}\,(v\omega)\Omega_0\,.$$

Appendix 3.

Here we derive (23) and (24), for the convenience of the reader. Formulae (23) are classical, see for example [A]. Inequality (24) is hardly new, but we were unable to find it in the literature. We recall the definitions

$$F(\kappa) = \int_0^{\pi/2} \frac{d\varphi}{\sqrt{1+\kappa\sin^2\varphi}}, \quad E(\kappa) = \int_0^{\pi/2} \sqrt{1+\kappa\sin^2\varphi}\, d\varphi\,. \tag{30}$$

The calculation of E' is straightforward:

$$E'(\kappa) = \frac{d}{d\kappa}\int_0^{\pi/2} \sqrt{1+\kappa\sin^2\varphi}\, d\varphi = \int_0^{\pi/2} \frac{\sin^2\varphi\, d\varphi}{2\sqrt{1+\kappa\sin^2\varphi}} = \frac{1}{2\kappa}\int_0^{\pi/2} \frac{1+\kappa\sin^2\varphi - 1}{\sqrt{1+\kappa\sin^2\varphi}}\, d\varphi = \frac{1}{2\kappa}(E-F)\,.$$

For F' we obtain

$$F'(\kappa) = \int_0^{\pi/2} -\frac{\sin^2\varphi\, d\varphi}{2(1+\kappa\sin^2\varphi)^{\frac{3}{2}}} = \int_0^{\pi/2} \frac{-1-\kappa\sin^2\varphi+1}{2\kappa(1+\kappa\sin^2\varphi)^{\frac{3}{2}}}\, d\varphi = -\frac{F}{2\kappa} + \frac{1}{2\kappa}\int_0^{\pi/2} \frac{d\varphi}{(1+\kappa\sin^2\varphi)^{\frac{3}{2}}}\, d\varphi\,.$$

To evaluate the last integral in terms of E, F, we set

$$\frac{1+\kappa}{1+\kappa\sin^2\varphi} = 1+\kappa\sin^2 t\ .$$

We calculate

$$\frac{d\varphi}{(1+\kappa\sin^2\varphi)^{\frac{3}{2}}} = -\frac{1}{1+\kappa}\sqrt{1+\kappa\sin^2 t}\, dt\ ,$$

which gives

$$\int_0^{\pi/2} \frac{d\varphi}{(1+\kappa\sin^2\varphi)^{\frac{3}{2}}}\, d\varphi = \frac{E}{1+\kappa}\,.$$

Hence

$$F' = \frac{1}{2\kappa}(\frac{E}{1+\kappa} - F)\ ,$$

as claimed. (One can also prove the identity by comparing the power series in κ.)

Let us now turn to the proof of inequality (24). We first note that

$$\frac{d}{d\kappa}\,\frac{E}{\sqrt{2+k}}=\frac{E'}{\sqrt{2+\kappa}}-\frac{E}{2(2+\kappa)^{\frac{3}{2}}}=\frac{1}{\sqrt{2+\kappa}}\left(\frac{E-F}{2\kappa}-\frac{E}{2(2+\kappa)}\right)=\frac{1}{\kappa(2+\kappa)^{\frac{3}{2}}}\left(E-(1+\frac{\kappa}{2})F\right)\,.$$

Hence (24) is equivalent to showing that

$$\frac{d}{d\kappa}\left(\frac{E}{\sqrt{2+\kappa}}\right)<0,\qquad \kappa>0\;. \tag{31}$$

In the second integral (30) which defines E we can write $\sin^2\varphi=\frac{1-\cos 2\varphi}{2}$ and set $2\varphi=\theta$ to obtain

$$\frac{E}{\sqrt{2+\kappa}}=\frac{1}{2\sqrt{2}}\int_0^{\pi}\sqrt{\frac{2+\kappa-\kappa\cos\theta}{2+\kappa}}\,d\theta=\frac{1}{2\sqrt{2}}\int_0^{\pi/2}\left(\sqrt{1-\tau\sin\theta}+\sqrt{1+\tau\sin\theta}\right)\,d\theta\,,$$

where

$$\tau=\frac{\kappa}{2+\kappa}\,.$$

We have

$$\frac{d}{d\tau}\int_0^{\pi/2}\left(\sqrt{1-\tau\sin\theta}+\sqrt{1+\tau\sin\theta}\right)\,d\theta=\frac{1}{2}\int_0^{\pi/2}\left(-\frac{\sin\theta}{\sqrt{1-\tau\sin\theta}}+\frac{\sin\theta}{\sqrt{1+\tau\sin\theta}}\right)\,d\theta\,.$$

As the last integral is obviously strictly negative for $0<\tau<1$ and τ is strictly increasing in κ, we have established (31), and hence (24) is proved.

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