

SECOND ORDER ELLIPTIC OPERATORS WITH COMPLEX BOUNDED MEASURABLE COEFFICIENTS IN L^p , SOBOLEV AND HARDY SPACES

STEVE HOFMANN, SVITLANA MAYBORODA AND ALAN MCINTOSH

ABSTRACT. Let L be a second order divergence form elliptic operator with complex bounded measurable coefficients. The operators arising in connection with L , such as the heat semigroup and Riesz transform, are not, in general, of Calderón-Zygmund type and exhibit behavior different from their counterparts built upon the Laplacian. The current paper aims at a thorough description of the properties of such operators in L^p , Sobolev, and some new Hardy spaces naturally associated to L .

First, we show that the known ranges of boundedness in L^p for the heat semigroup and Riesz transform of L , are sharp. In particular, the heat semigroup e^{-tL} need not be bounded in L^p if $p \notin [2n/(n+2), 2n/(n-2)]$. Then we provide a complete description of *all* Sobolev spaces in which L admits a bounded functional calculus, in particular, where e^{-tL} is bounded.

Secondly, we develop a comprehensive theory of Hardy and Lipschitz spaces associated to L , that serves the range of p beyond $[2n/(n+2), 2n/(n-2)]$. It includes, in particular, characterizations by the sharp maximal function and the Riesz transform (for certain ranges of p), as well as the molecular decomposition and duality and interpolation theorems.

CONTENTS

1. Introduction	3
1.1. Definition of H_L^p	5
1.2. Riesz Transform characterization of H_L^p	6
1.3. The Dual of H_L^p , $0 < p \leq 1$	8
1.4. The Dual of H_L^p , $1 < p < 2$	9
1.5. Sobolev spaces and fractional powers of L	10
2. The heat semigroup and functions of L in L^p .	13
2.1. Definitions and L^2 theory	13
2.2. L^p boundedness of the heat semigroup: sharp results	14
2.3. Off-diagonal estimates and $L^p - L^q$ bounds	16
3. Molecular decomposition and duality, $0 < p \leq 1$.	19
4. Square function characterizations and interpolation.	31
5. Riesz transform characterization of Hardy spaces.	37

The first and second authors were partially supported by NSF grants DMS 0801079 and 0758500, respectively. The third author was supported by the Australian Government through the Australian Research Council.

6. Sharp maximal function characterization.	46
7. Fractional powers of the operator L .	51
8. Functional calculus and fractional powers of L in smoothness spaces.	56
8.1. Functional calculus and fractional powers of L in H_L^p - BMO_L - Λ_L^α spaces	56
8.2. Classical scales of function spaces measuring smoothness	57
8.3. Weighted tent spaces	58
8.4. Hardy-Sobolev spaces associated to L : general theory	59
8.5. Hardy-Sobolev spaces associated to L : identifications with classical scales	61
8.6. Functional calculus and fractional powers of L in Sobolev and regular Hardy spaces	62
9. Appendix 1: Relationships between H_L^p and classical H^p	66
10. Appendix 2: Embedding of $H_L^p(\mathbb{R}^n)$ spaces into an ambient Banach space	72
References	74

1. INTRODUCTION

Let A be an $n \times n$ matrix with entries

$$(1.1) \quad a_{jk} : \mathbb{R}^n \longrightarrow \mathbb{C}, \quad j = 1, \dots, n, \quad k = 1, \dots, n,$$

satisfying the ellipticity condition

$$(1.2) \quad \lambda|\xi|^2 \leq \Re A\xi \cdot \bar{\xi} \quad \text{and} \quad |A\xi \cdot \bar{\zeta}| \leq \Lambda|\xi||\zeta|, \quad \forall \xi, \zeta \in \mathbb{C}^n,$$

for some constants $0 < \lambda \leq \Lambda < \infty$. For such matrices A , our aim in this paper is to present a detailed investigation of Hardy spaces and their duals associated to the second order divergence form operator

$$(1.3) \quad Lf := -\operatorname{div}(A\nabla f),$$

which we interpret in the usual weak sense via a sesquilinear form.

In the case that A is the $n \times n$ identity matrix (i.e., so that L is the usual Laplacian $\Delta := -\operatorname{div} \cdot \nabla$), this theory reduces to the classical Hardy space theory of Stein-Weiss [56] and Fefferman-Stein [31]. For more general operators L whose heat kernel satisfies a pointwise Gaussian upper bound, an adapted Hardy space theory has been introduced by Auscher, Duong and McIntosh [2], and by Duong and Yan, [25], [26]. In the absence of such pointwise kernel bounds, the theory has been developed more recently in [10] by Auscher, McIntosh and Russ (when L is the Laplace-Beltrami operator on a manifold with doubling measure), and in [39] by the first two authors of the present paper, for the complex divergence form elliptic operators considered here. In [10, 39], the pointwise Gaussian bounds are replaced by the weaker ‘‘Gaffney estimates’’ (cf. (2.21) and (2.24) below), whose L^2 version is a refined parabolic ‘‘Caccioppoli’’ inequality which may also be proved via integration by parts using only ellipticity and the divergence form structure of L . The present paper may be viewed in part as a sequel to [39], in which we extend results for the case $p = 1$ given there, to the case of general p (although we also obtain here some results, pertaining to the characterization of adapted Hardy spaces via Riesz transforms, that are new even in the case $p = 1$). In particular, it is in the nature of our present setting, in which pointwise kernel bounds may fail, that the Hardy space theory for $p > 1$ becomes non-trivial (i.e., the L -adapted H^p spaces may be strictly smaller than L^p , even when $p > 1$). We shall return to this point momentarily. We note also that general non-negative self-adjoint operators satisfying an L^2 Gaffney estimate have recently been treated in [37].

We now proceed to discuss some relevant history, and to present a more detailed overview of the paper. In [9], the authors solved a long-standing conjecture, known as the Kato problem, by identifying the domain of the square root of L . More precisely, they showed that the domain of \sqrt{L} is the Sobolev space $W^{1,2}(\mathbb{R}^n) = \{f \in L^2 : \nabla f \in L^2\}$ with

$$(1.4) \quad \|\sqrt{L}f\|_{L^2(\mathbb{R}^n)} \approx \|\nabla f\|_{L^2(\mathbb{R}^n)},$$

In particular, the Riesz transform $\nabla L^{-1/2}$ is bounded in $L^2(\mathbb{R}^n)$.

Since then, substantial progress has been made in the development of the L^p theory of elliptic operators of the type described above. Let us define

$$p_-(L) := \inf\{p : \nabla L^{-1/2} : L^p(\mathbb{R}^n) \longrightarrow L^p(\mathbb{R}^n)\}.$$

It is now known that $1 \leq p_-(L) < 2n/(n+2)$ (with $1 < p_-(L)$ for some L ; we shall return to the latter point momentarily), and that there exists $\varepsilon(L) > 0$ such that

$$(1.5) \quad \nabla L^{-1/2} : L^p(\mathbb{R}^n) \longrightarrow L^p(\mathbb{R}^n) \iff p_-(L) < p < 2 + \varepsilon(L),$$

(given (1.4) as a starting point, (1.5) with $p_-(L) < 2n/(n+2)$ is established by combining the results and methods of [38] or [16] with those of [7]; see also [6], [12], [13], [17] for related theory). Moreover, again given (1.4) as a starting point, one has the reverse inequality

$$(1.6) \quad \|\sqrt{L}f\|_{L^p(\mathbb{R}^n)} \lesssim \|\nabla f\|_{L^p(\mathbb{R}^n)}, \quad \text{for } (p_-(L))_* < p < (p_-(L^*))',$$

where in general $p_* := pn/(p+n)$ denotes the ‘‘lower’’ Sobolev exponent, and as usual $p' := p/(p-1)$ is the exponent dual to p . The case $p < 2$ of (1.6) is due to Auscher [7], while the case $p > 2$ is simply dual to the adjoint version of (1.5). Combining (1.5) and (1.6), we have that

$$(1.7) \quad \|\sqrt{L}f\|_{L^p(\mathbb{R}^n)} \approx \|\nabla f\|_{L^p(\mathbb{R}^n)} \iff p_-(L) < p < 2 + \varepsilon.$$

One of the main goals of the present paper is to understand the sense in which (1.7) extends to the range $p \leq p_-(L)$. This extension may be viewed as solving the Kato problem below the critical exponent $p_-(L)$. We discuss this question in more detail in subsection 1.2 below; the proofs are given in Section 5 (cf. Theorem 5.2).

Let us now discuss optimality of the range of p in (1.5) (hence also that in (1.7)), for the entire class of L under consideration. Even in the case of real symmetric coefficients, the upper bound cannot be improved, in general: for each $p > 2$, Kenig¹ has constructed an operator L whose Riesz transform is not bounded in L^p . In addition, the counterexamples in [49], [8], [24] showed that for some elliptic operator L satisfying (1.1)–(1.3) there is a $p \in (1, 2)$ such that the Riesz transform is not bounded in L^p ; i.e., for such L , one has $p_-(L) > 1$. Moreover, the latter fact permeates all the L^p results in the theory: as shown in [7], $p_-(L)$ is also the lower bound for the respective intervals of p for which the heat semigroup and the L -adapted square function (cf. (1.10) below) are L^p bounded, and for which the semigroup enjoys $L^p \rightarrow L^2$ off diagonal estimates. However, identification of the sharp lower bound $p_-(L)$ remained an open problem (posed, along with related questions, in [7], Conjecture 4.14, and in [5], Problem 1.4, Problem 1.5, Problem 1.13).

In Section 2 of the present paper, we observe that the example constructed by Frehse in [33] may be used to resolve these remaining sharpness issues, i.e., to show that $p_\pm(L) = 2n/(n \mp 2) \pm \varepsilon_\pm(L)$, where $(p_-(L), p_+(L))$ is the interior of the interval of L^p boundedness of the heat semigroup e^{-tL} , $t > 0$. More precisely, we have

$$(1.8) \quad \forall p \notin [2n/(n+2), 2], \exists L \text{ with } \nabla L^{-1/2} : L^p(\mathbb{R}^n) \not\rightarrow L^p(\mathbb{R}^n),$$

¹Kenig’s example is described in [13].

$$(1.9) \quad \forall p \notin [2n/(n+2), 2n/(n-2)], \exists L \text{ with } e^{-tL} : L^p(\mathbb{R}^n) \not\rightarrow L^p(\mathbb{R}^n).$$

It follows, in particular, that in dimensions $n \geq 3$, the kernel of the heat semigroup may fail to satisfy the pointwise Gaussian estimate

$$|K_t(x, y)| \leq Ct^{-n/2} e^{-c|x-y|^2/t}, \quad t > 0 \text{ and } x, y \in \mathbb{R}^n.$$

This solves an open problem in [13], p. 33.

Thus, in dimensions $n > 2$, the Riesz transform may fail to be bounded in L^p for some $p \in (1, 2)$, as may the heat semigroup e^{-tL} , $t > 0$, as well as the other natural operators associated with such L (e.g., square function, non-tangential maximal function). Consequently, in the case that the endpoint $p_-(L) > 1$, the L -adapted Riesz transforms, semigroup and square function cannot be bounded from the classical Hardy space H^1 into L^1 , since interpolation with the known L^2 bound would then produce a contradiction with (1.8), (1.9) (or with the analogous statement for the square function). These operators therefore lie beyond the scope of the Calderón-Zygmund theory and exhibit behavior different to their counterparts built upon the Laplacian.

By analogy to the classical theory then, this motivates the introduction of a family of L -adapted Hardy spaces H_L^p , **not** equal to L^p in the range $p \leq p_-(L)$, on which the L -adapted semigroup, Riesz transforms and square function are well behaved, and which comprise a complex interpolation scale including L^p for $p_-(L) < p < p_+(L)$. We note that the endpoint $p_-(L)$ plays a similar role to the exponent $p = 1$ in the classical theory.

In particular, in Section 5 we give a suitable Hardy space extension of (1.5) to the case $p \leq p_-(L)$ (the case $p = 1$ already appeared in [39]), and, in one of the main results of this paper, we present an appropriate converse, thus obtaining a Riesz transform characterization of L -adapted H^p spaces, for some range of p depending on n . As observed above, this characterization may be viewed as a sharp extension of the Kato square root estimate (1.4), and of its L^p version (1.7), to the endpoint $p_-(L)$ and below. In order to make these notions precise, we should first define our adapted H_L^p spaces.

1.1. Definition of H_L^p . The first step in the development of an L -adapted Hardy space theory, in the case that pointwise kernel bounds may fail², was taken in [39] (and independently in [10]), in which the authors considered the model case of $H_L^1(\mathbb{R}^n)$ and, on the dual side, the appropriate analogue of the space BMO . The definition of H_L^1 given in [39]³ (by means of an L -adapted square function) can be extended immediately to $0 < p \leq 2$ and with some additional care to $2 \leq p < \infty$ as well. To this end, consider the square function associated with the heat semigroup generated by L

$$(1.10) \quad Sf(x) = \left(\iint_{\Gamma(x)} |t^2 L e^{-t^2 L} f(y)|^2 \frac{dy dt}{t^{n+1}} \right)^{1/2}, \quad x \in \mathbb{R}^n,$$

²In the presence of **pointwise** Gaussian heat kernel bounds, an L -adapted H^1 and BMO theory was previously introduced by Duong and Yan [25], [26].

³and in [10] for H_L^p , $p \geq 1$.

where, as usual, $\Gamma(x) = \{(y, t) \in \mathbb{R}^n \times (0, \infty) : |x - y| < t\}$ is a non-tangential cone with vertex at $x \in \mathbb{R}^n$. Analogously to [39], we define the space $H_L^p(\mathbb{R}^n)$ for $0 < p \leq 2$ as the completion of $\{f \in L^2(\mathbb{R}^n) : Sf \in L^p(\mathbb{R}^n)\}$ in the norm

$$(1.11) \quad \|f\|_{H_L^p(\mathbb{R}^n)} := \|Sf\|_{L^p(\mathbb{R}^n)}.$$

For $2 < p < \infty$ we assign

$$(1.12) \quad H_L^p(\mathbb{R}^n) := \left(H_{L^*}^{p'}(\mathbb{R}^n)\right)^*,$$

where $1/p + 1/p' = 1$ and L^* is the adjoint operator to L . These spaces also have an appropriate square function characterization as will be discussed in Section 4.

1.2. Riesz Transform characterization of H_L^p . We shall show in Section 5 that the Riesz transforms are bounded from H_L^p into L^p , $0 < p < 2 + \varepsilon(L)$, and even into classical H^p , $n/(n+1) < p \leq 1$. Conversely, for some restricted range of p , we show that these estimates are reversible, thus obtaining a Riesz transform characterization of the corresponding H_L^p . Let us describe these results in more detail.

As preliminary steps, we establish two results that are also of independent interest: in Section 3, we shall obtain a molecular decomposition of H_L^p spaces, $0 < p \leq 1$, analogous to the classical atomic decompositions of Coifman [20] and Latter [46] and in Section 4, we observe that the spaces H_L^p form a complex interpolation scale, including L^p in the range $p_-(L) < p < p_+(L)$ (see (1.15)). As in the classical case, we are then able to use these fundamental properties of Hardy spaces to prove in Section 5 that

$$(1.13) \quad \nabla L^{-1/2} : H_L^p(\mathbb{R}^n) \rightarrow L^p(\mathbb{R}^n), \quad 0 < p < 2 + \varepsilon(L),$$

$$(1.14) \quad \nabla L^{-1/2} : H_L^p(\mathbb{R}^n) \rightarrow H^p(\mathbb{R}^n), \quad \frac{n}{n+1} < p \leq 1,$$

where $H^p(\mathbb{R}^n)$ denotes the classical Hardy space [31]. Observe that these results extend (1.5) to the range of p below the endpoint $p_-(L)$ (the case $p = 1$ has already appeared in [39]). The H_L^p spaces in (1.13)–(1.14) do not, in general, coincide with L^p or classical H^p (we recall that $H^p(\mathbb{R}^n) = L^p(\mathbb{R}^n)$ if $1 < p < \infty$). In fact, we can ascertain only that

$$(1.15) \quad H_L^p(\mathbb{R}^n) = L^p(\mathbb{R}^n), \quad p_-(L) < p < p_+(L),$$

$$(1.16) \quad L^2 \cap H_L^p \subset L^2 \cap H^p, \quad n/(n+1) < p \leq p_-(L),$$

$$(1.17) \quad L^p(\mathbb{R}^n)/\mathcal{N}_p(L) \hookrightarrow H_L^p(\mathbb{R}^n), \quad p \geq p_+(L),$$

where $\mathcal{N}_p(L)$ is the null space of L in $L^p(\mathbb{R}^n)$ (cf. Section 9 for details). In addition, the containments in (1.16)⁴ (resp. (1.17)) are *strict* if $p_-(L) > 1$ (resp. $p_+(L) < \infty$).

⁴We note that $L^2 \cap H_L^p$ is dense in $H_L^p(\mathbb{R}^n)$, so by (1.16) there is a natural “embedding” of $H_L^p(\mathbb{R}^n)$ into $H^p(\mathbb{R}^n)$ which extends the identity map on a dense subset. Intuitively then, one might expect that the stronger containment $H_L^p(\mathbb{R}^n) \subset H^p(\mathbb{R}^n)$ should hold in (1.16). In practice, however, matters appear to be more subtle, so we present a more detailed discussion of this matter, along with proofs of (1.15) – (1.17), in an Appendix, Section 9.

By contrast, when $L = \Delta$, the space $H^p_\Delta(\mathbb{R}^n)$ is the usual Hardy space for $0 < p \leq 1$ and L^p for $1 < p < \infty$. Hence, (1.13)–(1.14) recover the well-known mapping properties of $\nabla\Delta^{-1/2}$ in L^p and H^p .

Moreover, we have that $H^1_L = H^1$, and $H^p_L = L^p$ for $1 < p < \infty$ whenever the heat kernel of L satisfies a Gaussian upper bound and local Nash type Hölder continuity (as in (2.16–2.18)); indeed, in that case the square function (1.10) is a standard Hilbert space valued Calderón-Zygmund operator, which therefore maps $H^1(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$; whence it follows readily that $H^1(\mathbb{R}^n)$ embeds continuously into $H^1_L(\mathbb{R}^n)$, and thus $H^1(\mathbb{R}^n) = H^1_L(\mathbb{R}^n)$, by (1.16). The case $p > 1$ is obtained by interpolation and duality. The ‘‘Gaussian’’ property (2.16–2.18) holds always in dimensions $n = 1, 2$, and for real coefficients, it holds in all dimensions. However, as we mentioned earlier, it may fail for complex coefficients in dimensions $n \geq 3$.

We turn now to the matter of characterizing H^p_L , for some range of $p \leq p_-(L)$, via the Riesz transform operator $\nabla L^{-1/2}$. In the classical setting (i.e., $L = \Delta$), the Riesz transform provided the foundation for the development, beginning in [56] and [31], of the real variable theory of H^p , and furnished also a link between that theory and PDEs, via sub-harmonic functions. The classical Riesz transform characterization says that

$$(1.18) \quad f \in H^p(\mathbb{R}^n) \quad \text{if and only if} \quad f \in L^p(\mathbb{R}^n) \quad \text{and} \quad \nabla\Delta^{-1/2} f \in L^p(\mathbb{R}^n),$$

for all $(n-1)/n < p \leq 1$ (assuming some growth restriction at infinity when $p < 1$; see, e.g. [55], p. 123). There are analogous, but more complicated results involving higher order Riesz transforms when $p \leq (n-1)/n$. Apparently, no such characterization has been obtained for operators substantially different from the Laplacian (although we mention that some results in this direction have been obtained for lower order perturbations of the Laplacian [29, 30]).

