Square function and Riesz transform in non-integer dimensions

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Abstract

Following a recent paper [10] we show that the finiteness of square function associated with the Riesz transforms with respect to Hausdorff measure H^s implies that s is integer.

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1. Introduction

For a Borel mesure μ in $\mathbb{R}^m$ and $s \in (0, m]$ the s-Riesz transform of μ is defined as

$$R^s\mu(x) := \int \frac{x-y}{|x-y|^{s+1}}\, d\mu(y), \qquad x \notin \operatorname{supp}\mu, \tag{1}$$

and the truncated Riesz transform is given by

$$R^s_\varepsilon\,\mu(x) := \int_{|x-y|>\varepsilon} \frac{x-y}{|x-y|^{s+1}}\, d\mu(y), \quad R^s_{\varepsilon,\eta}\,\mu(x) := R^s_\eta\,\mu(x) - R^s_\varepsilon\,\mu(x), \tag{2}$$

where $x \in \mathbb{R}^m$, $\eta > \varepsilon > 0$.

Further, recall that the upper and lower s-dimensional densities of μ at x are given by

$$\theta^{s,*}_\mu(x) := \limsup_{r\to 0} \frac{\mu(B(x,r))}{r^s} \quad \text{and} \quad \theta^s_{\mu,*}(x) := \liminf_{r\to 0} \frac{\mu(B(x,r))}{r^s}, \tag{3}$$

respectively, where $B(x,r)$ is the ball of radius $r > 0$ centered at $x \in \mathbb{R}^m$.

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Preprint submitted to *May 4, 2009*

It has been proved in [6] and [7] that whenever $0 \le s \le 1$ and μ is a finite Radon measure with $0 < \theta^{s,*}_\mu(x) < \infty$, for $\mu -$ a.e. $x \in \mathbb{R}^m$, the condition

$$\sup_{\varepsilon>0} |R^s_\varepsilon \mu(x)| < \infty \qquad \mu - \text{a.e.}\, x \in \mathbb{R}^m, \tag{4}$$

implies that $s \in \mathbb{Z}$. Moreover, an analogous result has been obtained in [11] for all $0 \le s \le m$ under a stronger assumption that $0 < \theta^s_{\mu,*}(x) \le \theta^{s,*}_\mu(x) < \infty$. However, neither the curvature methods of [6], [7], nor the tangent measure techniques in [11] could be applied to establish that (4) implies $s \in \mathbb{Z}$ for all $0 \le s \le m$ assuming only $0 < \theta^{s,*}_\mu(x) < \infty$.

In [10] the authors proved that the latter statement holds if the condition (4) is substituted by the existence of the principal value $\lim_{\varepsilon\to 0} R^s_\varepsilon \mu(x)$, $\mu -$ a.e. $x \in \mathbb{R}^m$. In the present work we refine the techniques of [10] and establish the following result.

Theorem 1.1 *Let μ be a finite Radon measure in $\mathbb{R}^m$ with*

$$0 < \theta^{s,*}_\mu(x) < \infty \quad \text{for} \quad \mu - \text{a.e.}\, x \in \mathbb{R}^m. \tag{5}$$

Furthermore, assume that for some $s \in (0, m]$ the square function

$$S^s\mu(x) := \left(\int_0^\infty \left| R^s_{t,2t}\, \mu(x) \right|^2 \frac{dt}{t} \right)^{1/2} < \infty, \qquad \mu - \text{a.e.}\, x \in \mathbb{R}^m. \tag{6}$$

Then $s \in \mathbb{Z}$.

In fact, we also prove the following closely related result which is a strengthening of the main results in [10].

Theorem 1.2 *Let μ be a finite Radon measure in $\mathbb{R}^m$ satisfying* (5). *Furthermore, assume that for some $s \in (0, m]$ we have*

$$\lim_{\varepsilon\to 0} R^s_{\varepsilon,2\varepsilon}\, \mu(x) = 0 \qquad \mu - \text{a.e.}\, x \in \mathbb{R}^m. \tag{7}$$

Then $s \in \mathbb{Z}$.