Upon attempting to generalize the Riesz transform characterization to H^p_L spaces, one immediately encounters several difficulties. The original argument relied on the subharmonicity of small powers of the gradient of a harmonic function. No analogue of such a property exists (or even makes sense) in our context. In addition, that (1.18) holds only for the values of p close to 1 suggests that in our case, in which $H^p_L(\mathbb{R}^n)$ is strictly contained in $L^p(\mathbb{R}^n)$ if $p \leq p_-(L)$, the Riesz transform characterization should be proved for p close to $p_-(L)$. In fact, in Section 5 of this paper we show that

$$(1.19) \quad H^p_L(\mathbb{R}^n) = H^p_{L,\text{Riesz}}(\mathbb{R}^n), \quad \frac{p_-(L)n}{n + p_-(L)} < p < 2 + \varepsilon(L),$$

where for p in the stated range, $H^p_{L,\text{Riesz}}(\mathbb{R}^n)$ is defined as the completion of the set $\{f \in L^2(\mathbb{R}^n) : \nabla L^{-1/2} f \in H^p(\mathbb{R}^n)\}$, with respect to the norm

$$(1.20) \quad \|f\|_{H^p_{L,\text{Riesz}}(\mathbb{R}^n)} := \|\nabla L^{-1/2} f\|_{H^p(\mathbb{R}^n)}$$

(bearing in mind that classical $H^p(\mathbb{R}^n) = L^p(\mathbb{R}^n)$ if $p > 1$). The equivalence (1.19) amounts to proving that for $f \in L^2(\mathbb{R}^n)$,⁵

$$(1.21) \quad \|f\|_{H_L^p(\mathbb{R}^n)} \approx \|\nabla L^{-1/2} f\|_{L^p(\mathbb{R}^n)}, \quad \max\left\{1, \frac{p_-(L)n}{n+p_-(L)}\right\} < p < 2 + \varepsilon(L),$$

$$(1.22) \quad \|f\|_{H_L^p(\mathbb{R}^n)} \approx \|\nabla L^{-1/2} f\|_{H^p(\mathbb{R}^n)}, \quad \frac{p_-(L)n}{n+p_-(L)} < p \leq 1.$$

We note that (1.21) and (1.22) can be viewed as sharp extensions of the Kato square root estimate (1.4) to the endpoint $p_-(L)$ and below.⁶

Consequently, for this same range of p , (1.22) together with (1.18) imply that

$$(1.23) \quad \|f\|_{H_L^p(\mathbb{R}^n)} \approx \|\Delta^{1/2} L^{-1/2} f\|_{H^p(\mathbb{R}^n)} \approx \|\nabla L^{-1/2} f\|_{L^p(\mathbb{R}^n)} + \|\Delta^{1/2} L^{-1/2} f\|_{L^p(\mathbb{R}^n)},$$

for suitable f . This further leads to a new characterization of the classical Hardy spaces. Namely,

$$(1.24) \quad f \in H^1(\mathbb{R}^n) \\ \text{if and only if } \nabla L^{-1/2} f \in L^1(\mathbb{R}^n) \quad \text{and} \quad \Delta^{1/2} L^{-1/2} f \in L^1(\mathbb{R}^n),$$

for any operator L whose heat kernel satisfies Gaussian bounds.

Finally, we remark that in [48], the second named author has recently developed further the circle of ideas related to the Riesz transform characterization of $H_L^p(\mathbb{R}^n)$ to establish sharp L^p solvability results for the regularity problem for the equation $u_{tt} - Lu = 0$ in the half-space \mathbb{R}_+^{n+1} .

1.3. The Dual of H_L^p , $0 < p \leq 1$. Another important aspect of the theory is the identification of the duals of Hardy spaces, and the elaboration of their properties. In the classical setting, the duality result for $p = 1$ is the celebrated theorem of Fefferman [31]; the case $0 < p < 1$ was treated in one dimension by Duren, Romberg and Shields [28], and in general by Fefferman and Stein [31]. Just as H^1 provides a substitute for L^1 in harmonic analysis, so too does the dual of H^1 , the space of functions with bounded mean oscillation (*BMO*), substitute for L^∞ . Furthermore, the duals of H^p for $p < 1$ are Lipschitz spaces, whose norms measure fractional order smoothness. In our setting they can be introduced as follows.

Let α be a non-negative real number and $M \in \mathbb{N}$ be such that $M > \frac{1}{2}\left(\alpha + \frac{n}{2}\right)$. For $\varepsilon > 0$ we define the space $\mathbf{M}_{\alpha, L}^{\varepsilon, M}$ as the collection of all $\mu \in L^2(\mathbb{R}^n)$ such that μ belongs to the range of L^k in $L^2(\mathbb{R}^n)$, $k = 1, \dots, M$, and

$$\|\mu\|_{\mathbf{M}_{\alpha, L}^{\varepsilon, M}} \equiv \sup_{j \geq 0} 2^{j(n/2 + \alpha + \varepsilon)} \sum_{k=0}^M \|L^{-k} \mu\|_{L^2(S_j(Q_0))} < \infty,$$

⁵By definition, $H_L^p(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ is dense in $H_L^p(\mathbb{R}^n)$; similarly for $H_{L, \text{Riesz}}^p(\mathbb{R}^n)$.

⁶We remark also that the direction $\|f\|_{H_L^p(\mathbb{R}^n)} \lesssim \|\nabla L^{-1/2} f\|_{L^p(\mathbb{R}^n)}$ of (1.21) is a sharp version of the bound $\|f\|_{L^p(\mathbb{R}^n)} \lesssim \|\nabla L^{-1/2} f\|_{L^p(\mathbb{R}^n)}$, proved in [7] for the same range of p . Indeed, as mentioned above $H_L^p(\mathbb{R}^n)$ may be ‘‘strictly smaller’’ (in the sense of (1.16)) than $L^p(\mathbb{R}^n)$. We shall discuss this point in more detail in Sections 5 and 9.

where Q_0 is the unit cube centered at 0 and $S_j(Q_0)$, $j \in \mathbb{N}$, are the corresponding dyadic annuli (see (3.2)). We say that an element

$$(1.25) \quad f \in \cap_{\varepsilon>0} (\mathbf{M}_{\alpha,L}^{\varepsilon,M})^* =: \mathbf{M}_{\alpha,L}^{M,*}$$

belongs to the space $\Lambda_{L^*}^\alpha(\mathbb{R}^n)$ if ⁷

$$(1.26) \quad \|f\|_{\Lambda_{L^*}^\alpha(\mathbb{R}^n)} := \sup_Q \frac{1}{|Q|^{\alpha/n}} \left(\frac{1}{|Q|} \int_Q |(I - e^{-l(Q)^2 L^*})^M f(x)|^2 dx \right)^{1/2} < \infty,$$

where the supremum runs over all cubes $Q \subset \mathbb{R}^n$. Here and throughout the paper $|Q|$ stands for the Euclidean volume of the cube Q , and $l(Q)$ denotes its sidelength. For $\alpha > 0$ the spaces $\Lambda_{L^*}^\alpha(\mathbb{R}^n)$ are the analogues of the classical Lipschitz spaces,⁸ while the case $\alpha = 0$ corresponds to BMO . Accordingly, we denote $BMO_{L^*}(\mathbb{R}^n) := \Lambda_{L^*}^0(\mathbb{R}^n)$. We refer the reader to [39], where the authors also established some further properties of BMO_{L^*} such as a Carleson measure characterization and an analogue of the John-Nirenberg inequality. In addition, the authors showed in [39] that $(H_L^1)^* = BMO_{L^*}$. In Section 3 of the present paper, we extend this duality as follows:

$$(1.27) \quad (H_L^p(\mathbb{R}^n))^* = \Lambda_{L^*}^\alpha(\mathbb{R}^n), \quad 0 < p \leq 1, \quad \alpha = n(1/p - 1).$$

Moreover, the dual of $\Lambda_{L^*}^\alpha(\mathbb{R}^n)$, in turn, provides an ambient space for H_L^p , for the elements of H_L^p , $p < 1$, are not necessarily functions, they are linear functionals on $\Lambda_{L^*}^\alpha(\mathbb{R}^n)$ (recall that the elements of H^p are tempered distributions).

Finally, as we already mentioned, $H_\Delta^p(\mathbb{R}^n) = H^p(\mathbb{R}^n)$ for all $0 < p < \infty$, which reduces to $L^p(\mathbb{R}^n)$ when $p > 1$. Then, by duality, $BMO_\Delta(\mathbb{R}^n) = BMO(\mathbb{R}^n)$ and $\Lambda_\Delta^\alpha(\mathbb{R}^n) = \Lambda^\alpha(\mathbb{R}^n)$, the classical BMO and Lipschitz spaces. In general, one has only the proper inclusions (1.16) and on the dual side $BMO(\mathbb{R}^n) \subset BMO_L(\mathbb{R}^n)$, $\Lambda^\alpha(\mathbb{R}^n) \subset \Lambda_L^\alpha(\mathbb{R}^n)$ for $0 < \alpha < 1$.

1.4. The Dual of H_L^p , $1 < p < 2$. In the case $2 < p < \infty$, the spaces H_L^p were originally defined by the duality relationship (1.12). We shall give two intrinsic characterizations of these spaces: one, in Section 4 (cf. Corollary 4.21), in terms of square functions, analogous to (1.10)–(1.11), and another one, in Section 6, in terms of a variant of the sharp maximal function. The former characterization is a consequence of tent space duality, and is similar to the analogous results presented in [10]. The latter is new (although rooted in ideas of [31] and also [47]), and we discuss it in a bit more detail at this point.

⁷We note that in the presence of a pointwise Gaussian bound, similar spaces were previously introduced in the work of Duong and Yan [25, 26, 27]. We shall discuss this point in more detail at the end of this section.

⁸Indeed, for $\alpha > 0$, the norm in (1.26) is clearly modeled on the mean oscillation characterization, due to N. Meyers [52], of the classical homogeneous ‘‘Lip $_\alpha$ ’’ space $\Lambda^\alpha(\mathbb{R}^n)$. For $0 < \alpha < 1$, we define the latter to be the space of continuous functions modulo constants, for which the norm $\|\varphi\|_{\Lambda^\alpha(\mathbb{R}^n)} := \sup_{x \neq y} \frac{|\varphi(x) - \varphi(y)|}{|x - y|^\alpha} < \infty$.

Following [31] and [47], consider the operator

$$(1.28) \quad \mathcal{M}^\sharp f(x) := \sup_{Q \ni x} \left(\frac{1}{|Q|} \int_Q \left| (I - e^{-l(Q)^2 L})^M f(y) \right|^2 dy \right)^{1/2}, \quad x \in \mathbb{R}^n,$$

where $M \in \mathbb{N}$ and $\sup_{Q \ni x}$ is the supremum over all cubes in \mathbb{R}^n containing x . We shall refer to \mathcal{M}^\sharp as the sharp maximal operator and write \mathcal{M}_M^\sharp to underline the dependence on M whenever necessary. By definition, we have that $f \in \mathbf{M}_{0,L}^{M,*}$, $M > n/4$, belongs to the space $BMO_L(\mathbb{R}^n)$ if and only if $\mathcal{M}^\sharp f \in L^\infty(\mathbb{R}^n)$. In the current paper we show that an analogous characterization holds for all spaces in the Hardy-BMO scale when $p > 2$. That is, roughly speaking, for $2 < p < \infty$, we have $f \in H_L^p(\mathbb{R}^n)$ if and only if $\mathcal{M}_M^\sharp f \in L^p(\mathbb{R}^n)$, $M > n/4$, and

$$(1.29) \quad \|f\|_{H_L^p(\mathbb{R}^n)} \approx \|\mathcal{M}_M^\sharp f\|_{L^p(\mathbb{R}^n)}, \quad M > n/4.$$

We shall prove a precise version of this statement in Section 6.

1.5. Sobolev spaces and fractional powers of L . The last topic that we shall treat, in Sections 7 and 8, concerns the adapted H_L^p spaces and their relationship to the behavior of L in classical Sobolev spaces. In fact, we find a *complete range of all Sobolev spaces* which naturally interact with the operators associated to L , and one of the major ingredients in the argument is the Riesz transform characterization of H_L^p . Let us describe these results in more detail.

We first prove in Section 7 that the fractional powers of L satisfy

$$(1.30) \quad L^{-\alpha} : H_L^p(\mathbb{R}^n) \longrightarrow H_L^r(\mathbb{R}^n), \quad \alpha = \frac{1}{2} \left(\frac{n}{p} - \frac{n}{r} \right), \quad 0 < p < r < \infty,$$

thereby extending the mapping properties of $L^{-\alpha}$ in L^p (cf. [7]) to the range of p beyond $(p_-(L), p_+(L))$.

In Section 8, we then consider the action of operators associated to L in the classical Sobolev spaces. As is customary, we define the homogeneous Sobolev spaces $\dot{W}^{1,p}(\mathbb{R}^n)$, $1 \leq p < \infty$, to be the completion of $C_0^\infty(\mathbb{R}^n)$ in the seminorm

$$(1.31) \quad \|f\|_{\dot{W}^{1,p}(\mathbb{R}^n)} = \|\nabla f\|_{L^p(\mathbb{R}^n)}.$$

More generally (except for the case $p = 1$), we let $\dot{W}^{s,p}(\mathbb{R}^n)$, $1 < p < \infty$, denote the completion of $C_0^\infty(\mathbb{R}^n)$ in the seminorm

$$(1.32) \quad \|f\|_{\dot{W}^{s,p}(\mathbb{R}^n)} = \|\Delta^{s/2} f\|_{L^p(\mathbb{R}^n)}, \quad s > 0,$$

and set $\dot{W}^{-s,p}(\mathbb{R}^n) = (\dot{W}^{s,p}(\mathbb{R}^n))^*$.

Consider first the case $n \geq 5$. We prove that for any operator L defined in (1.1)–(1.3), for every function φ holomorphic in a certain sector of a complex plane Σ_μ^0 (the exact definitions will be given in the body of the paper), and for every $f \in \dot{W}^{\alpha,p}(\mathbb{R}^n)$,

$$(1.33) \quad \|\varphi(L)f\|_{\dot{W}^{\beta,q}(\mathbb{R}^n)} \leq C \left\| z^{\frac{\beta-\alpha}{2} + \frac{1}{2} \left(\frac{n}{p} - \frac{n}{q} \right)} \varphi \right\|_{L^\infty(\Sigma_\mu^0)} \|f\|_{\dot{W}^{\alpha,p}(\mathbb{R}^n)},$$

provided that the function $z \mapsto z^{\frac{\beta-\alpha}{2} + \frac{1}{2}(\frac{n}{p} - \frac{n}{q})} \varphi(z)$ belongs to $L^\infty(\Sigma_\mu^0)$ and the indices $\alpha, \beta, p \leq q$ are such that the points $(\beta, 1/q)$ and $(\alpha, 1/p)$ belong to the closed region \mathcal{R}_1 , depicted on Figure 1.

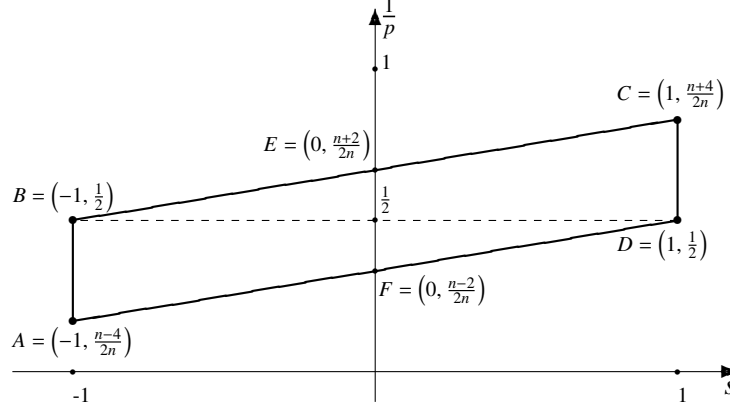


Figure 1 – the region \mathcal{R}_1 .

In particular, for every $t > 0$

$$(1.34) \quad e^{-tL} : \dot{W}^{\alpha,p}(\mathbb{R}^n) \longrightarrow \dot{W}^{\alpha,p}(\mathbb{R}^n), \text{ if } (\alpha, 1/p) \in \mathcal{R}_1,$$

and

$$(1.35) \quad L^{-s} : \dot{W}^{\alpha,p}(\mathbb{R}^n) \longrightarrow \dot{W}^{\beta,q}(\mathbb{R}^n), \text{ if } (\beta, 1/q) \in \mathcal{R}_1 \text{ and } (\alpha, 1/p) \in \mathcal{R}_1,$$

with $s = \frac{\beta-\alpha}{2} + \frac{1}{2}(\frac{n}{p} - \frac{n}{q})$, $p \leq q$.

The region \mathcal{R}_1 is closed and is also sharp, in the sense that for every pair α, p such that $(\alpha, 1/p) \notin \mathcal{R}_1$ there is an operator L for which the property (1.34) is not satisfied and hence, (1.33) is not generally satisfied.

Furthermore, all the results in (1.33)–(1.35) have analogues for $n \leq 4$. In this case $\frac{2n}{n+4} \leq 1$, and just as the classical Hardy spaces provide a natural extension of L^p to the range $p \leq 1$, so too do the Triebel-Lizorkin (or “ H^p Sobolev”) spaces $\dot{F}_s^{p,2}$ extend $\dot{W}^{s,p}$ in this range; i.e., the spaces $\dot{F}_s^{p,2}$ coincide with $\dot{W}^{s,p}$ when $p > 1$ and otherwise naturally extend the Sobolev scale to small values of p . We prove that

$$(1.36) \quad \|\varphi(L)f\|_{\dot{F}_\beta^{q,2}(\mathbb{R}^n)} \leq C \left\| z^{\frac{\beta-\alpha}{2} + \frac{1}{2}(\frac{n}{p} - \frac{n}{q})} \varphi \right\|_{L^\infty(\Sigma_\mu^0)} \|f\|_{\dot{F}_\alpha^{p,2}(\mathbb{R}^n)}, \quad \forall f \in \dot{F}_\alpha^{p,2}(\mathbb{R}^n),$$

provided that the function $z \mapsto z^{\frac{\beta-\alpha}{2} + \frac{1}{2}(\frac{n}{p} - \frac{n}{q})} \varphi(z)$ belongs to $L^\infty(\Sigma_\mu^0)$ and the indices $\alpha, \beta, p \leq q$ are such that the points $(\beta, 1/q)$ and $(\alpha, 1/p)$ belong to the region \mathcal{R}_2 , depicted on Figure 2. In particular, the analogues of (1.34)–(1.35) hold in this context as well. Moreover, all the results are once again sharp, in the sense that for every point outside of the region \mathcal{R}_2 even the heat semigroup is not necessarily bounded in the corresponding Triebel-Lizorkin space.

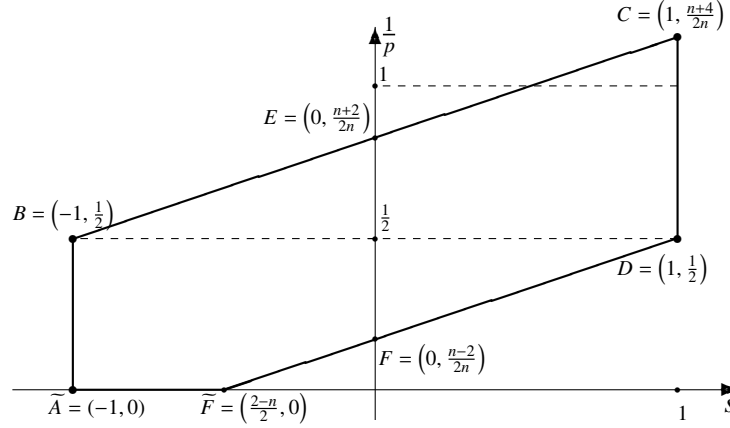


Figure 1 – the region \mathcal{R}_2 .

The study of the properties of the operators associated to L in Sobolev spaces stems from the work of P. Auscher in [7]. Our results extend the theorems in [7] in several directions: to the range of p beyond the range of L^p -boundedness of the heat semigroup (i.e. to the cases $p < p_-(L) < 2n/(n+2)$ and $p > p_+(L) > 2n/(n-2)$), and in particular to $p \leq 1$, and are accompanied by the negative results which lead to sharpness of the obtained range of indices. In particular, we resolve the conjecture posed at the end of Section 6 in [7].

The results we describe in this paper generalize most of the important aspects of the real variable Hardy space theory to a context in which the standard tools of the Calderón-Zygmund theory are not applicable. Besides the aforementioned works [10] and [39], some properties of the Hardy and BMO spaces associated with different operators were introduced previously in [11], [25], [26], [60].

In particular, we note that the theory of L -adapted H^1 and BMO spaces, including an appropriate analogue of Fefferman’s duality theorem, originates in the work of Duong and Yan [25], [26] who treated the case that the associated heat kernel satisfies a pointwise Gaussian bound. Their BMO norm is the same as that in (1.26), with $\alpha = 0$ and $M = 1$, and they have also considered Morrey-Campanato type spaces corresponding to the case $\alpha > 0$ [27]. As we have observed above, the theory and techniques of the present paper, which we develop in the absence of pointwise kernel bounds, assuming instead decay estimates of “Gaffney” type, are necessarily somewhat different.

We note also that, while this manuscript was in preparation, we learned that some of the results presented here in the case $0 < p < 1$ have been obtained independently by R. Jiang and D. Yang [43]. As mentioned above, the case $p = 1$ was already treated in [39] (and in [10], in a somewhat different context). Our main

results in the case $p > 1$, as well as our Riesz transform characterization (1.19), appear to be unique to this paper.⁹

Acknowledgements. A part of this work was conducted during the second author's visit to the Centre for Mathematics and its Applications at the Australian National University, and during the third author's visit to the University of Missouri. The authors thank the faculty and staff of those institutions for their warm hospitality. We also thank Pascal Auscher for reading a draft of the manuscript and for offering several helpful suggestions. In addition, the third named author thanks Prof. Auscher and Andrew Morris for illuminating discussions.

Finally, we thank Dachun Yang for making a preliminary version of his joint work with R. Jiang [43] available to us while this manuscript was in preparation. As mentioned above, their results and ours, which overlap substantially in the case $0 < p < 1$, have been obtained independently, but we have incorporated a simplification introduced in [43] in the proof of the duality result for $0 < p < 1$ (cf. Step II of Theorem 3.52 below). Our original proof here had been based on the more complicated argument in [39].

2. THE HEAT SEMIGROUP AND FUNCTIONS OF L IN L^p .

2.1. Definitions and L^2 theory. Let L be a second order elliptic operator satisfying (1.1)–(1.3) viewed as an accretive operator in $L^2(\mathbb{R}^n)$. There exists some $\omega \in [0, \pi/2)$ such that the operator L is of type ω on $L^2(\mathbb{R}^n)$. In particular, $-L$ generates a complex semigroup which extends to an analytic semigroup $\{e^{-zL}\}_{z \in \Sigma_{\pi/2-\omega}^0}$ on $L^2(\mathbb{R}^n)$. Here

$$(2.1) \quad \Sigma_{\mu}^0 := \{z \in \mathbb{C} \setminus \{0\} : |\arg z| < \mu\}, \quad \mu \in (0, \pi).$$

Furthermore, L has bounded holomorphic functional calculus on $L^2(\mathbb{R}^n)$ (see [50] and [1]). To be more precise, let us define

$$(2.2) \quad H^{\infty}(\Sigma_{\mu}^0) := \{\psi : \Sigma_{\mu}^0 \rightarrow \mathbb{C} : \psi \text{ is analytic and } \|\psi\|_{L^{\infty}(\Sigma_{\mu}^0)} < \infty\},$$

$$\Psi_{\sigma, \tau}(\Sigma_{\mu}^0) := \{\psi : \Sigma_{\mu}^0 \rightarrow \mathbb{C} : \psi \text{ is analytic and}$$

$$(2.3) \quad |\psi(\xi)| \leq C \inf\{|\xi|^{\sigma}, |\xi|^{-\tau}\} \text{ for every } \xi \in \Sigma_{\mu}^0\}.$$

Alternatively, one can say that

$$(2.4) \quad \psi \in \Psi_{\sigma, \tau}(\Sigma_{\mu}^0) \iff \psi \in H^{\infty}(\Sigma_{\mu}^0) \text{ and } |\psi(\xi)| \leq C \frac{|\xi|^{\sigma}}{1+|\xi|^{\sigma+\tau}}, \quad \sigma, \tau > 0.$$

Whenever $\psi \in H^{\infty}(\Sigma_{\mu}^0)$

$$(2.5) \quad \|\psi(L)f\|_{L^2(\mathbb{R}^n)} \leq C \|\psi\|_{L^{\infty}(\Sigma_{\mu}^0)} \|f\|_{L^2(\mathbb{R}^n)} \quad \text{for every } f \in L^2(\mathbb{R}^n).$$

Let $\Psi(\Sigma_{\mu}^0) := \cup_{\sigma, \tau > 0} \Psi_{\sigma, \tau}(\Sigma_{\mu}^0)$. If $\psi \in \Psi(\Sigma_{\mu}^0)$ then $\psi(L)$ can be represented as

$$(2.6) \quad \psi(L) = \int_{\Gamma_+} e^{-zL} \eta_+(z) dz + \int_{\Gamma_-} e^{-zL} \eta_-(z) dz,$$

⁹Although as mentioned above, our tent space/square function definition of adapted H^p spaces with $p > 1$ follows that given in [10].

where

$$(2.7) \quad \eta_{\pm}(z) = \frac{1}{2\pi i} \int_{\gamma_{\pm}} e^{\xi z} \psi(\xi) d\xi, \quad z \in \Gamma_{\pm},$$

and $\Gamma_{\pm} = \mathbb{R}^+ e^{\pm i(\pi/2-\theta)}$, $\gamma_{\pm} = \mathbb{R}^+ e^{\pm i\nu}$, $\omega < \theta < \nu < \mu < \pi/2$. In general, $\psi \in H^{\infty}(\Sigma_{\mu}^0)$ can be defined using (2.6)–(2.7) and a limiting procedure (see [7], §3.3, and references therein).

Finally, let us introduce

$$(2.8) \quad \Psi'_{\sigma,\tau}(\Sigma_{\mu}^0) = \{\psi : \Sigma_{\mu}^0 \rightarrow \mathbb{C} : \psi \text{ is analytic and there are some } \sigma, \tau, C > 0 \text{ such that } |\psi(\xi)| \leq C \sup\{|\xi|^{\sigma}, |\xi|^{-\tau}\} \text{ for every } \xi \in \Sigma_{\mu}^0\}.$$

For every $\psi \in \Psi'_{\sigma,\tau}$ one can define an unbounded operator $\psi(L)$ on $L^2(\mathbb{R}^n)$ following the procedure in [50]. In particular, the fractional powers of L arise in this way.

2.2. L^p boundedness of the heat semigroup: sharp results. Following [7], let us denote by $\mathcal{J}(L)$ the maximal interval of exponents $p \in [1, \infty]$ for which the heat semigroup $\{e^{-tL}\}_{t>0}$ is L^p -bounded and let us write $\text{int } \mathcal{J}(L) = (p_-(L), p_+(L))$. It was proved in [7] that

$$(2.9) \quad p_-(L) < \frac{2n}{n+2} \quad \text{and} \quad p_+(L) > \frac{2n}{n-2},$$

for L as in (1.1)–(1.3), and that $p_-(L)$ is also the lower bound for the interval of p for which $\nabla L^{-1/2} : L^p \rightarrow L^p$ (hence this notation is consistent with that in Section 1). We shall show that the bounds in (2.9) are sharp, in the following sense.

Proposition 2.10. *Given any \tilde{p}_- with $1 \leq \tilde{p}_- < \frac{2n}{n+2}$ there exists an operator L such that the heat semigroup $\{e^{-tL}\}_{t>0}$ is not bounded in $L^{\tilde{p}_-}$. And similarly, given any \tilde{p}_+ with $\frac{2n}{n-2} < \tilde{p}_+ \leq \infty$, there exists an operator L such that the heat semigroup $\{e^{-tL}\}_{t>0}$ is not bounded in $L^{\tilde{p}_+}$.*

Proof. We argue as in [13], but using the example of [33] rather than that of [49].

Let $n \geq 3$. By [33], for every $q < n/2$ and $\lambda > 0$, there is an $n \times n$ matrix $A = A(q, \lambda)$ satisfying (1.1)–(1.2) and such that

$$(2.11) \quad u = \frac{x_1}{|x|^q} e^{i\lambda \ln|x|}$$

is a classical solution of the equation $Lu = -\text{div}(A\nabla u) = 0$ in $\mathbb{R}^n \setminus \{0\}$, and is a weak solution globally in \mathbb{R}^n .

More precisely, A has a form

$$(2.12) \quad A = \left\{ (\alpha + i)\delta_{jk} + \beta \frac{x_j x_k}{|x|^2} \right\}_{j,k=1}^n,$$

where $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{C}$ are some constants. For any fixed $\alpha \in \mathbb{R}$, $\lambda \neq 0$, $q \neq 0$ there exists $\beta = \beta(\alpha, q, \lambda)$ (explicitly written in [33]) such that u in (2.11) solves the equation $-\text{div}(A\nabla u) = 0$, and moreover, for $q < n/2$, $\lambda > 0$, $\alpha > 0$ sufficiently small and $\beta = \beta(\alpha, q, \lambda)$, the corresponding matrix A satisfies the ellipticity conditions.

Now let us return to the properties of the heat semigroup. First of all, take some $\phi \in C_0^{\infty}(\mathbb{R}^n)$, supported in the unit ball B_1 , such that $\phi = 1$ in the ball of radius

1/2 centered at the origin. Then $\nabla\phi \in C_0^\infty(B_1)$ and $\nabla\phi = 0$ in a neighborhood of 0. Since the only singularity of u (and of A) is at 0, we have

$$(2.13) \quad L(u\phi) = -\operatorname{div}(A\nabla(u\phi)) = -\operatorname{div}(Au\nabla\phi) - A\nabla u \cdot \nabla\phi =: f \in C_0^\infty(B_1),$$

where the second equality follows from the fact that $Lu = 0$.

Fix some $\tilde{p}_+ > \frac{2n}{n-2}$ and assume that the heat semigroup $\{e^{-tL}\}_{t>0}$ is bounded in $L^{\tilde{p}_+}$ for an operator L . Then, according to [7], we have

$$(2.14) \quad L^{-1} : L^p(\mathbb{R}^n) \longrightarrow L^r(\mathbb{R}^n), \quad n/p - n/r = 2,$$

provided $r < \tilde{p}_+$ and $p > p_-(L)$. But since $p_-(L)$ is always smaller than $\frac{2n}{n+2}$, (2.14) is valid for any $\frac{2n}{n-2} \leq r < \tilde{p}_+$.

The function $f \in C_0^\infty(B_1)$ in the right-hand side of (2.13) belongs, in particular, to all L^p spaces, $1 \leq p \leq \infty$, and therefore, by (2.14) the solution

$$(2.15) \quad L^{-1}f = u\phi \text{ must belong to all } L^r, \quad \frac{2n}{n-2} \leq r < \tilde{p}_+.$$

However, $u\phi = u$ in a neighborhood of the origin and u given by (2.11) does not belong to L^r when $r(1-q) + n < 0$. We can take $\varepsilon > 0$ sufficiently small so that $2n/(n-2-2\varepsilon) < \tilde{p}_+$ and take $q = n/2 - \varepsilon$. Then $u\phi \notin L^r$ for any $r > 2n/(n-2-2\varepsilon)$ which contradicts (2.15).

Since $p_-(L) = (p_+(L^*))'$, this computation also shows that assuming boundedness of $\{e^{-tL}\}_{t>0}$ in $L^{\tilde{p}_-}$ for all L will lead to a contradiction. \square

Let L be a divergence form elliptic operator with complex bounded coefficients given by (1.1)–(1.3). Let $K_t(x, y)$, $t > 0$, $x, y \in \mathbb{R}^n$, denote the Schwartz kernel of the heat semigroup generated by L . We say that it satisfies the Gaussian property if for each $t > 0$ the kernel $K_t(x, y)$ is Hölder continuous in x and y and there exist some constants $C, c, \alpha > 0$ such that for every $x, y, h \in \mathbb{R}^n$

$$(2.16) \quad |K_t(x, y)| \leq \frac{C}{t^{n/2}} e^{-\frac{|x-y|^2}{ct}},$$

$$(2.17) \quad |K_t(x, y) - K_t(x+h, y)| \leq \frac{C}{t^{n/2}} \left(\frac{|h|}{t^{1/2} + |x-y|} \right)^\alpha e^{-\frac{|x-y|^2}{ct}},$$

$$(2.18) \quad |K_t(x, y) - K_t(x, y+h)| \leq \frac{C}{t^{n/2}} \left(\frac{|h|}{t^{1/2} + |x-y|} \right)^\alpha e^{-\frac{|x-y|^2}{ct}},$$

whenever $2|h| \leq t^{1/2} + |x-y|$. For every elliptic operator defined in (1.1)–(1.3) the heat kernel satisfies the Gaussian bounds in dimensions $n = 1, 2$, and for every elliptic operator with real coefficients this property holds in all dimensions. It was known that in general the Gaussian bounds may fail in dimensions $n \geq 5$. Whether or not they necessarily hold when $n = 3, 4$ has been an open problem (see, e.g., [13], §1.2 and the Remark on p. 33). The Corollary below answers this question to the negative.

Corollary 2.19. *Let $n \geq 3$. There exists an elliptic operator L given by (1.1)–(1.3) such that the kernel of the heat semigroup generated by L does not satisfy (2.16). In particular, for such L the Gaussian property does not hold.*

Proof. The estimate (2.16) implies that the integral kernel $G(x, y)$, $x, y \in \mathbb{R}^n$, of the operator $L^{-1} = \int_0^\infty e^{-tL} dt$ is controlled by $C|x - y|^{2-n}$. Hence, (2.14) holds, which yields as before a contradiction. An analogous argument was used in [13], §1.3. \square

Corollary 2.20. *For each $p < \frac{2n}{n+2}$ and each $p > 2$ there exists L such that $\nabla L^{-1/2}$ is not bounded in L^p .*

Proof. The counterexample for $p > 2$ is due to C. Kenig. The case $p < \frac{2n}{n+2}$ follows from Proposition 2.10 along with the fact, proved in [7] and noted above, that the lower endpoint of the interval of boundedness of Riesz transform coincides with the the lower endpoint of the interval of boundedness of the heat semigroup. \square

2.3. Off-diagonal estimates and $L^p - L^q$ bounds. We say that a family of operators $\{S_t\}_{t>0}$ satisfies L^2 off-diagonal estimates (“Gaffney estimates”) if there are some constants $c, C > 0$ such that for arbitrary closed sets $E, F \subset \mathbb{R}^n$

$$(2.21) \quad \|S_t f\|_{L^2(F)} \leq C e^{-\frac{\text{dist}(E,F)^2}{ct}} \|f\|_{L^2(E)},$$

for every $t > 0$ and every $f \in L^2(\mathbb{R}^n)$ supported in E . Similarly, a family $\{S_z\}_{z \in \Sigma_\mu^0}$, $0 < \mu < \pi/2$, satisfies L^2 off-diagonal estimates in z if the analogue of (2.21) holds with $|z|$ in place of t on the right-hand side. For example, if $0 < \mu < \pi/2 - \omega$, the families $\{e^{-zL}\}_{z \in \Sigma_\mu^0}$ and $\{(zL)^k e^{-zL}\}_{z \in \Sigma_\mu^0}$, $k = 1, 2, \dots$, satisfy L^2 off-diagonal estimates in z (see [7], §3.3). For later reference we record the following result.

Lemma 2.22. ([38]) *If two families of operators, $\{S_t\}_{t>0}$ and $\{T_t\}_{t>0}$, satisfy Gaffney estimates (2.21) then so does $\{S_t T_t\}_{t>0}$. Moreover, there exist $c, C > 0$ such that for arbitrary closed sets $E, F \subset \mathbb{R}^n$*

$$(2.23) \quad \|S_s T_t f\|_{L^2(F)} \leq C e^{-\frac{\text{dist}(E,F)^2}{c \max\{t,s\}}} \|f\|_{L^2(E)},$$

for all $t, s > 0$ and all $f \in L^2(\mathbb{R}^n)$ supported in E .

A family of operators $\{S_t\}_{t>0}$ satisfies $L^p - L^q$ off-diagonal estimates, $1 < p, q < \infty$, if for arbitrary closed sets $E, F \subset \mathbb{R}^n$

$$(2.24) \quad \|S_t f\|_{L^q(F)} \leq C t^{\frac{1}{2}(\frac{n}{q} - \frac{n}{p})} e^{-\frac{\text{dist}(E,F)^2}{ct}} \|f\|_{L^p(E)},$$

for every $t > 0$ and every $f \in L^p(\mathbb{R}^n)$ supported in E .

Lemma 2.25. ([7]) *For every p and q such that $p_-(L) < p \leq q < p_+(L)$ the family $\{e^{-tL}\}_{t>0}$ satisfies $L^p - L^q$ off-diagonal estimates. In particular, the operator e^{-tL} , $t > 0$, maps $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ with norm controlled by $C t^{\frac{1}{2}(\frac{n}{q} - \frac{n}{p})}$.*

Lemma 2.26. *Assume that for some $1 \leq r \leq 2$ the family $\{e^{-tL}\}_{t>0}$ satisfies $L^r - L^2$ off-diagonal estimates. Then the family $\{tLe^{-tL}\}_{t>0}$ also satisfies $L^r - L^2$ off-diagonal estimates and the operators e^{-tL} , tLe^{-tL} , $t > 0$, are bounded from $L^r(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$ with norms bounded by $C t^{\frac{1}{2}(\frac{n}{2} - \frac{n}{r})}$, and from $L^r(\mathbb{R}^n)$ to $L^r(\mathbb{R}^n)$ with norms independent of t .*

Proof. The fact that $L^r - L^2$ off-diagonal estimates implies boundedness in $L^r(\mathbb{R}^n)$ is rather standard, see e.g., [7], Proposition Lemma 4.3, or [17].

Next, recall that the ellipticity of L implies analyticity of the semigroup e^{-zL} , acting on L^2 , in a sector containing the positive real axis. Hence, the family of operators $\{tLe^{-tL}\}_{t>0}$ satisfies $L^2 - L^2$ off-diagonal estimates and in particular is bounded in $L^2(\mathbb{R}^n)$. We can combine this information with the properties of the heat semigroup to deduce that $tLe^{-tL} = 2\left(\frac{t}{2}Le^{-\frac{t}{2}L}\right)e^{-\frac{t}{2}L}$, $t > 0$, also satisfies $L^r - L^2$ off-diagonal estimates and is $L^r - L^2$ bounded. \square

We say that a family of operators $\{S_t\}_{t>0}$ satisfies L^2 off-diagonal estimates of order N , $N > 0$, $N \in \mathbb{R}$, if there is a constant $C > 0$ such that for arbitrary closed sets $E, F \subset \mathbb{R}^n$

$$(2.27) \quad \|S_t f\|_{L^2(F)} \leq C \min \left\{ 1, \frac{t}{\text{dist}(E, F)^2} \right\}^N \|f\|_{L^2(E)},$$

for every $t > 0$ and every $f \in L^2(\mathbb{R}^n)$ supported in E .

Lemma 2.28. *Let $\mu \in (\omega, \pi/2)$, $\psi \in \Psi_{\sigma, \tau}(\Sigma_\mu^0)$ for some $\sigma, \tau > 0$, and $f \in H^\infty(\Sigma_\mu^0)$. Then the family of operators $\{\psi(tL)f(L)\}_{t>0}$ satisfies L^2 off-diagonal estimates of order σ , with the constant controlled by $\|f\|_{L^\infty(\Sigma_\mu^0)}$.*

Proof. Recall the representation formulas (2.6), (2.7). We use them for the function $\psi(tL)f(L)$, $t > 0$. First of all,

$$(2.29) \quad |\eta_\pm(z)| \leq \frac{C}{t} \int_{\gamma_\pm} |\psi(t\xi)| |f(\xi)| d(t\xi) \\ \leq \frac{C}{t} \|f\|_{L^\infty(\Sigma_\mu^0)} \int_{\gamma_\pm} \frac{|t\xi|^\sigma}{1 + |t\xi|^{\sigma+\tau}} d(t\xi) \leq \frac{C}{t} \|f\|_{L^\infty(\Sigma_\mu^0)},$$

for all $z \in \Gamma_\pm$, in particular, for z with $|z| \leq t$.