This circle of problems goes back, in particular, to the work of David and Semmes [1], [2], where the authors showed, under certain assumptions on the measure μ, that the L^2 boundedness of a large class of singular integral operators implies that s is an integer and μ is uniformly rectifiable, that is, the support of μ contains "large pieces of Lipschitz graphs" – see [1], [2] for details. The ultimate goal, which seems to be out of reach at the moment, is to prove that a similar conclusion holds purely under the assumption that the Riesz transform is bounded in L^2, i.e., that the Riesz transform alone encodes the geometric information about the underlying measure. The achievements in [5], [12], [4] showed that the L^2-boundedness of the Riesz transform, suitably interpreted, is almost equivalent to the condition (4). However, under the assumption (4) the problem seems to be just as challenging. In both cases the question has only been resolved for $s = 1$ ([3], [8], [9]), by the methods involving curvature of measures.

In this vein, we would like to point out that by Khinchin's inequality (6) can be viewed *almost* as a condition

$$\mathbb{E} \left| \sum_{k\in\mathbb{Z}} \varepsilon_k R^s_{2^{-k},2^{-k+1}}\, \mu(x) \right| < \infty \qquad \mu - \text{a.e.}\, x \in \mathbb{R}^m, \tag{8}$$

where ε_k are independent random variables taking the values -1 and 1 with probability $1/2$ each. Therefore, in order to guarantee $s \in \mathbb{Z}$, it is sufficient to assume only that the singular integrals of the type $\sum_{k=0}^\infty \varepsilon_k R^s_{2^{-k},2^{-k+1}}\, \mu(x)$ are uniformly bounded.

Finally, the s dimensional Hausdorff measure H^s of a set E with $0 < H^s(E) < \infty$ satisfies the condition (5), and hence, the results of Theorems 1.1 and 1.2 remain valid in this context.

2. Preliminary estimates

Our proof largely relies on the estimates for a slightly modified version of the Riesz transform that were obtained in [10]. To be precise, let us consider the operator

$$R_\varepsilon^{s,\varphi}\,\mu(x) := \int \varphi\left(\frac{|x-y|^2}{\varepsilon^2}\right) \frac{x-y}{|x-y|^{s+1}}\,d\mu(y), \qquad \varepsilon > 0, \tag{9}$$

where $\varphi = \varphi_\rho$ is a C^2 function depending on the parameter $\rho \in (0, 1/2)$, to be determined below, with $\operatorname{supp}\varphi \subset [0, 1+\rho+2\rho^2]$ and such that

(i) $\varphi(r) = r^{\frac{s+1}{2}}$ for $0 \le r \le 1$, $\qquad \varphi(r) = -\frac{r}{\rho} + 1 + \rho + \frac{1}{\rho}$ for $1+\rho^2 \le r \le 1+\rho^2+\rho$,

(ii) $|\varphi(r)| \le C$, $|\varphi'(r)| \le 1/\rho$, $|\varphi''(r)| \le C(\rho)$ for all $r > 0$.

Analogously to [10], we start with a set

$$\begin{aligned} F_\delta := \{x \in \mathbb{R}^m :\ & \mu(B(x,r))/r^s \le 2\theta_\mu^{s,*}(x) \ \text{ for } \ r \le r_0, \quad \theta_\mu^{s,*}(x) \le C_0, \\ & \text{and } |R_\varepsilon^{s,\varphi}\,\mu(x) - R_{2\varepsilon}^{s,\varphi}\,\mu(x)| \le \delta \ \text{ for all } \ 0 < \varepsilon < \varepsilon_0\}, \end{aligned} \tag{10}$$

where $0 < \delta < 1$ and C_0, r_0, ε_0 are some positive constants.

One can see that both the condition (6) and (7) imply that

$$\lim_{\varepsilon \to 0} |R_\varepsilon^{s,\varphi}\,\mu(x) - R_{2\varepsilon}^{s,\varphi}\,\mu(x)| = 0 \qquad \mu - \text{a.e.}\, x \in \mathbb{R}^n. \tag{11}$$