When $|z| > t$ we break $\eta_\pm(z)$ into two integrals: one over $\{\xi \in \gamma_\pm : |\xi| \leq 1/t\}$ (called J_1) and the second one over $\{\xi \in \gamma_\pm : |\xi| \geq 1/t\}$ (called J_2). Then

$$(2.30) \quad J_1 \leq C \|f\|_{L^\infty(\Sigma_\mu^0)} \int_{\xi \in \gamma_\pm : |\xi| \leq 1/t} e^{-|z||\xi|} |t\xi|^\sigma d\xi \\ \leq \frac{C}{|z|} \|f\|_{L^\infty(\Sigma_\mu^0)} \frac{t^\sigma}{|z|^\sigma} \int_0^\infty e^{-\rho} \rho^\sigma d\rho \leq \frac{C}{t} \|f\|_{L^\infty(\Sigma_\mu^0)} \left(\frac{t}{|z|}\right)^{\sigma+1},$$

and

$$(2.31) \quad J_2 \leq \|f\|_{L^\infty(\Sigma_\mu^0)} \int_{\xi \in \gamma_\pm : |\xi| \geq 1/t} |z\xi|^{-\sigma-1} |t\xi|^{-\tau} d\xi \\ \leq C \|f\|_{L^\infty(\Sigma_\mu^0)} \left(\frac{t}{|z|}\right)^{\sigma+1} t^{-\tau-\sigma-1} \int_{\xi \in \gamma_\pm : |\xi| \geq 1/t} |\xi|^{-\sigma-1-\tau} d\xi \\ \leq \frac{C}{t} \|f\|_{L^\infty(\Sigma_\mu^0)} \left(\frac{t}{|z|}\right)^{\sigma+1}.$$

Hence,

$$(2.32) \quad |\eta_{\pm}(z)| \leq \frac{C}{t} \|f\|_{L^\infty(\Sigma_\mu^0)} \min \left\{ 1, \left(\frac{t}{|z|} \right)^{\sigma+1} \right\}, \quad \forall z \in \Gamma_{\pm}.$$

Armed with this estimate, we proceed to the bounds on $\psi(tL)f(L)$, $t > 0$. Take some $g \in L^2(\mathbb{R}^n)$ supported in a closed set E . Then for any closed set $F \subset \mathbb{R}^n$

$$(2.33) \quad \|\psi(tL)f(L)g\|_{L^2(F)} \leq \int_{\Gamma_+} \|e^{-zL}g\|_{L^2(F)} |\eta_+(z)| dz + \int_{\Gamma_-} \|e^{-zL}g\|_{L^2(F)} |\eta_-(z)| dz.$$

Further,

$$(2.34) \quad \begin{aligned} \int_{\Gamma_{\pm}} \|e^{-zL}g\|_{L^2(F)} |\eta_{\pm}(z)| dz &\leq C \|g\|_{L^2(E)} \int_{\Gamma_{\pm}} e^{-\frac{\text{dist}(E,F)^2}{c|z|}} |\eta_{\pm}(z)| dz \\ &\leq C \|f\|_{L^\infty(\Sigma_\mu^0)} \|g\|_{L^2(E)} \int_{\Gamma_{\pm}} e^{-\frac{\text{dist}(E,F)^2}{c|z|}} \min \left\{ 1, \left(\frac{t}{|z|} \right)^{\sigma+1} \right\} \frac{1}{t} dz. \end{aligned}$$

Now we split the last integral in (2.34) according to whether $|z| \leq t$ or $|z| \geq t$, and denote the corresponding parts of it by I_1 and I_2 , respectively. Then

$$(2.35) \quad I_1 = \int_{z \in \Gamma_{\pm}; |z| \leq t} e^{-\frac{\text{dist}(E,F)^2}{c|z|}} \frac{1}{t} dz \leq e^{-\frac{\text{dist}(E,F)^2}{ct}}.$$

On the other hand,

$$(2.36) \quad I_2 = \int_{z \in \Gamma_{\pm}; |z| \geq t} e^{-\frac{\text{dist}(E,F)^2}{c|z|}} \left(\frac{t}{|z|} \right)^{\sigma+1} \frac{1}{t} dz.$$

If $t \geq \text{dist}(E, F)^2$, we obtain the bound

$$(2.37) \quad I_2 \leq \int_{z \in \Gamma_{\pm}; |z| \geq t} \left(\frac{t}{|z|} \right)^{\sigma+1} \frac{1}{t} dz \leq C.$$

If $t \leq \text{dist}(E, F)^2$, then

$$(2.38) \quad \begin{aligned} I_2 &\leq \int_{z \in \Gamma_{\pm}; t \leq |z| \leq \text{dist}(E,F)^2} \left(\frac{|z|}{\text{dist}(E, F)^2} \right)^N \left(\frac{t}{|z|} \right)^{\sigma+1} \frac{1}{t} dz \\ &\quad + \int_{z \in \Gamma_{\pm}; |z| \geq \text{dist}(E,F)^2} \left(\frac{t}{|z|} \right)^{\sigma+1} \frac{1}{t} dz, \end{aligned}$$

for any $N > 0$. Let us take $N > \sigma$. Then

$$(2.39) \quad \begin{aligned} I_2 &\leq C \left(\frac{1}{\text{dist}(E, F)^2} \right)^N t^\sigma \text{dist}(E, F)^{2(N-\sigma)} + C \left(\frac{t}{\text{dist}(E, F)^2} \right)^\sigma \\ &\leq C \left(\frac{t}{\text{dist}(E, F)^2} \right)^\sigma. \end{aligned}$$

This finishes the proof of the Proposition. \square

3. MOLECULAR DECOMPOSITION AND DUALITY, $0 < p \leq 1$.

To begin, we would like to make a few comments regarding the well-definedness and the nature of the space $\Lambda_{L^*}^\alpha(\mathbb{R}^n)$, $\alpha \geq 0$. Let $M \in \mathbb{N}$, $M > \frac{1}{2}(\alpha + \frac{n}{2})$. First, $(I - e^{-t^2 L^*})^M f$, $t \in \mathbb{R}$, is globally well defined in the sense of distributions for every $f \in \mathbf{M}_{\alpha,L}^{M,*}$, and belongs to L_{loc}^2 . Indeed, if $\varphi \in L^2(Q)$ for some cube Q , it follows from the Gaffney estimate (2.24) that $(I - e^{-t^2 L})^M \varphi \in \mathbf{M}_{\alpha,L}^{\varepsilon,M}$ for every $\varepsilon > 0$ (with the norm depending on $t, \ell(Q), \text{dist}(Q, 0)$). Thus,

$$(3.1) \quad \langle (I - e^{-t^2 L^*})^M f, \varphi \rangle \equiv \langle f, (I - e^{-t^2 L})^M \varphi \rangle \leq C_{t, \ell(Q), \text{dist}(Q, 0)} \|f\|_{(\mathbf{M}_{\alpha,L}^{\varepsilon,M})^*} \|\varphi\|_{L^2(Q)}.$$

Since Q was arbitrary, the claim follows. Therefore, the norm in (1.26) is well-defined for such f . Furthermore, the elements of $\mathbf{M}_{\alpha,L}^{\varepsilon,M}$ are, modulo translation, dilation and normalization, the molecules of the corresponding Hardy spaces. The details are as follows.

For a cube $Q \subset \mathbb{R}^n$, by $S_i(Q)$, $i = 0, 1, 2, \dots$, we denote the dyadic annuli based on Q , i.e.

$$(3.2) \quad S_0(Q) := Q \quad \text{and} \quad S_i(Q) := 2^i Q \setminus 2^{i-1} Q \quad \text{for } i = 1, 2, \dots,$$

where $2^i Q$ is the cube with the same center as Q and sidelength $2^i \ell(Q)$. Let $0 < p \leq 1$, $\varepsilon > 0$, and $M \in \mathbb{N}$. We will always assume the above restrictions on ε and M , and typically, given p , unless otherwise stated we will take $M > \frac{n}{2}(\frac{1}{p} - \frac{1}{2})$. A function $m \in L^2(\mathbb{R}^n)$ is called an (H_L^p, ε, M) -molecule¹⁰ if it belongs to the range of L^k in $L^2(\mathbb{R}^n)$, for each $k = 1, \dots, M$, and there exists a cube $Q \subset \mathbb{R}^n$ such that

$$(3.3) \quad \|(\ell(Q)^{-2} L^{-1})^k m\|_{L^2(S_i(Q))} \leq (2^i \ell(Q))^{\frac{n}{2} - \frac{n}{p}} 2^{-i\varepsilon}, \quad i = 0, 1, 2, \dots, \quad k = 0, 1, \dots, M.$$

Observe that for $k = 0$ the estimate (3.3) is the usual size control condition and for $k = 1, \dots, M$ the condition (3.3) is a quantitative version of the requirement that $m \in R(L^k)$, which in turn is analogous to the classical requirement of vanishing moments.

We are now able to define a molecular H_L^p space, which we shall eventually show is equivalent to the space H_L^p defined via square functions.

Definition 3.4. Let $0 < p \leq 1$, and fix $\varepsilon > 0$. The Hardy space $H_{L, \text{mol}, M}^p(\mathbb{R}^n)$ is defined as follows. We say that $f = \sum \lambda_j m_j$, where $\{\lambda_j\}_{j=0}^\infty \in \ell^p$, is a molecular $(H_L^p, 2, \varepsilon, M)$ -representation (of f) if each m_j is an (H_L^p, ε, M) -molecule, and the sum converges in $L^2(\mathbb{R}^n)$. Set

$$\mathbb{H}_{L, \text{mol}, M}^p(\mathbb{R}^n) = \left\{ f : f \text{ has a molecular } (H_L^p, 2, \varepsilon, M)\text{-representation} \right\},$$

with the “norm” (it is a true norm only when $p = 1$), given by

$$\|f\|_{\mathbb{H}_{L, \text{mol}, M}^p(\mathbb{R}^n)} =$$

¹⁰Molecules have been introduced in the classical setting corresponding to $L = -\Delta$ in [57]; see also [22].

$$\inf \left\{ \left(\sum_{j=0}^{\infty} |\lambda_j|^p \right)^{1/p} : f = \sum_{j=0}^{\infty} \lambda_j m_j \text{ is a molecular } (H_L^p, 2, \varepsilon, M)\text{-representation} \right\}.$$

The space $H_{L,mol,M}^p(\mathbb{R}^n)$ is then defined as the completion of $\mathbb{H}_{L,mol,M}^p(\mathbb{R}^n)$ with respect to the metric induced by $\|f\|_{\mathbb{H}_{L,mol,M}^p(\mathbb{R}^n)}^p$.

We note that this approach to the definition of adapted H^p spaces has also been used in [37], at least in the case $p = 1$. We also remark that this approach, in the case $p = 1$, was implicit in [39], but with a more complicated formulation in which L^2 convergence of the molecular sums was achieved constructively, by means of an explicit truncation in scale.

Eventually, we shall see that any fixed choice of $M > \frac{n}{2}(\frac{1}{p} - \frac{1}{2})$ and $\varepsilon > 0$, yields the same space. Indeed, more generally, we will show that the ‘‘square function’’ and ‘‘molecular’’ H^p spaces are equivalent, if the parameter $M > \frac{n}{2}(\frac{1}{p} - \frac{1}{2})$. In fact, we shall prove

Theorem 3.5. *Let $0 < p \leq 1$. Suppose that $M > \frac{n}{2}(\frac{1}{p} - \frac{1}{2})$ and that $\varepsilon > 0$. Then $H_{L,mol,M}^p(\mathbb{R}^n) = H_L^p(\mathbb{R}^n)$. Moreover,*

$$\|f\|_{H_{L,mol,M}^p(\mathbb{R}^n)} \approx \|f\|_{H_L^p(\mathbb{R}^n)},$$

where the implicit constants depend only on M, n, p, ε and ellipticity.

Consequently, one may write simply $H_{L,mol}^p(\mathbb{R}^n)$ in place of $H_{L,mol,M}^p(\mathbb{R}^n)$, when $M > \frac{n}{2}(\frac{1}{p} - \frac{1}{2})$, and for any fixed $\varepsilon > 0$, as these spaces are all equivalent. Moreover, we could also define $(H_L^p, q, \varepsilon, M)$ -molecules as $m \in L^q(\mathbb{R}^n)$ belonging to the range of L^k in $L^q(\mathbb{R}^n)$, $k = 1, \dots, M$, and satisfying the estimates

$$(3.6) \quad \|(l(Q)^{-2}L^{-1})^k m\|_{L^q(S_i(Q))} \leq C (2^i l(Q))^{\frac{n}{q} - \frac{n}{p}} 2^{-i\varepsilon},$$

$$i = 0, 1, 2, \dots, \quad k = 0, 1, \dots, M,$$

These would also yield the same $H_L^p(\mathbb{R}^n)$ spaces provided $p_-(L) < q < p_+(L)$. We omit the details here, although we do note that a proof is given in [39], [40] in the case $p = 1$.

We now proceed to the proof of Theorem 3.5. The basic strategy is as follows: by density, it is enough to show that

$$(3.7) \quad \mathbb{H}_{L,mol,M}^p(\mathbb{R}^n) = L^2(\mathbb{R}^n) \cap H_L^p(\mathbb{R}^n), \quad M > \frac{n}{2} \left(\frac{1}{p} - \frac{1}{2} \right)$$

with equivalence of norms. The proof of this fact proceeds in two steps.

Step 1: $\mathbb{H}_{L,mol,M}^p(\mathbb{R}^n) \subseteq L^2(\mathbb{R}^n) \cap H_L^p(\mathbb{R}^n)$, if $M > \frac{n}{2}(1/p - 1/2)$.

Step 2: $H_L^p(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) \subseteq \mathbb{H}_{L,mol,M}^p(\mathbb{R}^n)$, for every $M \in \mathbb{N}$.

We take these in order. The conclusion of Step 1 is an immediate consequence of the following pair of Lemmata.

Lemma 3.8. Fix $M \in \mathbb{N}$, and suppose that $0 < p \leq 1$. Assume that T is a linear operator, or a non-negative sublinear operator, satisfying the weak-type (2,2) bound

$$(3.9) \quad \mu\{x \in \mathbb{R}^n : |Tf(x)| > \eta\} \leq C_T \eta^{-2} \|f\|_{L^2(\mathbb{R}^n)}^2, \quad \forall \eta > 0,$$

and that for every (H_L^p, ε, M) -molecule m , we have

$$(3.10) \quad \|Tm\|_{L^p(\mathbb{R}^n)} \leq C$$

with constant C independent of m . Then T is bounded from $\mathbb{H}_{L,mol,M}^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$, and

$$\|Tf\|_{L^p(\mathbb{R}^n)} \leq C \|f\|_{\mathbb{H}_{L,mol,M}^p(\mathbb{R}^n)}.$$

Consequently, by density, T extends to a bounded operator from $H_{L,mol,M}^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$.

Lemma 3.11. Let m be an (H_L^p, ε, M) -molecule, with $0 < p \leq 1$, $M > \frac{n}{2}(\frac{1}{p} - \frac{1}{2})$ and $\varepsilon > 0$. Then there is a constant C_0 depending only on p, ε, M, n and ellipticity such that

$$\|Sm\|_p \leq C_0,$$

where S denotes the square function defined in (1.10).

Indeed, given Lemma 3.11, we may apply Lemma 3.8 with $T = S$ to obtain

$$\|f\|_{H_L^p(\mathbb{R}^n)} := \|Sf\|_{L^p(\mathbb{R}^n)} \leq C \|f\|_{\mathbb{H}_{L,mol,M}^p(\mathbb{R}^n)},$$

whence Step 1 follows.

To finish Step 1, it therefore suffices to prove the two Lemmata.

Proof of Lemma 3.8. Let $f \in \mathbb{H}_{L,mol,M}^p(\mathbb{R}^n)$, where $f = \sum \lambda_j m_j$ is a molecular $(H_L^p, 2, \varepsilon, M)$ -representation such that

$$\|f\|_{\mathbb{H}_{L,mol,M}^p(\mathbb{R}^n)}^p \approx \sum_{j=0}^{\infty} |\lambda_j|^p.$$

Since the sum converges in L^2 (by definition), and since T is of weak-type (2, 2), we have that at almost every point,

$$(3.12) \quad |T(f)| \leq \sum_{j=0}^{\infty} |\lambda_j| |T(m_j)|.$$

Indeed, for every $\eta > 0$, we have that, if $f^N := \sum_{j>N} \lambda_j m_j$, then,

$$\begin{aligned} \mu\left\{|T(f)| - \sum_{j=0}^{\infty} |\lambda_j| |T(m_j)| > \eta\right\} &\leq \limsup_{N \rightarrow \infty} \mu\left\{|T(f^N)| > \eta\right\} \\ &\leq C_T \eta^{-2} \limsup_{N \rightarrow \infty} \|f^N\|_2 = 0, \end{aligned}$$

from which (3.12) follows. In turn, (3.12) and (3.10) imply the desired L^p bound for Tf , since $0 < p \leq 1$. \square

Proof of Lemma 3.11. Fix a cube Q , and let m be an (H_L^p, ε, M) -molecule, adapted to Q , with $0 < p \leq 1$, $M > \frac{n}{2}(\frac{1}{p} - \frac{1}{2})$ and $\varepsilon > 0$. In particular, we have that for each $k \in \{0, 1, \dots, M\}$,

$$(3.13) \quad \|(\ell(Q)^2 L)^{-k} m\|_{L^2(\mathbb{R}^n)} \leq C_k |Q|^{1/2-1/p}.$$

Hence, by Hölder's inequality and the L^2 boundedness of S , we have that

$$\|S m\|_{L^p(16Q)} \leq C |Q|^{1/p-1/2} \|S m\|_{L^2(\mathbb{R}^n)} \leq C.$$

Writing now $\|S m\|_p^p = \|S m\|_{L^p(16Q)}^p + \sum_{j=5}^{\infty} \|S m\|_{L^p(S_j(B))}^p$, where we recall that $S_j(Q) := 2^j Q \setminus 2^{j-1} Q$, we see that it is enough to prove that

$$(3.14) \quad \|S m\|_{L^2(S_j(Q))} \leq C 2^{-j\alpha} |2^j Q|^{1/2-1/p},$$

for some $\alpha > 0$ and for each $j \geq 5$. To this end, we write

$$\begin{aligned} \|S m\|_{L^2(S_j(Q))}^2 &= \int_{S_j(Q)} \int_0^\infty \int_{|x-y|<t} \left| (t^2 L e^{-t^2 L} m)(y) \right|^2 \frac{dy dt}{t^{n+1}} dx \\ &= \int_{S_j(Q)} \int_0^{2^{\theta(j-5)\ell(Q)}} \int_{|x-y|<t} + \int_{S_j(Q)} \int_{2^{\theta(j-5)\ell(Q)}}^\infty \int_{|x-y|<t} =: I + II, \end{aligned}$$

where $\theta \in (0, 1)$ will be chosen momentarily. Then by Fubini's theorem, the definition of an (H_L^p, ε, M) -molecule (cf. (3.3)), the uniform L^2 boundedness of $t^{2K} L^K e^{-t^2 L}$ for each non-negative integer K , and (3.13), setting $b := L^{-M} m$, we have

$$\begin{aligned} II &\leq \int_{2^{\theta(j-5)\ell(Q)}}^\infty \int_{\mathbb{R}^n} \left| (t^{2(M+1)} L^{M+1} e^{-t^2 L} b)(y) \right|^2 dy \frac{dt}{t^{4M+1}} \\ &\leq C (2^{\theta j \ell(Q)})^{-4M} \|b\|_{L^2(\mathbb{R}^n)}^2 \leq C 2^{-j(4\theta M + n(1-2/p))} 2^{jn(1-2/p)} |Q|^{1-2/p} \\ &= C 2^{-j(4\theta M - n(2/p-1))} |2^j Q|^{1-2/p}. \end{aligned}$$

Taking square roots, and choosing θ sufficiently close to 1, we obtain (3.14) for the contribution of the term II , with $\alpha = (2\theta M - n(1/p - 1/2)) > 0$.

We now treat the term I . We set

$$\widetilde{S}_j(Q) := 2^{j+1} Q \setminus 2^{j-2} Q, \quad \widehat{S}_j(Q) := 2^{j+2} Q \setminus 2^{j-3} Q,$$

and observe that, by Fubini's Theorem

$$\begin{aligned} I &\leq \int_0^{2^{\theta(j-5)\ell(Q)}} \int_{\widetilde{S}_j(Q)} \left| (t^2 L e^{-t^2 L} m)(y) \right|^2 dy \frac{dt}{t} \\ &\lesssim \int_0^{2^{\theta(j-5)\ell(Q)}} \int_{\widetilde{S}_j(Q)} \left| (t^2 L e^{-t^2 L} (1_{2^{j-3}Q} m))(y) \right|^2 dy \frac{dt}{t} \\ &\quad + \int_0^{2^{\theta(j-5)\ell(Q)}} \int_{\widetilde{S}_j(Q)} \left| (t^2 L e^{-t^2 L} (1_{\widehat{S}_j(Q)} m))(y) \right|^2 dy \frac{dt}{t} \end{aligned}$$

$$\begin{aligned}
& + \int_0^{2^{\theta(j-5)\ell(Q)}} \int_{\widehat{S}_j(Q)} \left| \left(t^2 L e^{-t^2 L} (1_{\mathbb{R}^n \setminus 2^{j+2} Q} m) \right) (y) \right|^2 dy \frac{dt}{t} \\
& \qquad \qquad \qquad =: I_1 + I_2 + I_3.
\end{aligned}$$

By the L^2 boundedness of S and the definition of a molecule (cf. (3.3)) and the doubling property,

$$\sqrt{I_2} \leq C \|m\|_{\widehat{S}_j(Q)} \leq C 2^{-j\varepsilon} |2^j Q|^{1/2-1/p},$$

which is (3.14) for the contribution of I_2 . For the other two terms, we have that by the Gaffney estimates (cf. subsection 2.3),

$$\begin{aligned}
I_1 + I_3 & \leq C \|m\|_{L^2(\mathbb{R}^n)}^2 \int_0^{2^{\theta(j-5)\ell(Q)}} \exp\left(\frac{-(2^j \ell(Q))^2}{c t^2}\right) \frac{dt}{t} \\
& \leq C_N \|m\|_{L^2(\mathbb{R}^n)}^2 \int_0^{2^{\theta(j-5)\ell(Q)}} \left(\frac{t}{2^j \ell(Q)}\right)^N \frac{dt}{t} \leq C_N |Q|^{2(1/2-1/p)} 2^{N(\theta-1)j},
\end{aligned}$$

where we have used (3.13) in the last step, and N is at our disposal. Having fixed $\theta < 1$ above, we may now choose N so large that $N(1-\theta) \geq 4M > 2n(1/p - 1/2)$, to obtain in turn the desired bound

$$I_1 + I_3 \leq C |2^j Q|^{2(1/2-1/p)} 2^{-j(4M-2n(1/p-1/2))},$$

whence (3.14) follows. \square

This concludes Step 1. We now turn to Step 2.

Our goal is to show that every $f \in L^2(\mathbb{R}^n) \cap H_L^p(\mathbb{R}^n)$ has a molecular $(H_L^p, 2, \varepsilon, M)$ -representation, with appropriate quantitative control of the coefficients. To this end, we follow the (nowadays) standard tent space approach of [21], as adapted to the present setting in the case $p = 1$ in [10] (cf. [37] and [43], as well as the earlier work [25]); yet another (somewhat more complicated) adaptation of the methods of [21] was used in [39], [40].

Let us begin by recalling some basic facts from [21]. First, for $0 < p < \infty$, the tent spaces on $\mathbb{R}_+^{n+1} = \mathbb{R}^n \times (0, \infty)$ are defined by

$$(3.15) \quad T^p(\mathbb{R}_+^{n+1}) := \{F : \mathbb{R}_+^{n+1} \rightarrow \mathbb{C}; \|F\|_{T^p(\mathbb{R}_+^{n+1})} := \|\mathcal{A}F\|_{L^p(\mathbb{R}^n)} < \infty\},$$

where

$$(3.16) \quad \mathcal{A}F(x) = \left(\iint_{\Gamma(x)} |F(y, t)|^2 \frac{dy dt}{t^{n+1}} \right)^{1/2}, \quad x \in \mathbb{R}^n.$$

In addition, the case $p = \infty$ may be handled as follows. For $F : \mathbb{R}_+^{n+1} \rightarrow \mathbb{C}$ let

$$(3.17) \quad CF(x) := \sup_{B \ni x} \left(\frac{1}{|B|} \iint_{\widehat{B}} |F(y, t)|^2 \frac{dy dt}{t} \right)^{1/2}, \quad x \in \mathbb{R}^n,$$

where B stands for a ball in \mathbb{R}^n and

$$(3.18) \quad \widehat{B} := \{(x, t) \in \mathbb{R}^n \times (0, \infty) : \text{dist}(x, {}^c B) \geq t\}.$$

For $p = \infty$, we then have

$$(3.19) \quad T^\infty(\mathbb{R}_+^{n+1}) := \{F : \mathbb{R}_+^{n+1} \rightarrow \mathbb{C}; \|F\|_{T^\infty(\mathbb{R}_+^{n+1})} := \|CF\|_{L^\infty(\mathbb{R}^n)} < \infty\}.$$

Moreover, according to [21],

$$(3.20) \quad \|CF\|_{L^p(\mathbb{R}^n)} \approx \|\mathcal{A}F\|_{L^p(\mathbb{R}^n)} = \|F\|_{T^p(\mathbb{R}_+^{n+1})}, \quad 2 < p < \infty.$$

The tent spaces satisfy the natural duality and interpolation properties:

$$(3.21) \quad (T^q(\mathbb{R}_+^{n+1}))^* = T^{q'}(\mathbb{R}_+^{n+1}), \quad 1/q + 1/q' = 1, \quad 1 < q < \infty,$$

and also $(T^1(\mathbb{R}_+^{n+1}))^* = T^\infty(\mathbb{R}_+^{n+1})$; moreover,

$$(3.22) \quad [T^{p_0}(\mathbb{R}_+^{n+1}), T^{p_1}(\mathbb{R}_+^{n+1})]_\theta = T^p(\mathbb{R}_+^{n+1}), \quad 1/p = (1-\theta)/p_0 + \theta/p_1, \quad 0 < \theta < 1,$$

for $0 < p_0 < p_1 \leq +\infty$. We will later discuss the precise meaning of the complex interpolation in (3.22) and provide references (see the proof of Lemma 4.24 and the preceding discussion).

It has been proved in [21] that every $F \in T^p(\mathbb{R}_+^{n+1})$, $0 < p \leq 1$ has an atomic decomposition. For future reference, we record this result below. We first define the notion of a $T^p(\mathbb{R}_+^{n+1})$ -atom.

Definition 3.23. Let $0 < p \leq 1$. A measurable function A on \mathbb{R}_+^{n+1} is said to be a T^p -atom if there exists a cube $Q \subset \mathbb{R}^n$ such that A is supported in the ‘‘Carleson box’’

$$R_Q := Q \times (0, \ell(Q)),$$

and

$$(3.24) \quad \left(\iint_{R_Q} |A(x, t)|^2 \frac{dx dt}{t} \right)^{1/2} \leq |Q|^{\frac{1}{2} - \frac{1}{p}}.$$

Proposition 3.25. [21] Let $0 < p \leq 1$. For every element $F \in T^p(\mathbb{R}_+^{n+1})$, there exist a numerical sequence $\{\lambda_j\}_{j=0}^\infty \subset \ell^p$ and a sequence of T^p -atoms $\{A_j\}_{j=0}^\infty$ such that

$$(3.26) \quad F = \sum_{j=0}^\infty \lambda_j A_j \quad \text{in } T^p(\mathbb{R}_+^{n+1}) \text{ and a.e. in } (\mathbb{R}_+^{n+1}).$$

Moreover,

$$\sum_{j=0}^\infty |\lambda_j|^p \approx \|F\|_{T^p(\mathbb{R}_+^{n+1})}^p,$$

where the implicit constants depend only on dimension.

Finally, if $F \in T^p(\mathbb{R}_+^{n+1}) \cap T^2(\mathbb{R}_+^{n+1})$, then the decomposition (3.26) also converges in $T^2(\mathbb{R}_+^{n+1})$.

Proof. Except for the final part of the proposition, concerning T^2 convergence, this is proved in [21], and we refer the reader to that paper for the proof. The T^2 convergence is only implicit there, so we shall sketch the proof here. To this end, we first note that

$$(3.27) \quad \|F\|_{T^2(\mathbb{R}_+^{n+1})}^2 := \int_{\mathbb{R}^n} (\mathcal{A}F)^2 dx = \int_{\mathbb{R}^n} \int_0^\infty \int_{|x-y|<t} |F(y,t)|^2 \frac{dydt}{t^{n+1}} d(x) \\ \approx \int_0^\infty \int_{\mathbb{R}^n} |F(y,t)|^2 \frac{dydt}{t}$$

Suppose now that $F \in T^p \cap T^2$. We recall that, in the constructive proof of the decomposition (3.26) in [21], one has that

$$\lambda_j A_j = F 1_{S_j},$$

where $\{S_j\}$ is a collection of sets which are pairwise disjoint (up to sets of measure zero), and whose union covers \mathbb{R}_+^{n+1} . Thus, by (3.27),

$$\left\| \sum_{j>N} \lambda_j A_j \right\|_{T^2(\mathbb{R}_+^{n+1})}^2 \approx \int_0^\infty \int_{\mathbb{R}^n} \left| \sum_{j>N} 1_{S_j} F(y,t) \right|^2 \frac{dydt}{t} \\ = \sum_{j>N} \iint_{S_j} |F|^2 \frac{dydt}{t} \rightarrow 0,$$

as $N \rightarrow \infty$, where we have used disjointness of the sets S_j and dominated convergence. It therefore follows that $F = \sum \lambda_j A_j$ in T^2 . \square

Now, given $M \geq 1$, we define an operator $\pi_{M,L}$, acting initially on T^2 , as follows:

$$(3.28) \quad \pi_{M,L}(F) := \int_0^\infty (t^2 L)^{M+1} e^{-t^2 L} F(\cdot, t) \frac{dt}{t}.$$

By a standard duality argument involving well known quadratic estimates for L^* , one obtains that the improper integral converges weakly in L^2 , and that

$$(3.29) \quad \|\pi_{M,L}(F)\|_{L^2(\mathbb{R}^n)} \lesssim \|F\|_{T^2(\mathbb{R}_+^{n+1})}, \quad M \geq 0,$$

where the implicit bound depends only on M , ellipticity and dimension.

Following [21], we now observe that $\pi_{M,L}$ essentially maps T^p atoms into H_L^p -molecules. We have:

Lemma 3.30. *Suppose that A is a $T^p(\mathbb{R}_+^{n+1})$ -atom associated to a cube $Q \subset \mathbb{R}^n$ (or more precisely, to its Carleson box R_Q). Then for each integer $M \geq 1$, and every $\varepsilon > 0$, there is a uniform constant $C_{\varepsilon,M}$ such that $C_{\varepsilon,M}^{-1} \pi_{M,L}(A)$ is an (H_L^p, ε, M) -molecule associated to Q .*

Proof. Fix a cube Q and let A be a $T^p(\mathbb{R}_+^{n+1})$ -atom associated to R_Q , so that (3.24) holds. We set

$$m := \pi_{M,L}(A) = L^M b,$$

where

$$b := \int_0^\infty t^{2M} t^2 L e^{-t^2 L} (A(\cdot, t)) \frac{dt}{t},$$

and we need to establish that m satisfies (3.3). We first prove an L^2 estimate which in particular yields the desired bound “near” Q . Let $g \in L^2(\mathbb{R}^n)$. Then for every $k = 0, 1, \dots, M$ we have

$$\begin{aligned}
(3.31) \quad & \left| \int_{\mathbb{R}^n} (\ell(Q)^2 L)^k b(x) g(x) dx \right| \\
&= \left| \lim_{\delta \rightarrow 0} \int_{\mathbb{R}^n} \left(\int_{\delta}^{1/\delta} \ell(Q)^{2k} L^k t^{2M} t^2 L e^{-t^2 L} (A(\cdot, t))(x) \frac{dt}{t} \right) g(x) dx \right| \\
&= \left| \int_{R_Q} A(x, t) \ell(Q)^{2k} (L^*)^k t^{2M} t^2 L^* e^{-t^2 L^*} g(x) \frac{dx dt}{t} \right| \\
&\leq \ell(Q)^{2M} |Q|^{1/2-1/p} \left(\int_{R_Q} |(t^2 L^*)^{k+1} e^{-t^2 L^*} g(x)|^2 \frac{dx dt}{t} \right)^{1/2}.
\end{aligned}$$

Here, the third line is obtained by using the compactness of the t interval to interchange the order of integration, and the fourth line by using that A is a T^p -atom supported in R_Q (so that $0 < t < \ell(Q)$ and (3.24) holds) and the fact that $k \leq M$. In turn, by standard square function estimates for L^* , (3.31) is bounded by

$$C \ell(Q)^{2M} |Q|^{1/2-1/p} \|g\|_{L^2(\mathbb{R}^n)}.$$

Specializing to the case that g is supported in $2Q$, and taking a supremum over all such g with $\|g\|_{L^2(2Q)} = 1$, we then have the bound

$$\|(\ell(Q)^2 L)^k b\|_{L^2(2Q)} \leq C \ell(Q)^{2M} |Q|^{1/2-1/p}, \quad k = 0, 1, \dots, M,$$

which is clearly equivalent to the cases $i = 0, 1$ of (3.3).

Now for $i \geq 2$, let g be supported in $S_i(Q)$, with $\|g\|_{L^2(S_i(Q))} = 1$. Applying the Gaffney estimate to dx integral in the last line in (3.31), and taking a supremum over all such g , we find that

$$\begin{aligned}
\|(\ell(Q)^2 L)^k b\|_{L^2(S_i(Q))} &\leq C \ell(Q)^{2M} |Q|^{1/2-1/p} \int_0^{\ell(Q)} e^{-(2^i \ell(Q)/t)^2} \frac{dt}{t} \\
&\leq C_N 2^{-iN} \ell(Q)^{2M} |Q|^{1/2-1/p},
\end{aligned}$$

for every $N \in \mathbb{N}$ and each $k = 0, 1, \dots, M$. The molecular bound (3.3) follows, for every choice of $\varepsilon > 0$. \square

We are now ready to establish the molecular decomposition of $H_L^p(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$. Our proof here is based on the approach in [10]¹¹. A similar approach, also following [10], is taken in [37] and in [43]. As mentioned above, a more complicated method was used in [39, 40].

Proposition 3.32. *Let $0 < p \leq 1$ and $M \geq 1$. If $f \in H_L^p(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, then there exist a family of (H_L^p, ε, M) -molecules $\{m_j\}_{j=0}^\infty$ and a sequence of numbers $\{\lambda_j\}_{j=0}^\infty \subset \ell^p$ such that f can be represented in the form $f = \sum_{j=0}^\infty \lambda_j m_j$, with the sum converging in $L^2(\mathbb{R}^n)$, and*

$$\|f\|_{\mathbb{H}_{L, mol, M}^p(\mathbb{R}^n)}^p \leq C \sum_{j=0}^\infty |\lambda_j|^p \leq C \|f\|_{H_L^p(\mathbb{R}^n)}^p,$$

¹¹In particular, it is the idea of [10], in the case $p = 1$, to exploit the fact that a T^p -atomic decomposition, of an element in $T^p \cap T^2$, converges also in T^2 .

where C is independent of f . In particular,

$$(3.33) \quad H_L^p(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) \subseteq \mathbb{H}_{L,mol,M}^p(\mathbb{R}^n).$$

Proof. Let $f \in H_L^p(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, and set

$$F(\cdot, t) := t^2 L e^{-t^2 L} f.$$

We note that $F \in T^2(\mathbb{R}_+^{n+1}) \cap T^p(\mathbb{R}_+^{n+1})$, by standard quadratic estimates for L and the definition of $H_L^p(\mathbb{R}^n)$. Therefore, by Proposition 3.25, we have that

$$(3.34) \quad F = \sum \lambda_j A_j,$$

where each A_j is a T^p -atom, the sum converges in both $T^p(\mathbb{R}_+^{n+1})$ and $T^2(\mathbb{R}_+^{n+1})$, and

$$(3.35) \quad \sum |\lambda_j|^p \leq C \|F\|_{T^p(\mathbb{R}_+^{n+1})}^p = C \|f\|_{H_L^p(\mathbb{R}^n)}^p.$$

Also, by L^2 -functional calculus ([50]), we have the ‘‘Calderón reproducing formula’’

$$(3.36) \quad f = c_M \pi_{M,L}(t^2 L e^{-t^2 L} f) = c_M \pi_{M,L}(F) = c_M \sum \lambda_j \pi_{M,L}(A_j),$$

where by (3.29) and the T^2 convergence of the decomposition in (3.34), the last sum converges in $L^2(\mathbb{R}^n)$. Moreover, by Lemma 3.30, for every $M \geq 1$, we have that up to multiplication by some harmless constant, each $m_j := c_M \pi_{M,L}(A_j)$ is an (H_L^p, ε, M) -molecule. Consequently, the last sum in (3.36) is a molecular $(H_L^p, 2, \varepsilon, M)$ -representation, so that $f \in \mathbb{H}_{L,mol,M}^p(\mathbb{R}^n)$, and by (3.35) we have

$$\|f\|_{\mathbb{H}_{L,mol,M}^p(\mathbb{R}^n)} \leq C \|f\|_{H_L^p(\mathbb{R}^n)}.$$

□

Step 2 is now complete. This concludes the proof of Theorem 3.5. □

We next discuss duality for the spaces $H_L^p(\mathbb{R}^n)$ with $0 < p \leq 1$.

If m is an (H_L^p, ε, M) -molecule, then $m \in \mathbf{M}_{n(1/p-1),L}^{\varepsilon,M}$ (this follows from the fact that, given any two cubes Q_1 and Q_2 , there exists integers K_1 and K_2 , depending upon $\ell(Q_1)$, $\ell(Q_2)$ and $\text{dist}(Q_1, Q_2)$, such that $2^{K_1} Q_1 \supseteq Q_2$ and $2^{K_2} Q_2 \supseteq Q_1$), and the converse is also true (up to a normalization). Therefore, $g(m) := \langle g, m \rangle$ is well-defined for every (H_L^p, ε, M) -molecule m and every $g \in \Lambda_{L^*}^{n(1/p-1)}(\mathbb{R}^n)$. Moreover, the following estimate holds.

Lemma 3.37. *Suppose $0 < p \leq 1$, $\varepsilon > 0$, $M > \frac{n}{2}(\frac{1}{p} - \frac{1}{2})$. Then*

$$(3.38) \quad |g(m)| \leq C \|g\|_{\Lambda_{L^*}^{n(1/p-1)}(\mathbb{R}^n)}$$

for every $g \in \Lambda_{L^*}^{n(1/p-1)}(\mathbb{R}^n)$ (in the case $p = 1$ we set $\Lambda_{L^*}^0 := BMO_{L^*}$) and every (H_L^p, ε, M) -molecule m .

Proof. The case $p = 1$ was proved in [39], so we now suppose that $p < 1$. For every $x \in \mathbb{R}^n$

$$(3.39) \quad m(x) = 2^{-M} \left(l(Q)^{-2} \int_{l(Q)}^{\sqrt{2}l(Q)} s ds \right)^M m(x),$$

and

$$(3.40) \quad \int_{l(Q)}^{\sqrt{2}l(Q)} s ds = \int_{l(Q)}^{\sqrt{2}l(Q)} s(I - e^{-s^2L})^M ds + \sum_{k=1}^M C_{k,M} \int_{l(Q)}^{\sqrt{2}l(Q)} s e^{-ks^2L} ds.$$

where $C_{k,M} \in \mathbb{R}$ are some constants depending on k and M only. Going further,

$$(3.41) \quad \begin{aligned} 2kL \int_{l(Q)}^{\sqrt{2}l(Q)} s e^{-ks^2L} ds &= - \int_{l(Q)}^{\sqrt{2}l(Q)} \partial_s e^{-ks^2L} ds = e^{-kl(Q)^2L} - e^{-2kl(Q)^2L} \\ &= e^{-kl(Q)^2L} (I - e^{-kl(Q)^2L}) = e^{-kl(Q)^2L} (I - e^{-l(Q)^2L}) \sum_{j=0}^{k-1} e^{-jl(Q)^2L}. \end{aligned}$$

Applying the procedure outlined in (3.40)–(3.41) M times, we arrive at the following formula

$$(3.42) \quad \begin{aligned} m &= 2^{-M} \left(l(Q)^{-2} \int_{l(Q)}^{\sqrt{2}l(Q)} s(I - e^{-s^2L})^M ds \right. \\ &\quad \left. + \sum_{k=1}^M C_{k,M} l(Q)^{-2} L^{-1} e^{-kl(Q)^2L} (I - e^{-l(Q)^2L}) \sum_{j=0}^{k-1} e^{-jl(Q)^2L} \right)^M m. \end{aligned}$$

Let

$$(3.43) \quad m_{N_i} := l(Q)^{-2N_i} L^{-N_i} m, \quad 0 \leq N_i \leq M.$$

Then

$$(3.44) \quad \begin{aligned} g(m) &= C_{1,1} \int_{\mathbb{R}^n} (I - e^{-l(Q)^2L^*})^M g(x) T_{1,1}^{l(Q)} m_M(x) dx \\ &\quad + \sum_{i=1}^{(M+1)^M-1} C_{i,2} \int_{\mathbb{R}^n} \left(l(Q)^{-2} \int_{l(Q)}^{\sqrt{2}l(Q)} s(I - e^{-s^2L^*})^M g(x) ds \right) T_{i,2}^{l(Q)} m_{N_i}(x) dx, \end{aligned}$$

where $C_{i,k}$ are some constants, $T_{i,k}^{l(Q)}$ are some operator families satisfying Gaffney estimates (2.21) with $t \approx l(Q)^2$, and the integrals on the right-hand side are interpreted analogously to (3.1). More precisely, each $T_{i,k}$ is a composition of operators of the form (3.41) and operators coming from

$$(3.45) \quad l(Q)^{-2} \int_{l(Q)}^{\sqrt{2}l(Q)} s(I - e^{-s^2L})^M ds.$$

However, according to (3.41)–(3.42), the latter can be written as a constant plus an operator in (3.41), modulo the factor $l(Q)^{-2}L^{-1}$. The negative powers of $l(Q)^2L$

are absorbed in m_{N_i} . Hence, each $T_{i,k}$ is a constant (possibly, zero) plus a linear combination of the terms in the form $e^{-t^2 L}$ with $t \approx l(Q)^2$.