Indeed,

$$R_\varepsilon^{s,\varphi}\,\mu(x) = \int \int_{0<t<\frac{|x-y|^2}{\varepsilon^2}} \varphi'(t)\,dt\, \frac{x-y}{|x-y|^{s+1}}\,d\mu(y) = \int_0^{1+\rho+2\rho^2} \varphi'(t) R^s_{\varepsilon\sqrt{t}}\,\mu(x)\,dt, \tag{12}$$

so that (7) directly gives (11). Furthermore, (12) entails that

$$|R_\varepsilon^{s,\varphi}\,\mu(x) - R_{2\varepsilon}^{s,\varphi}\,\mu(x)| \le C(\rho) \int_0^{1+\rho+2\rho^2} |R^s_{\varepsilon\sqrt{t},2\varepsilon\sqrt{t}}\,\mu(x)|\,dt \le C(\rho) \left(\int_0^{\varepsilon\sqrt{1+\rho+2\rho^2}} |R^s_{u,2u}\,\mu(x)|^2\, \frac{du}{u}\right)^{1/2},$$

where we used the change of variables $u := \varepsilon\sqrt{t}$ and Hölder's inequality. Hence, (6) also leads to (11).

Therefore, for sufficiently small ε_0 and r_0 and sufficiently large C_0 the set F_δ has $\mu(F_\delta) > 0$. Note that $\mu(B(x,r)) \le M r^s$ for all $x \in F_\delta$, $r > 0$ and $M = \max\{2C_0, \mu(\mathbb{R}^m)/r_0^s\}$.

Let $\theta^s(x,r)$ denote the average s-dimensional density of the ball $B(x,r)$, $x \in \mathbb{R}^m$, $r > 0$, that is, $\theta^s(x,r) := \mu(B(x,r))/r^s$. We start with the following estimates.

Proposition 2.1 *[10] Assume that for some $C' > 0$, $r > 0$ and $x_0 \in \mathbb{R}^m$ we have $\mu(B(x_0,r)) \ge C' r^s$, and denote by n the biggest integer strictly smaller than s. Then for a sufficiently small ρ (depending on s only) and any $\tau \in (0, 1/20)$ there exists a constant $\omega_0 = C(C', M, \rho)\tau^{-s\frac{1}{\log_4(1+\rho^2/4)}}$, there exists $\varepsilon \in \left(\frac{r}{\tau}, \omega_0 \frac{r}{\tau}\right]$ and a set of points $y_0, ..., y_{n+1} \in B(x_0,r) \cap F_\delta$ such that*

$$\theta^s(y_0, 4\varepsilon) \le C(\rho)\,\theta^s(y_0,\varepsilon), \qquad \theta^s(y_0,\varepsilon) \ge C'\tau^s/2, \tag{13}$$

and

$$\sum_{j=1}^{n+1} |R_\varepsilon^{s,\varphi}\,\mu(y_j) - R_\varepsilon^{s,\varphi}\,\mu(y_0)| + \theta^s(y_0, 3\varepsilon)\frac{r^2}{\varepsilon^2} \ge C(C', M, s)(n+1-s)\, r\, \frac{\theta^s(y_0,\varepsilon)}{\varepsilon}. \tag{14}$$

3. The proof of the main result

We will argue by contradiction. We initially assume that $s \notin \mathbb{Z}$, and then show that the estimate in (14) is accompanied by the corresponding bound from above in terms of δ, r, τ and ε. Ultimately, choosing r, δ, τ sufficiently small leads to a contradiction. Observe that in the case $s \in \mathbb{Z}$ the lower bound in (14) is degenerate, and hence, such an argument could not be constructed.

Set

$$\delta := \frac{\tau^{s+2}}{\omega_0} = C(M,\rho)\, \tau^{s+2+s\,\frac{1}{\log_4(1+\rho^2/4)}}, \tag{15}$$

where the constant $C(M,\rho)$ is equal to the reciprocal of $C(C', M, \rho)$ from the definition of ω_0 corresponding to $C' = 1/2$. Note that M depends on r_0 and C_0 in the definition of F_δ, however, the choice of r_0 and C_0 is determined solely by the properties of μ and can be made independent of δ.