Applying the Cauchy-Schwartz inequality, we deduce that

$$(3.46) \quad |g(m)| \leq C \|g\|_{\Lambda_{L^*}^{n(1/p-1)}(\mathbb{R}^n)} |Q|^{\frac{1}{p}-\frac{1}{2}} \sum_{j=0}^{\infty} 2^{j\frac{n}{2}} \sum_{i,k} \|T_{i,k}^{l(Q)} m_{N_i}\|_{L^2(S_j(Q))}.$$

If $j \leq 3$, then

$$(3.47) \quad \|T_{i,k}^{l(Q)} m_{N_i}\|_{L^2(S_j(Q))} \leq C \|m_{N_i}\|_{L^2(\mathbb{R}^n)} \leq C |Q|^{\frac{1}{2}-\frac{1}{p}},$$

for i and k as above. If $j \geq 3$, we split

$$(3.48) \quad m_{N_i} = m_{N_i} \chi_{\widetilde{S}_j(Q)} + m_{N_i} \chi_{\mathbb{R}^n \setminus \widetilde{S}_j(Q)},$$

where

$$(3.49) \quad \widetilde{S}_j(Q) = 2^{j+1}Q \setminus 2^{j-2}Q.$$

Then

$$(3.50) \quad \left\| T_{i,k}^{l(Q)} \left(m_{N_i} \chi_{\widetilde{S}_j(Q)} \right) \right\|_{L^2(S_j(Q))} \leq C \left\| m_{N_i} \chi_{\widetilde{S}_j(Q)} \right\|_{L^2(\mathbb{R}^n)} \leq C 2^{j(\frac{n}{2}-\frac{n}{p}-\varepsilon)} |Q|^{\frac{1}{2}-\frac{1}{p}},$$

by the definition of molecule, and

$$(3.51) \quad \left\| T_{i,k}^{l(Q)} \left(m_{N_i} \chi_{\mathbb{R}^n \setminus \widetilde{S}_j(Q)} \right) \right\|_{L^2(S_j(Q))} \leq C e^{-\frac{(2^j l(Q))^2}{c l(Q)^2}} \|m_{N_i}\|_{L^2(\mathbb{R}^n)} \leq C 2^{-jN} |Q|^{\frac{1}{2}-\frac{1}{p}},$$

for a number N arbitrarily large. Inserting the results into (3.46), we finish the proof of (3.38). \square

We are now ready to state our duality results generalizing [31, 28, 25].

Theorem 3.52. *Suppose $0 < p \leq 1$. Then*

$$(3.53) \quad (H_L^p(\mathbb{R}^n))^* = \Lambda_{L^*}^{n(1/p-1)}(\mathbb{R}^n) \text{ if } p < 1, \quad \text{and} \quad (H_L^1(\mathbb{R}^n))^* = BMO_{L^*}(\mathbb{R}^n).$$

Proof. The statement about the duality of H_L^1 and BMO_{L^*} was proved in [39]. Therefore we consider here only the case $p < 1$.

Step I. We start with the left-to-the-right inclusion.

Assume that g is a linear functional on $H_L^p(\mathbb{R}^n)$. Then for every $f \in H_L^p(\mathbb{R}^n)$

$$(3.54) \quad |g(f)| \leq \|g\| \|f\|_{H_L^p(\mathbb{R}^n)}.$$

Theorem 3.5, in particular, implies that every (H_L^p, ε, M) -molecule belongs to H_L^p and $\|m\|_{H_L^p} \leq C$. Hence,

$$(3.55) \quad |g(m)| \leq C \|g\|.$$

However, if $\mu \in \mathbf{M}_{n(1/p-1)}^{\varepsilon, M}$ with norm 1, then μ is a (p, ε, M) -molecule adapted to Q_0 . Therefore, by (3.55), g defines a linear functional on $\mathbf{M}_{n(1/p-1)}^{\varepsilon, M}$. It remains to prove that the norm (1.26), understood in the sense of (3.1), is finite. To do this,

it would be enough to show that for every $\varphi \in L^2(Q)$ such that $\|\varphi\|_{L^2(Q)} = 1$ the function

$$(3.56) \quad \frac{1}{|Q|^{\alpha/n+1/2}} (I - e^{-l(Q)^2 L})^M \varphi, \quad \alpha = n \left(\frac{1}{p} - 1 \right), \quad M > \frac{n}{2} \left(\frac{1}{p} - \frac{1}{2} \right),$$

is a (p, ε, M) -molecule (then the claim follows from (3.55)).

Since φ is supported in Q , by Gaffney estimates

$$(3.57) \quad \begin{aligned} \frac{1}{|Q|^{\alpha/n+1/2}} \|(I - e^{-l(Q)^2 L})^M \varphi\|_{L^2(S_j(Q))} &\leq C \frac{1}{|Q|^{\alpha/n+1/2}} \sum_{k=0}^M \|e^{-kl(Q)^2 L} \varphi\|_{L^2(S_j(Q))} \\ &\leq \frac{C}{|Q|^{\alpha/n+1/2}} e^{-\frac{\text{dist}(S_j(Q), Q)^2}{cl(Q)^2}} \|\varphi\|_{L^2(Q)} \leq \frac{C 2^{-jN}}{|Q|^{\alpha/n+1/2}} = \frac{C 2^{-jN}}{|Q|^{1/p-1/2}}, \end{aligned}$$

for every $j \in \mathbb{N}$ and $N \in \mathbb{N}$ arbitrarily large. Similarly, for $k = 1, \dots, M$

$$(3.58) \quad \begin{aligned} &\frac{1}{|Q|^{\alpha/n+1/2}} \|(l(Q)^{-2} L^{-1})^k (I - e^{-l(Q)^2 L})^M \varphi\|_{L^2(S_j(Q))} \\ &= \frac{1}{|Q|^{\alpha/n+1/2}} \left\| (l(Q)^{-2} L^{-1})^k \left(\int_0^{l(Q)} \partial_t e^{-t^2 L} dt \right)^k (I - e^{-l(Q)^2 L})^{M-k} \varphi \right\|_{L^2(S_j(Q))} \\ &= \frac{1}{|Q|^{\alpha/n+1/2}} \left\| \left(\int_0^{l(Q)} \frac{2t}{l(Q)^2} e^{-t^2 L} dt \right)^k (I - e^{-l(Q)^2 L})^{M-k} \varphi \right\|_{L^2(S_j(Q))} \\ &\leq \frac{C}{|Q|^{\alpha/n+1/2}} e^{-\frac{\text{dist}(S_j(Q), Q)^2}{cl(Q)^2}} \|\varphi\|_{L^2(Q)} \leq \frac{C 2^{-jN}}{|Q|^{1/p-1/2}}, \end{aligned}$$

where we employed Lemma 2.22 for the next-to-the-last inequality. As before, $N \in \mathbb{N}$ can be taken arbitrarily large, and that finishes the argument.

Step II. Let us now turn to the right-to-the-left inclusion in (3.53). Let $g \in \Lambda_{L^*}^{n(1/p-1)}(\mathbb{R}^n)$. We note that the mapping

$$\mathcal{L}_g(f) := \langle g, f \rangle,$$

may be defined initially (by virtue of Lemma 3.37) when f is a finite linear combination of (H_L^p, ε, M) -molecules, with $M > (n/2)(1/p - 1/2)$, and by the density, in $H_L^p(\mathbb{R}^n)$, of the collection of all such f , it is enough to establish the *a priori* bound

$$(3.59) \quad |\mathcal{L}_g(f)| \leq C \|g\|_{\Lambda_{L^*}^{n(1/p-1)}(\mathbb{R}^n)} \|f\|_{\mathbb{H}_{L, \text{mol}, M}^p(\mathbb{R}^n)},$$

for some uniform constant C , whenever f is such a finite linear combination. Indeed, in that case, \mathcal{L}_g extends by continuity to a continuous linear functional on $H_L^p(\mathbb{R}^n)$.

Our proof of (3.59) is based in part on the approach in [39], but we shall incorporate a simplification to that approach, which was introduced in [43]. As above, let $g \in \Lambda_{L^*}^\alpha(\mathbb{R}^n)$, $\alpha = n(1/p - 1)$, and let f be a finite linear combination of (H_L^p, ε, M) -molecules, with $M > (n/2)(1/p - 1/2)$. We begin by noting that the following two facts, first proved in [39] in the case $p = 1$ (equivalently, $\alpha = 0$), may be extended

to the case $0 < p < 1$ ($\alpha > 0$) mutatis mutandi, and we omit the details. First, as in [39], Lemma 8.3, we have that

$$(3.60) \quad \sup_Q \frac{1}{|Q|^{1+2\alpha/n}} \iint_{R_Q} |(t^2 L^*)^M e^{-t^2 L^*} g|^2 \frac{dxdt}{t} \leq C \|g\|_{\Lambda_{L^*}^\alpha(\mathbb{R}^n)}^2;$$

second, as in [39], Lemma 8.4, for f, g as above, the following Calderón reproducing formula is valid:

$$(3.61) \quad \langle g, f \rangle = C_M \lim_{\delta \rightarrow 0} \int_{\delta}^{1/\delta} \int_{\mathbb{R}^n} (t^2 L^*)^M e^{-t^2 L^*} g(x) \overline{t^2 L e^{-t^2 L} f(x)} \frac{dxdt}{t}.$$

At this point we follow [43]. Since $t^2 L e^{-t^2 L} f \in T^p$, we may invoke the result of [21] to obtain the decomposition

$$t^2 L e^{-t^2 L} f = \sum \lambda_j A_j,$$

where each A_j is a T^p atom, supported in a Carleson box R_{Q_j} , and where $\{\lambda_j\} \in \ell^p$, with

$$(3.62) \quad \left(\sum |\lambda_j|^p \right)^{1/p} \lesssim \|F\|_{T^p} \approx \|f\|_{\mathbb{H}_{L, mol, M}^p(\mathbb{R}^n)}.$$

Using (3.61), we then have

$$\begin{aligned} |\langle g, f \rangle| &\leq C \sum |\lambda_j| \iint_{\mathbb{R}_+^{n+1}} |(t^2 L^*)^M e^{-t^2 L^*} g(x)| |A_j(x, t)| \frac{dxdt}{t} \\ &\leq C \sum |\lambda_j| \left(\iint_{R_{Q_j}} |(t^2 L^*)^M e^{-t^2 L^*} g(x)|^2 \frac{dxdt}{t} \right)^{1/2} |Q_j|^{\frac{1}{2} - \frac{1}{p}} \\ &\leq C \sum |\lambda_j| \|g\|_{\Lambda_{L^*}^\alpha}, \end{aligned}$$

where in the second inequality we have used the definition of a T^p -atom (cf. (3.24)), and in the last inequality we have used (3.60) with $\alpha = n(1/p - 1)$. The desired bound (3.59) now follows from (3.62), since $p < 1$. \square

4. SQUARE FUNCTION CHARACTERIZATIONS AND INTERPOLATION.

Recall the square function definition of Hardy spaces given in (1.10)–(1.11). In fact, there is certain flexibility in the choice of the square function which gives an equivalent norm in $H_L^p(\mathbb{R}^n)$. It is possible to replace $\psi(\xi) = \xi e^{-\xi}$, $\xi = t^2 L$, in (1.10) by another function of ξ with holomorphic extension to an open sector of the complex plane, provided it has enough decay at zero and infinity. One way to see this is to re-prove the molecular decomposition of Hardy spaces, this time using a square function based on ψ , Lemma 2.28 and quadratic estimates in [50].

Now we present a different approach, via the connection with the tent spaces (cf. (3.15), (3.16)), again using fundamentally the ideas of [21]. In a different context this has been done in [10]. Here we will follow a similar path, pointing out the aspects which are particular to our setting.

Let $\omega < \mu < \pi/2$ and $\psi \in \Psi(\Sigma_\mu^0)$. According to the quadratic estimates in [50] the operator

$$(4.1) \quad Q_{\psi,L} f(x,t) := \psi(t^2 L) f(x), \quad (x,t) \in \mathbb{R}_+^{n+1},$$

is bounded from $L^2(\mathbb{R}^n)$ to $T^2(\mathbb{R}_+^{n+1})$. Then for every $\psi \in \Psi(\Sigma_\mu^0)$ the operator

$$(4.2) \quad \pi_{\psi,L} F(x) := \int_0^\infty \psi(t^2 L) F(x,t) \frac{dt}{t}, \quad x \in \mathbb{R}^n,$$

is well-defined for all $F \in T^2(\mathbb{R}_+^{n+1})$ and bounded from $T^2(\mathbb{R}_+^{n+1})$ to $L^2(\mathbb{R}^n)$ by duality. Indeed, the operator $\pi_{\psi,L}$ is the adjoint of the operator Q_{ψ,L^*} , and vice versa. In the sequel, for the sake of notational convenience, we shall sometimes omit the subscript L , and write merely Q_ψ , π_ψ when there is no chance of confusion.

Finally, for $\psi, \tilde{\psi} \in \Psi(\Sigma_\mu^0)$ and $f \in H^\infty(\Sigma_\mu^0)$ let $Q^f := Q_\psi \circ f \circ \pi_{\tilde{\psi}}$, i.e.,

$$(4.3) \quad Q^f F(x,s) := \int_0^\infty (\psi(s^2 L) f(L) \tilde{\psi}(t^2 L) F(\cdot, t))(x) \frac{dt}{t}, \quad (x,s) \in \mathbb{R}_+^{n+1}.$$

Then it follows from the observations above that Q^f is bounded in $T^2(\mathbb{R}_+^{n+1})$, with the norm bounded by $\|f\|_{L^\infty(\Sigma_\mu^0)}$. We will sometimes write Q in place of Q^f when $f = 1$.

Proposition 4.4. *Let $\mu \in (\omega, \pi/2)$. Then for every $\psi, \tilde{\psi} \in \Psi(\Sigma_\mu^0)$ and $f \in H^\infty(\Sigma_\mu^0)$ the operator Q^f originally defined on $T^2(\mathbb{R}_+^{n+1})$ extends by continuity to a bounded operator on $T^p(\mathbb{R}_+^{n+1})$ provided that either*

(1) $0 < p \leq 2$, $\psi \in \Psi_{\alpha,\beta}(\Sigma_\mu^0)$, $\tilde{\psi} \in \Psi_{\beta,\alpha}(\Sigma_\mu^0)$, or

(2) $2 \leq p < \infty$, $\psi \in \Psi_{\beta,\alpha}(\Sigma_\mu^0)$, $\tilde{\psi} \in \Psi_{\alpha,\beta}(\Sigma_\mu^0)$,

where $\alpha > 0$, $\beta > \frac{n}{2} (\max\{\frac{1}{p}, 1\} - \frac{1}{2})$. Moreover,

$$(4.5) \quad \|Q^f F\|_{T^p(\mathbb{R}_+^{n+1})} \leq C \|f\|_{L^\infty(\Sigma_\mu^0)} \|F\|_{T^p(\mathbb{R}_+^{n+1})}, \quad \text{for all } F \in T^p(\mathbb{R}_+^{n+1}).$$

One of the main ingredients in the proof of Proposition 4.4 is the following Lemma (cf. Lemma 3.7 in [10] and Lemma 2.22).

Lemma 4.6. *Let $\mu \in (\omega, \pi/2)$ and $\sigma_1, \sigma_2, \tau_1, \tau_2 > 0$. Suppose further that $\psi \in \Psi_{\sigma_1, \tau_1}(\Sigma_\mu^0)$, $\tilde{\psi} \in \Psi_{\sigma_2, \tau_2}(\Sigma_\mu^0)$ and $f \in H^\infty(\Sigma_\mu^0)$. Then for any $0 < a < \min\{\sigma_1, \tau_2\}$ and $0 < b < \min\{\sigma_2, \tau_1\}$ there is a family of operators $T_{s,t}$, $s, t > 0$ such that*

$$(4.7) \quad \psi(sL) f(L) \tilde{\psi}(tL) = \min \left\{ \left(\frac{s}{t} \right)^a, \left(\frac{t}{s} \right)^b \right\} T_{s,t},$$

where

(1) $\{T_{s,t}\}_{s \leq t}$ satisfy the L^2 off-diagonal estimates in t of order $\sigma_2 + a$ uniformly in $s \leq t$,

(2) $\{T_{s,t}\}_{t \leq s}$ satisfy the L^2 off-diagonal estimates in s of order $\sigma_1 + b$ uniformly in $t \leq s$,

with the constants bounded by $\|f\|_{L^\infty(\Sigma_\mu^0)}$.

Proof. Let us consider first $s \leq t$. Then

$$(4.8) \quad \psi(sL) f(L) \tilde{\psi}(tL) = \left(\frac{s}{t} \right)^a (sL)^{-a} \psi(sL) f(L) (tL)^a \tilde{\psi}(tL) =: \left(\frac{s}{t} \right)^a T_{s,t}.$$

The function $(s\xi)^{-a}\psi(s\xi)f(\xi)$, $\xi \in \Sigma_\mu^0$, belongs to $H^\infty(\Sigma_\mu^0)$ and

$$(4.9) \quad \|(s\xi)^{-a}\psi(s\xi)f(\xi)\|_{L^\infty(\Sigma_\mu^0)} \leq C\|f\|_{L^\infty(\Sigma_\mu^0)},$$

with the constant C independent of $s > 0$. Hence, by Lemma 2.28 the operators $\{T_{s,t}\}_{s \leq t}$ satisfy the L^2 off-diagonal estimates in t of order $\sigma_2 + a$ uniformly in $s \leq t$, with the constant bounded by $\|f\|_{L^\infty(\Sigma_\mu^0)}$. The case $s \geq t$ follows analogously, and their combination proves the Lemma. \square

Proof of Proposition 4.4. Let $0 < p \leq 2$. Using the Lemma above for any a, b such that $0 < a < \alpha$ and $0 < b < \beta$ one can write

$$(4.10) \quad Q^f F(x, s) = \int_0^\infty \min\left\{\left(\frac{s}{t}\right)^{2a}, \left(\frac{t}{s}\right)^{2b}\right\} T_{s^2, t^2} F(\cdot, t)(x) \frac{dt}{t}, \quad (x, s) \in \mathbb{R}_+^{n+1},$$

where

- (1) $\{T_{s,t}\}_{s \leq t}$ satisfy the L^2 off-diagonal estimates in t of order $\beta + a$ uniformly in $s \leq t$,
- (2) $\{T_{s,t}\}_{t \leq s}$ satisfy the L^2 off-diagonal estimates in s of order $\alpha + b$ uniformly in $t \leq s$.

with the constant bounded by $\|f\|_{L^\infty(\Sigma_\mu^0)}$. Note that the constants a, b can be chosen so that both $\alpha + b > \frac{n}{2} \left(\max\{\frac{1}{p}, 1\} - \frac{1}{2}\right)$ and $\beta + a > \frac{n}{2} \left(\max\{\frac{1}{p}, 1\} - \frac{1}{2}\right)$. Then there exist some $M > \frac{n}{2} \left(\max\{\frac{1}{p}, 1\} - \frac{1}{2}\right)$, $M \in \mathbb{R}$, and some $C > 0$ such that for arbitrary closed sets $E, F \subset \mathbb{R}^n$

$$(4.11) \quad \|T_{s^2, t^2} g\|_{L^2(F)} \leq C\|f\|_{L^\infty(\Sigma_\mu^0)} \min\left\{1, \frac{\max\{t, s\}}{\text{dist}(E, F)}\right\}^{2M} \|g\|_{L^2(E)},$$

for every $s, t > 0$ and every $g \in L^2(\mathbb{R}^n)$ supported in E .

The remainder of the proof follows the same path as that of Theorem 4.9 in [10]. Suppose first that $p \leq 1$. By density of $T^2(\mathbb{R}_+^n) \cap T^p(\mathbb{R}_+^n)$ in $T^p(\mathbb{R}_+^n)$ it is enough to establish an *a priori* estimate for $F(x, t) \in T^2(\mathbb{R}_+^n) \cap T^p(\mathbb{R}_+^n)$. We may then use the atomic decomposition of tent spaces in [21] (cf. Proposition 3.25 above) to reduce (4.5) to the atomic estimate

$$(4.12) \quad \|Q^f A\|_{T^p(\mathbb{R}_+^n)} \leq C\|f\|_{L^\infty(\Sigma_\mu^0)} \text{ uniformly for } T^p(\mathbb{R}_+^n)\text{-atoms } A.$$

Then one breaks down $Q^f A$ into a part close to the support of A and a part away from the support of A . Close to the support we use the boundedness of Q^f in $T^2(\mathbb{R}_+^n)$, and away from the support we use (4.11). The details can be recovered carefully following an analogous argument in [10]. Then the case $1 < p \leq 2$ follows by interpolation and the case $2 \leq p < \infty$ is obtained by duality. \square

Proposition 4.13. *Let $\mu \in (\omega, \pi/2)$ and $\psi \in \Psi(\Sigma_\mu^0)$. The operator $Q_{\psi, L}$ originally defined on $L^2(\mathbb{R}^n)$ by the formula (4.1) extends to a bounded operator*

$$(4.14) \quad Q_{\psi, L} : H_L^p(\mathbb{R}^n) \longrightarrow T^p(\mathbb{R}_+^{n+1}),$$

provided that

$$\text{either } (1) \ 0 < p \leq 2, \psi \in \Psi_{\alpha, \beta}(\Sigma_\mu^0), \quad \text{or } (2) \ 2 \leq p < \infty, \psi \in \Psi_{\beta, \alpha}(\Sigma_\mu^0),$$

where $\alpha > 0$ and $\beta > \frac{n}{2} \left(\max\{\frac{1}{p}, 1\} - \frac{1}{2} \right)$.

The operator $\pi_{\psi,L}$ defined on $T^2(\mathbb{R}_+^{n+1})$ by means of (4.2) extends to a bounded operator

$$(4.15) \quad \pi_{\psi,L} : T^p(\mathbb{R}_+^{n+1}) \longrightarrow H_L^p(\mathbb{R}^n),$$

provided that

$$\text{either (1) } 0 < p \leq 2, \psi \in \Psi_{\beta,\alpha}(\Sigma_\mu^0), \quad \text{or (2) } 2 \leq p < \infty, \psi \in \Psi_{\alpha,\beta}(\Sigma_\mu^0),$$

where $\alpha > 0$ and $\beta > \frac{n}{2} \left(\max\{\frac{1}{p}, 1\} - \frac{1}{2} \right)$.

Remark. Before proving the Proposition, we note that for $\psi, \tilde{\psi} \in \Psi(\Sigma_\mu^0)$ such that $\int_0^\infty \psi(t)\tilde{\psi}(t) \frac{dt}{t} = 1$, we have the following Calderón reproducing formula:

$$(4.16) \quad \pi_\psi \circ Q_{\tilde{\psi}} = \pi_{\tilde{\psi}} \circ Q_\psi = I \quad \text{in } L^2(\mathbb{R}^n).$$

Moreover, for every non-trivial $\psi \in \Psi(\Sigma_\mu^0)$, such $\tilde{\psi}$ can be found, for example, taking

$$(4.17) \quad \tilde{\psi}(z) := \bar{\psi}(z) \left(\int_0^\infty |\psi(t)|^2 \frac{dt}{t} \right)^{-1}, \quad z \in \Sigma_\mu^0.$$

Proof of Proposition 4.13. Let $\psi_0(z) = ze^{-z}$, $z \in \Sigma_\mu^0$. Then the boundedness of the corresponding Q_{ψ_0} in (4.14) for $0 < p \leq 2$ follows directly from the definitions of $H_L^p(\mathbb{R}^n)$, $0 < p \leq 2$, and $T^p(\mathbb{R}_+^{n+1})$.

Now take any $\psi \in \Psi_{\beta,\alpha}$ and $0 < p \leq 2$. For every $F \in T^p(\mathbb{R}_+^{n+1}) \cap T^2(\mathbb{R}_+^{n+1})$

$$(4.18) \quad \|\pi_\psi F\|_{H_L^p(\mathbb{R}^n)} = \|Q_{\psi_0} \circ \pi_\psi F\|_{T^p(\mathbb{R}_+^{n+1})},$$

and due to Proposition 4.4 the last expression above is controlled by $\|F\|_{T^p(\mathbb{R}_+^{n+1})}$. Then (4.15) follows by a density argument.

Next, let $\psi \in \Psi_{\alpha,\beta}(\Sigma_\mu^0)$, $0 < p \leq 2$. Since L^2 is dense in H_L^p , it is enough to prove that

$$(4.19) \quad \|Q_\psi f\|_{T^p(\mathbb{R}_+^{n+1})} \leq C \|f\|_{H_L^p(\mathbb{R}^n)}, \quad \text{for every } f \in H_L^p \cap L^2.$$

By definition $Q_{\psi_0} f \in T^p(\mathbb{R}_+^{n+1})$ for every $f \in H_L^p \cap L^2$. Let M be the smallest integer larger than $\frac{n}{2} \left(\max\{\frac{1}{p}, 1\} - \frac{1}{2} \right)$ and $\tilde{\psi}_0(\xi) := \xi^M e^{-\xi}$, $\xi \in \Sigma_\mu^0$. Then $\int_0^\infty \psi_0(t)\tilde{\psi}_0(t) \frac{dt}{t} = C_M$, and hence, by (4.16) we have

$$(4.20) \quad f = \frac{1}{C_M} \pi_{\tilde{\psi}_0} \circ Q_{\psi_0} f \quad \text{for } f \in L^2.$$

Note that $\tilde{\psi}_0 \in \Psi_{M,N}$ for every $N > 0$. Therefore,

$$\|Q_\psi f\|_{T^p(\mathbb{R}_+^{n+1})} = C \|Q_\psi \circ \pi_{\tilde{\psi}_0} \circ Q_{\psi_0} f\|_{T^p(\mathbb{R}_+^{n+1})} \leq C \|Q_{\psi_0} f\|_{T^p(\mathbb{R}_+^{n+1})} = C \|f\|_{H_L^p(\mathbb{R}^n)},$$

where the inequality is a consequence of Proposition 4.4.

For $p > 2$ we use the duality between the operators π and Q . □

Remark. We would like to mention that in [41] the authors developed an alternative approach to (4.14).

Remark. The results of the Proposition 4.13 lead to an alternative molecular decomposition of Hardy spaces, defining molecules as the images of the atoms of tent spaces under π_ψ for appropriate ψ (cf. [10]).

Remark. The tent spaces have an appropriate counterpart when $p = \infty$ and the results of Proposition 4.13 extend to this case as well (see [39]).

Proposition 4.13, in particular, provides the square function characterization for the Hardy spaces H_L^p with $p > 2$, which were originally defined by duality (1.12).

Corollary 4.21. *Let ψ be a nontrivial function satisfying either*

- (1) $0 < p \leq 2$, $\psi \in \Psi_{\alpha,\beta}(\Sigma_\mu^0)$, or
- (2) $2 \leq p < \infty$, $\psi \in \Psi_{\beta,\alpha}(\Sigma_\mu^0)$,

where $\alpha > 0$ and $\beta > \frac{n}{2} \left(\max\{\frac{1}{p}, 1\} - \frac{1}{2} \right)$. Define $H_{\psi,L}^p(\mathbb{R}^n)$ to be the completion of the space

$$\mathbb{H}_{\psi,L}^p(\mathbb{R}^n) := \{f \in L^2(\mathbb{R}^n) : Q_{\psi,L}f \in T^p(\mathbb{R}_+^{n+1})\},$$

with respect to the norm

$$(4.22) \quad \|f\|_{H_{\psi,L}^p(\mathbb{R}^n)} := \|Q_{\psi,L}f\|_{T^p(\mathbb{R}_+^{n+1})} = \left\| \left(\iint_{\Gamma(\cdot)} |\psi(t^2L)f(y)|^2 \frac{dydt}{t^{n+1}} \right)^{1/2} \right\|_{L^p(\mathbb{R}^n)}.$$

Then $H_L^p(\mathbb{R}^n) = H_{\psi,L}^p$, with equivalence of norms.

Proof. For $0 < p \leq 2$, by the definitions it is enough to establish equality of the dense spaces $L^2(\mathbb{R}^n) \cap H_L^p(\mathbb{R}^n)$ and $\mathbb{H}_{\psi,L}^p(\mathbb{R}^n)$, with equivalence of norms. One direction is precisely the estimate (4.19) above. The opposite direction is proved in exactly the same way as (4.19), by simply interchanging the roles of ψ and ψ_0 , and observing that the reproducing formula (4.20) is still valid (with a different constant), for the same choice of $\tilde{\psi}_0$, but with ψ_0 replaced by ψ . We omit the routine details.

The case $2 < p < \infty$ is slightly more involved. We begin by claiming that $L^2(\mathbb{R}^n) \cap H_L^p(\mathbb{R}^n)$ is dense in $H_L^p(\mathbb{R}^n)$ (this fact is immediate by definition only for the range $0 < p \leq 2$). To prove the claim, let χ_K denote the characteristic function of the set $\{(x, t) \in \mathbb{R}_+^{n+1} : |x| < K, 1/K < t < K\}$, so that for $F \in T^p$, $2 < p < \infty$, we have that $F_K := F\chi_K \in T^2 \cap T^p$, and also that $F_K \rightarrow F$ in T^p . Now given $\psi \in \Psi_{\beta,\alpha}(\Sigma_\mu^0)$, choose $\tilde{\psi} \in \Psi_{\alpha,\beta}(\Sigma_\mu^0)$ satisfying the reproducing formula (4.16). Then by (4.14) and (4.15), the reproducing formula extends to $H_{L^*}^{p'}(\mathbb{R}^n)$ (since $L^2 \cap H_{L^*}^{p'}$ is dense in the latter space), and thus by duality to $H_L^p(\mathbb{R}^n)$. Consequently, given $f \in H_L^p(\mathbb{R}^n)$, $2 < p < \infty$, we may write

$$f = \pi_{\tilde{\psi},L}^- \circ Q_{\psi,L}f = \lim_{K \rightarrow \infty} \pi_{\tilde{\psi},L}^- \left((Q_{\psi,L}f)\chi_K \right),$$

where by our previous remarks and (4.15), the limit exists in $H_L^p(\mathbb{R}^n)$. Moreover, $F_K := (Q_{\psi,L}f)\chi_K \in T^2 \cap T^p$, so that $\pi_{\tilde{\psi},L}F_K \in L^2(\mathbb{R}^n) \cap H_L^p(\mathbb{R}^n)$. Thus, the claimed density holds.

Therefore, it is enough to prove that $L^2(\mathbb{R}^n) \cap H_L^p(\mathbb{R}^n) = \mathbb{H}_{\psi,L}^p(\mathbb{R}^n)$, with equivalence of norms. One direction follows immediately from (4.14). We now proceed to establish the other direction, namely that for $f \in \mathbb{H}_{\psi,L}^p(\mathbb{R}^n)$, we have

$$\|f\|_{H_L^p(\mathbb{R}^n)} \lesssim \|Q_{\psi,L}f\|_{T^p(\mathbb{R}_+^{n+1})}.$$

In turn, by the definition of $H_L^p(\mathbb{R}^n)$, $2 < p < \infty$, as a dual space, it is enough to show that for $g \in L^2(\mathbb{R}^n) \cap H_{L^*}^{p'}(\mathbb{R}^n)$, we have

$$(4.23) \quad \left| \int_{\mathbb{R}^n} f \bar{g} \right| \lesssim \|Q_{\psi,L}f\|_{T^p(\mathbb{R}_+^{n+1})} \|g\|_{H_{L^*}^{p'}(\mathbb{R}^n)}.$$

To this end, given $\psi \in \Psi_{\beta,\alpha}(\Sigma_\mu^0)$, as above we choose $\tilde{\psi} \in \Psi_{\alpha,\beta}(\Sigma_\mu^0)$ satisfying the reproducing formula (4.16), so that

$$\begin{aligned} \left| \int_{\mathbb{R}^n} f \bar{g} \right| &= \left| \int_{\mathbb{R}^n} \pi_{\tilde{\psi},L} \circ Q_{\psi,L} f \bar{g} \right| \\ &\leq \|Q_{\psi,L}f\|_{T^p(\mathbb{R}_+^{n+1})} \|Q_{\tilde{\psi},L^*}g\|_{T^{p'}(\mathbb{R}_+^{n+1})} \\ &\leq C \|Q_{\psi,L}f\|_{T^p(\mathbb{R}_+^{n+1})} \|g\|_{H_{L^*}^{p'}(\mathbb{R}^n)}, \end{aligned}$$

as desired, where in the last step we have used (4.19). \square

Let us now turn to the interpolation property. One of the most important features of the classical Hardy spaces lies in the fact that they form a complex interpolation scale including, in particular, $L^p(\mathbb{R}^n)$ for some values of p (in fact, $1 < p < \infty$). It has to be mentioned that Calderón's original method of complex interpolation was defined for Banach spaces and could not be immediately extended to the case when the underlying spaces were only quasi-Banach ($p < 1$). One reason for that is a possible failure of the maximum modulus principle in quasi-Banach spaces. Over the years there have been developed several approaches to this issue (see, in particular, [18], [42], [23], [36] regarding the classical Hardy spaces). Here we are going to employ an extension of the complex interpolation method to analytically convex spaces described in [44], [45].

Lemma 4.24. *For each $0 < \theta < 1$ and $0 < p_0, p_1 < +\infty$,*

$$(4.25) \quad \left[H_L^{p_0}(\mathbb{R}^n), H_L^{p_1}(\mathbb{R}^n) \right]_\theta = H_L^p(\mathbb{R}^n), \quad \text{where } 1/p = (1-\theta)/p_0 + \theta/p_1,$$

and

$$(4.26) \quad \left[H_L^{p_0}(\mathbb{R}^n), BMO_L(\mathbb{R}^n) \right]_\theta = H_L^p(\mathbb{R}^n),$$

$$0 < \theta < 1, \quad 0 < p_0 < +\infty, \quad 1/p = (1-\theta)/p_0.$$

Proof. The proof of (4.25) is a combination of an analogous result for the tent spaces and Proposition 4.13. First of all, (3.22) holds for all $0 < p_0 < p_1 \leq +\infty$ (this is stated in [21], Proposition 6, p. 326; complete details are given in [19]). On

the other hand, by Proposition 4.13, if $0 < p < \infty$, Hardy spaces are the retracts of the corresponding tent spaces, i.e. there exists an operator mapping any tent space to the corresponding Hardy space and having the right inverse (actually, this is also true for $p = \infty$, if we designate $BMO_L(\mathbb{R}^n) =: H_L^\infty(\mathbb{R}^n)$; the proof is implicit in [39], however, we shall not need to make explicit use of this fact in the sequel). More precisely, given any pair $0 < p_0 < p_1 < \infty$, we can take $\psi \in \Psi_{\beta,\beta}$, where $\beta > \frac{n}{2} \left(\max\{\frac{1}{p_0}, 1\} - \frac{1}{2} \right)$ and $\tilde{\psi} \in \Psi_{\beta,\beta}$ as in (4.17). Then for all p between p_0 and p_1 the operator π_ψ maps T^p to H_L^p , and $Q_{\tilde{\psi}} : H_L^p \rightarrow T^p$ is its right inverse. Therefore, (3.22) implies (4.25) once we make sure that $T^{p_0}(\mathbb{R}^{n+1}) + T^{p_1}(\mathbb{R}_+^{n+1})$ is analytically convex (see Lemma 7.11 in [45]). This, however, follows from Theorem 7.9 in [45] and the discussion in [19]. The space BMO_L can then be incorporated by duality and Wolff's reiteration theorem [59], once we have shown that, given any fixed $p_0 > 0$, there is some large ambient Banach space into which every $H_L^p(\mathbb{R}^n)$, $p_0 \leq p < \infty$, and also $BMO_L(\mathbb{R}^n)$, may be continuously embedded. We shall establish the existence of such an ambient space in an appendix (cf. Section 10 below). \square

5. RIESZ TRANSFORM CHARACTERIZATION OF HARDY SPACES.

Let us recall that for a given operator L the interval $(p_-(L), p_+(L))$ is the interior of the interval of L^p -boundedness of the heat semigroup and $2 + \varepsilon(L)$ is an upper bound for the interval of L^p -boundedness of the Riesz transform. As pointed out in the introduction, we have

$$(5.1) \quad \nabla L^{-1/2} : L^p(\mathbb{R}^n) \longrightarrow L^p(\mathbb{R}^n) \iff p_-(L) < p < 2 + \varepsilon(L),$$

and the bounds $p_-(L) < \frac{2n}{n+2}$, $\varepsilon(L) > 0$ are sharp in the sense of Corollary 2.20. In the present section we aim to extend (5.1) to other values of p , passing to the Hardy H_L^p spaces, and to prove the reverse estimate for a certain range of p , thus establishing for such p the equivalence of the spaces $H_L^p(\mathbb{R}^n)$ and $H_{L,\text{Riesz}}^p(\mathbb{R}^n)$ (cf. (1.11) and (1.20)) Our main result in this section is the following.

Theorem 5.2. *Let $1 < r \leq 2$ be such that the family $\{e^{-tL}\}_{t>0}$ satisfies $L^r - L^2$ off-diagonal estimates. We then have*

$$(5.3) \quad H_L^p(\mathbb{R}^n) = H_{L,\text{Riesz}}^p(\mathbb{R}^n), \quad \frac{rn}{n+r} < p < 2 + \varepsilon(L)$$

Moreover, we have the following equivalence of norms:

$$(5.4) \quad \|f\|_{H_L^p(\mathbb{R}^n)} \approx \|\nabla L^{-1/2} f\|_{L^p(\mathbb{R}^n)}, \quad \max\left\{1, \frac{rn}{n+r}\right\} < p < 2 + \varepsilon(L)$$

and if $rn/(n+r) \leq 1$, then

$$(5.5) \quad \|f\|_{H_L^p(\mathbb{R}^n)} \approx \|\nabla L^{-1/2} f\|_{H^p(\mathbb{R}^n)}, \quad \frac{rn}{n+r} < p \leq 1.$$

Remark. Note that, in particular, (5.4) holds for every p such that $\max\left\{1, \frac{p_-(L)n}{n+p_-(L)}\right\} < p < 2 + \varepsilon(L)$, and if $\frac{p_-(L)n}{n+p_-(L)} < 1$, then (5.5) holds for every p such that $\frac{p_-(L)n}{n+p_-(L)} < p \leq 1$.

The proof of the Theorem will be split into Propositions 5.6–5.34. Let us start with the case $p \leq 1$. For the sake of notational convenience, given $p \in (0, 1]$, we shall throughout this section fix $M > (n/2)(1/p - 1/2)$ and $\varepsilon > 0$ (recall that, as we have seen, any such choice leads to an equivalent H_L^p space), and we may therefore refer to (H_L^p, ε, M) -molecules simply as H_L^p -molecules. The first result concerns the boundedness of the Riesz transform.