Going further, fix ε_0 and take

$$r < \varepsilon_0 \delta = \varepsilon_0\, \frac{\tau^{s+2}}{\omega_0} = C(M,\rho)\, \varepsilon_0\, \tau^{s+2+s\,\frac{1}{\log_4(1+\rho^2/4)}} \quad \text{such that} \quad \mu(B(x_0,r)\cap F_\delta) \geq r^s/2. \tag{16}$$

Now that r and δ are fixed, we invoke Proposition 2.1, find the points $y_0, \ldots, y_{n+1}$ and choose $\varepsilon \in \left(\frac{r}{\tau}, \omega_0\, \frac{r}{\tau}\right]$ such that (13) and (14) are satisfied. However, for every $x, z \in B(x_0,r)\cap F_\delta$, $x \in \mathbb{R}^m$ we have

$$|R^{s,\varphi}_\varepsilon\, \mu(x) - R^{s,\varphi}_\varepsilon\, \mu(z)| \leq C(\rho) M\delta + C\delta \log \frac{r}{\delta\varepsilon}, \tag{17}$$

whenever $2\varepsilon < \frac{r}{\delta} < \varepsilon_0$. Indeed, a direct calculation shows that for $\eta \in \left[\frac{r}{2\delta}, \frac{r}{\delta}\right]$

$$|R^{s,\varphi}_\eta\, \mu(x) - R^{s,\varphi}_\eta\, \mu(z)| \leq C(\rho)\, (r/\delta)^{-s-1}\ |z-x|\, \mu(B(x_0, 4r/\delta)) \leq C(\rho) M\delta. \tag{18}$$

Then we can choose $\eta \in \left[\frac{r}{2\delta}, \frac{r}{\delta}\right]$ such that $\eta = 2^k\varepsilon$ for some $k \in \mathbb{N}$, so that

$$\begin{aligned}
&|R^{s,\varphi}_\varepsilon\, \mu(x) - R^{s,\varphi}_\varepsilon\, \mu(z)| \\
&\quad \leq |R^{s,\varphi}_\varepsilon\, \mu(x) - R^{s,\varphi}_\eta\, \mu(x)| + |R^{s,\varphi}_\eta\, \mu(x) - R^{s,\varphi}_\eta\, \mu(z)| + |R^{s,\varphi}_\eta\, \mu(z) - R^{s,\varphi}_\varepsilon\, \mu(z)| \\
&\quad \leq C(\rho) M\delta + C \sup_{x\in F_\delta}\ \sup_{1\leq i\leq k} |R^{s,\varphi}_{2^i\varepsilon}\, \mu(x) - R^{s,\varphi}_{2^{i-1}\varepsilon}\, \mu(x)| \log \frac{r}{\delta\varepsilon} \leq C(\rho) M\delta + C\delta \log \frac{r}{\delta\varepsilon}.
\end{aligned} \tag{19}$$

Therefore, (14) is complemented by the estimate

$$\sum_{j=1}^{n+1} |R^{s,\varphi}_\varepsilon\, \mu(y_j) - R^{s,\varphi}_\varepsilon\, \mu(y_0)| + \theta^s(y_0, 3\varepsilon)\frac{r^2}{\varepsilon^2} \leq C(M,\rho)\left(\delta + \delta \log \frac{r}{\varepsilon\delta} + \theta^s(y_0,\varepsilon)\frac{r^2}{\varepsilon^2}\right), \tag{20}$$

where we used (17) and (13). Now combining (14) with (20) and dividing both sides by r/ε we arrive at the estimate

$$\theta^s(y_0,\varepsilon) \leq C(M,s,\rho)\left(\frac{\delta\varepsilon}{r} + \frac{\delta\varepsilon}{r}\log\frac{r}{\varepsilon\delta} + \theta^s(y_0,\varepsilon)\frac{r}{\varepsilon}\right). \tag{21}$$

According to our choice of δ and r,

$$\frac{\delta\varepsilon}{r} \leq \frac{\tau^{s+2}}{\omega_0}\,\frac{\omega_0}{\tau} = \tau^{s+1} \leq C\tau\, \theta^s(y_0,\varepsilon) \quad \text{and} \quad \frac{r}{\varepsilon} \leq \tau. \tag{22}$$

Now (21) and (22) give the bound

$$\theta^s(y_0,\varepsilon) \leq C(M,s,\rho)\left(\tau + \tau^{1-\alpha}\right)\theta^s(y_0,\varepsilon), \qquad \forall\, \alpha > 0, \tag{23}$$

which for $\tau > 0$ sufficiently small leads to a contradiction. □

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