Proposition 5.6. *For every p such that $\frac{n}{n+1} < p \leq 1$, there is a constant C depending only on n, p and ellipticity (and our fixed choices of M and ε), such that the Riesz transform $\nabla L^{-1/2}$, defined initially on $L^2 \cap H_L^p(\mathbb{R}^n) = \mathbb{H}_{L, \text{mol}, M}^p(\mathbb{R}^n)$ (cf. (3.7)), satisfies*

$$(5.7) \quad \|\nabla L^{-1/2} f\|_{H^p(\mathbb{R}^n)} \leq C \|f\|_{\mathbb{H}_{L, \text{mol}, M}^p(\mathbb{R}^n)},$$

and therefore extends to a bounded operator $\nabla L^{-1/2} : H_L^p(\mathbb{R}^n) \rightarrow H^p(\mathbb{R}^n)$.

Proof. We begin by recalling that the classical Hardy spaces can be characterized via a molecular decomposition (see, e.g., [22]). Our Theorem 3.5 with $L = -\Delta$ provides one such characterization, but a more traditional version is as follows.

The function $m \in L^2(\mathbb{R}^n)$ is an H^p -molecule, $0 < p \leq 1$, if it satisfies (3.3) for $k = 0$ and

$$(5.8) \quad \int_{\mathbb{R}^n} x^\alpha m(x) dx = 0, \quad 0 \leq |\alpha| \leq \tilde{M},$$

for some $\tilde{M} \in \mathbb{N} \cup \{0\}$ such that $\tilde{M} \geq [n(1/p - 1)]$, with $[\gamma]$ denoting the integer part of $\gamma \in \mathbb{R}$. Given $p \in (0, 1]$, fix some \tilde{M} as above. Then the classical real variable Hardy space can be realized as

$$(5.9) \quad H^p(\mathbb{R}^n) = \left\{ \sum_{i=0}^{\infty} \lambda_j m_j : \{\lambda_j\}_{j=0}^{\infty} \in \ell^p \text{ and } m_j \text{ are } H^p\text{-molecules} \right\},$$

with

$$\|f\|_{H^p(\mathbb{R}^n)} \approx \inf \left\{ \left(\sum_{j=0}^{\infty} |\lambda_j|^p \right)^{1/p} \right\},$$

where the infimum runs over all decompositions $f = \sum_{j=0}^{\infty} \lambda_j m_j$, converging in the space of tempered distributions \mathcal{S}' , such that $\{\lambda_j\}_{j=0}^{\infty} \in \ell^p$ and each m_j is an H^p molecule. We do not know if this particular version of molecular decomposition is explicitly stated anywhere but it readily follows from the classical arguments (see [22], [57], [34]).

Having these facts at hand, we first show that the Riesz transform maps H_L^p -molecules into H^p -molecules. Let $m \in L^2(\mathbb{R}^n)$ be an H_L^p -molecule associated with some cube $Q \in \mathbb{R}^n$ (and $M > \frac{n}{2}(\frac{1}{p} - \frac{1}{2})$, $\varepsilon > 0$ fixed as above). Then

$$(5.10) \quad \|\nabla L^{-1/2} m\|_{L^2(2Q)} \leq C \|m\|_{L^2(\mathbb{R}^n)} \leq C l(Q)^{n/2-n/p},$$

using boundedness of $\nabla L^{-1/2}$ in $L^2(\mathbb{R}^n)$. Next, for $i \geq 2$

$$\|\nabla L^{-1/2} m\|_{L^2(\mathcal{S}_i(Q))} \leq \|\nabla L^{-1/2} (I - e^{-l(Q)^2 L})^M m\|_{L^2(\mathcal{S}_i(Q))}$$

$$(5.11) \quad +\|\nabla L^{-1/2}[I - (I - e^{-l(Q)^2 L})^M]m\|_{L^2(S_i(Q))} =: I + II.$$

According to Theorem 3.2 in [39] (see also Lemma 2.2 in [38]), for all closed sets E, F in \mathbb{R}^n with $\text{dist}(E, F) > 0$, if $f \in L^2(\mathbb{R}^n)$ is supported in E , then

$$(5.12) \quad \|\nabla L^{-1/2}(I - e^{-tL})^M f\|_{L^2(F)} \leq C \left(\frac{t}{\text{dist}(E, F)^2}\right)^M \|f\|_{L^2(E)}, \quad \forall t > 0,$$

$$(5.13) \quad \|\nabla L^{-1/2}(tLe^{-tL})^M f\|_{L^2(F)} \leq C \left(\frac{t}{\text{dist}(E, F)^2}\right)^M \|f\|_{L^2(E)}, \quad \forall t > 0.$$

Therefore,

$$(5.14) \quad \begin{aligned} I &\leq \|\nabla L^{-1/2}(I - e^{-l(Q)^2 L})^M(m\chi_{2^{i-2}Q})\|_{L^2(S_i(Q))} \\ &\quad + \|\nabla L^{-1/2}(I - e^{-l(Q)^2 L})^M(m\chi_{\mathbb{R}^n \setminus 2^{i-2}Q})\|_{L^2(S_i(Q))} \\ &\leq C2^{-2Mi}\|m\|_{L^2(2^{i-2}Q)} + C\|m\|_{L^2(\mathbb{R}^n \setminus 2^{i-2}Q)} \\ &\leq C2^{-2Mi}l(Q)^{n/2-n/p} + C(2^i l(Q))^{n/2-n/p} 2^{-i\epsilon}. \end{aligned}$$

Since $M > \frac{n}{2}(\frac{1}{p} - \frac{1}{2})$, the estimate (5.14) implies

$$(5.15) \quad I \leq C(2^i l(Q))^{n/2-n/p} 2^{-i\epsilon},$$

where $\epsilon = \min\{\epsilon, 2M - n/p + n/2\} > 0$.

Turning to the second part of (5.11), we observe that

$$(5.16) \quad \begin{aligned} &\|\nabla L^{-1/2}[I - (I - e^{-l(Q)^2 L})^M]m\|_{L^2(\mathbb{R}^n)} \leq C \sup_{1 \leq k \leq M} \|\nabla L^{-1/2} e^{-kl(Q)^2 L} m\|_{L^2(\mathbb{R}^n)} \\ &\leq C \sup_{1 \leq k \leq M} \left\| \nabla L^{-1/2} \left(\frac{k}{M} l(Q)^2 L e^{-\frac{k}{M} l(Q)^2 L} \right)^M (l(Q)^{-2} L^{-1})^M m \right\|_{L^2(\mathbb{R}^n)}. \end{aligned}$$

This allows to employ the argument above, using (5.13) in place of (5.12), to prove an analogue of (5.15) for the expression II .

Finally, the vanishing moment condition (5.8) is satisfied, since

$$(5.17) \quad \int_{\mathbb{R}^n} \nabla L^{-1/2} m(x) dx = 0,$$

and one can take $\tilde{M} = 0$ when $p > \frac{n}{n+1}$.

So far, we have established that Riesz transform maps H_L^p -molecules into H^p -molecules for $p \in (\frac{n}{n+1}, 1]$. Let us now show that this implies the desired estimate (5.7). To this end, let $f \in \mathbb{H}_{L, mol, M}^p(\mathbb{R}^n)$, so that by definition we may select an L^2 convergent molecular decomposition $f = \sum_{i=0}^{\infty} \lambda_i m_i$, where each m_i is an H_L^p -molecule, such that

$$\|f\|_{\mathbb{H}_{L, mol, M}^p(\mathbb{R}^n)} \approx \left(\sum_{i=0}^{\infty} |\lambda_i|^p \right)^{1/p}.$$

By the L^2 convergence of the sum, we have that

$$\nabla L^{-1/2} f = \sum \lambda_i (\nabla L^{-1/2} m_i) =: \sum \lambda_i \tilde{m}_i,$$

where by the preceding argument each \tilde{m}_i is a classical H^p -molecule, and where the last sum also converges in L^2 (hence in \mathcal{S}'). The bound (5.7) then follows immediately by the molecular characterization of classical H^p . This finishes the proof. \square

Proposition 5.18. *Let $1 < r \leq 2$ be such that the family $\{e^{-tL}\}_{t>0}$ satisfies $L^r - L^2$ off-diagonal estimates. Then for every $p \leq 1$ such that $p > \frac{rn}{n+r}$*

$$(5.19) \quad \|h\|_{H^p_L(\mathbb{R}^n)} \leq C \|\nabla L^{-1/2} h\|_{H^p(\mathbb{R}^n)}$$

for every $h \in L^2(\mathbb{R}^n) \cap H^p_{L,Riesz}(\mathbb{R}^n)$. In particular, if $\frac{p-(L)n}{n+p-(L)} < 1$, then (5.19) holds for every p such that $\frac{p-(L)n}{n+p-(L)} < p \leq 1$.

Remark: Combining Propositions 5.6 and 5.18, we therefore obtain (1.22), for $f \in L^2(\mathbb{R}^n)$, and thus by density, we obtain (1.19) in the case $p \leq 1$.

Proof. Let $h \in H^p_{L,Riesz}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, and set

$$f := L^{-1/2} h.$$

Since, in particular, $h \in L^2(\mathbb{R}^n)$, we have that f is well defined: indeed, the solution of the Kato square root problem [9] (cf. (1.4)), implies that $f \in \dot{W}^{1,2}(\mathbb{R}^n)$ (cf. (1.31)).

Let us denote

$$(5.20) \quad S_1 h(x) := \left(\iint_{\Gamma(x)} |t \sqrt{L} e^{-t^2 L} h(y)|^2 \frac{dy dt}{t^{n+1}} \right)^{1/2}, \quad x \in \mathbb{R}^n.$$

Then by Corollary 4.21

$$(5.21) \quad \|S_1 h\|_{L^p(\mathbb{R}^n)} \approx \|h\|_{H^p_L(\mathbb{R}^n)}, \quad 0 < p \leq 2.$$

Hence, matters are reduced to proving the estimate

$$(5.22) \quad \|S_1 \sqrt{L} f\|_{L^p(\mathbb{R}^n)} \leq C \|\nabla f\|_{H^p(\mathbb{R}^n)}, \quad \frac{rn}{n+r} < p \leq 1.$$

Let us recall the ‘‘Hardy-Sobolev’’ spaces

$$(5.23) \quad H^{1,p}(\mathbb{R}^n) := \{f \in \mathcal{S}'(\mathbb{R}^n)/\mathbb{C} : \nabla f \in H^p(\mathbb{R}^n)\},$$

where $\mathcal{S}'(\mathbb{R}^n)/\mathbb{C}$ is the space of tempered distributions modulo constants. The space $H^{1,p}(\mathbb{R}^n)$ may be identified with the corresponding Triebel-Lizorkin spaces (see, e.g., [51] or Section 8.2 of the current paper), and thus admits an atomic decomposition [32]. Specifically, a function a satisfying

$$(5.24) \quad \text{supp } a \subset Q, \quad \|\nabla a\|_{L^2(\mathbb{R}^n)} \leq l(Q)^{n/2-n/p},$$

is called an $H^{1,p}$ -atom, $n/(n+1) < p \leq 1$ (as usual, for smaller p one has to impose an extra vanishing moment condition). Then

$$(5.25) \quad H^{1,p}(\mathbb{R}^n) = \left\{ \sum_{i=0}^{\infty} \lambda_j a_j : \{\lambda_j\}_{j=0}^{\infty} \in \ell^p \text{ and } a_j \text{ are } H^{1,p}\text{-atoms} \right\},$$

with the series understood in the sense of convergence in $\mathcal{S}'(\mathbb{R}^n)/\mathbb{C}$, and

$$\|f\|_{H^{1,p}(\mathbb{R}^n)} \approx$$

$$\inf\left\{\left(\sum_{j=0}^{\infty} |\lambda_j|^p\right)^{1/p} : f = \sum_{j=0}^{\infty} \lambda_j m_j, \{\lambda_j\}_{j=0}^{\infty} \in \ell^p \text{ and } a_j \text{ are } H^{1,p}\text{-atoms}\right\}.$$

We now claim that it is enough to show that

$$(5.26) \quad \|S_1 \sqrt{L} a\|_{L^p(\mathbb{R}^n)} \leq C, \quad \text{for every } H^{1,p}\text{-atom } a, \quad p > \frac{rn}{n+r}, \quad p \leq 1,$$

where C is a constant not depending on a . To see that (5.26) suffices to obtain the conclusion of the Proposition, we proceed as follows. We note that in the standard constructive tent space proof of the atomic decomposition of $H^{1,p}$, one obtains, much as in the proof of Step 2 of Theorem 3.5 above, that for f in the dense subspace $\dot{W}^{1,2}(\mathbb{R}^n) \cap H^{1,p}(\mathbb{R}^n)$, there is a decomposition $f = \sum \lambda_j a_j$, converging in $\dot{W}^{1,2}(\mathbb{R}^n)$, where each a_j is an $H^{1,p}$ -atom, and where

$$\sum |\lambda_j|^p \lesssim \|\nabla f\|_{H^p(\mathbb{R}^n)}^p.$$

By the solution of the Kato square root problem [9] (cf. (1.4)), and the L^2 boundedness of the square function S_1 , we have that

$$S_1 \sqrt{L} : \dot{W}^{1,2}(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n),$$

so using the $\dot{W}^{1,2}$ convergence of the atomic sum, we obtain that pointwise *a.e.*,

$$S_1 \sqrt{L} f \leq \sum |\lambda_j| S_1 \sqrt{L} a_j.$$

Thus, (5.26) implies (5.22).

It remains to prove (5.26). For $j \in \mathbb{N} \cup \{0\}$ let $\mathcal{R}(S_j(Q)) := \bigcup_{x \in S_j(Q)} \Gamma(x)$ be a saw-tooth region based on $S_j(Q) \subset \mathbb{R}^n$. Then

$$\begin{aligned} \|S_1 \sqrt{L} a\|_{L^p(\mathbb{R}^n)}^p &\leq \sum_{j=0}^{\infty} (2^j l(Q))^{n(1-\frac{p}{2})} \left(\int_{S_j(Q)} \iint_{\Gamma(x)} |t L e^{-t^2 L} a(y)|^2 \frac{dy dt}{t^{n+1}} dx \right)^{\frac{p}{2}} \\ &\leq C \sum_{j=3}^{\infty} (2^j l(Q))^{n(1-\frac{p}{2})} \left(\iint_{\mathcal{R}(S_j(Q))} |t L e^{-t^2 L} a(y)|^2 \frac{dy dt}{t} \right)^{\frac{p}{2}} \\ &\quad + C l(Q)^{n(1-\frac{p}{2})} \|S_1 \sqrt{L} a\|_{L^2(4Q)}^p \\ &\leq C \sum_{j=3}^{\infty} (2^j l(Q))^{n(1-\frac{p}{2})} \left(\int_{\mathbb{R}^n \setminus Q_{j-2}} \int_0^{\infty} |t^2 L e^{-t^2 L} a(y)|^2 \frac{dy dt}{t^3} \right)^{\frac{p}{2}} \\ &\quad + C \sum_{j=3}^{\infty} (2^j l(Q))^{n(1-\frac{p}{2})} \left(\int_{Q_{j-2}} \int_{c2^j l(Q)}^{\infty} |t^2 L e^{-t^2 L} a(y)|^2 \frac{dy dt}{t^3} \right)^{\frac{p}{2}} \\ (5.27) \quad &+ C l(Q)^{n(1-\frac{p}{2})} \|S_1 \sqrt{L} a\|_{L^2(4Q)}^p =: I + II + III. \end{aligned}$$

Then, since S_1 is bounded in $L^2(\mathbb{R}^n)$,

$$(5.28) \quad III \leq Cl(Q)^{n(1-\frac{p}{2})} \|\sqrt{L}a\|_{L^2(\mathbb{R}^n)}^p \leq Cl(Q)^{n(1-\frac{p}{2})} \|\nabla a\|_{L^2(\mathbb{R}^n)}^p \leq C.$$

Going further, observe that

$$(5.29) \quad \|a\|_{L^2(Q)} \leq l(Q)^{n/2-n/p+1},$$

for every $H^{1,p}$ -atom a by (5.24), Sobolev inequality and Hölder inequality. Then Lemma 2.26, (5.29) and another application of Hölder inequality imply that

$$(5.30) \quad \begin{aligned} II &\leq C \sum_{j=3}^{\infty} (2^j l(Q))^{n(1-\frac{p}{2})} \left(\int_{c2^j l(Q)}^{\infty} t^{2(\frac{n}{2}-\frac{n}{r})} \frac{dt}{t^3} \right)^{\frac{p}{2}} \|a\|_{L^r(Q)}^p \\ &\leq C \sum_{j=3}^{\infty} (2^j l(Q))^{n(1-\frac{p}{2})} (2^j l(Q))^{p(\frac{n}{2}-\frac{n}{r}-1)} \|a\|_{L^r(Q)}^p \leq C, \end{aligned}$$

provided $p > \frac{rn}{n+r}$.

Finally, in order to handle I , we split the integral in t into two parts, corresponding to $0 < t < 2^j l(Q)$ and $t \geq 2^j l(Q)$, respectively. The second part can be estimated closely following the argument in (5.30). As for the first one,

$$(5.31) \quad \begin{aligned} &\sum_{j=3}^{\infty} (2^j l(Q))^{n(1-\frac{p}{2})} \left(\int_{\mathbb{R}^n \setminus Q_{j-2}} \int_0^{2^j l(Q)} |t^2 L e^{-t^2 L} a(y)|^2 \frac{dy dt}{t^3} \right)^{\frac{p}{2}} \\ &\leq C \sum_{j=3}^{\infty} (2^j l(Q))^{n(1-\frac{p}{2})} \left(\int_0^{2^j l(Q)} t^{2(\frac{n}{2}-\frac{n}{r})} e^{-\frac{(2^j l(Q))^2}{ct^2}} \frac{dt}{t^3} \right)^{\frac{p}{2}} \|a\|_{L^r(\mathbb{R}^n)}^p \leq C, \end{aligned}$$

using $L^r - L^2$ off-diagonal estimates. This completes the proof. \square

Now we turn to the case $p > 1$.

Proposition 5.32. *The Riesz transform of the operator L satisfies*

$$(5.33) \quad \nabla L^{-1/2} : H_L^p(\mathbb{R}^n) \longrightarrow L^p(\mathbb{R}^n) \quad \text{for} \quad 1 < p < 2 + \varepsilon(L).$$

Proof. Since $p_+(L) \geq 2 + \varepsilon(L)$ (see [7], §4.4), the property (5.1) and (1.15) yield (5.33) for $p_-(L) < p < 2 + \varepsilon(L)$. Then the full range of p in (5.33) can be achieved by interpolation (Lemma 4.24) with the result of Proposition 5.6. \square

Proposition 5.34. *Let $1 < r \leq 2$ be such that the family $\{e^{-tL}\}_{t>0}$ satisfies $L^r - L^2$ off-diagonal estimates. Then for all p satisfying $\max\{1, \frac{rn}{n+r}\} < p < p_+(L)$,*

$$(5.35) \quad \|h\|_{H_L^p(\mathbb{R}^n)} \leq C \|\nabla L^{-1/2} h\|_{L^p(\mathbb{R}^n)},$$

for every $h \in L^2(\mathbb{R}^n) \cap H_{L,Riesz}^p(\mathbb{R}^n)$.

In particular, (5.35) holds for every p such that $\max\{1, \frac{p_-(L)n}{n+p_-(L)}\} < p < 2 + \varepsilon(L)$.

Remark. This Proposition is a sharpened version of [7], Proposition 5.10: in the latter, the left hand side of (5.35) is replaced by the L^p norm. Our proof is based on the circle of ideas developed in [7], but the estimates we seek are somewhat more

delicate, since H_L^p is “strictly smaller” than L^p (in the sense of Proposition 9.1 (ii) below) in the range $1 < p \leq p_-(L)$.

Proof. Step I. By (5.1) applied to L^* , and a standard duality argument using the fact that $(p_-(L^*))' = p_+(L)$, we have

$$(5.36) \quad \|h\|_{L^p(\mathbb{R}^n)} \leq C \|\nabla L^{-1/2} h\|_{L^p(\mathbb{R}^n)}, \quad 2 \leq p < p_+(L),$$

in which range of p we have $H_L^p(\mathbb{R}^n) = L^p(\mathbb{R}^n)$ (cf. Appendix, Section 9). Therefore we may suppose that $p < 2$.

We claim that it is enough to show that for each r as above,

$$(5.37) \quad S_1 \sqrt{L} : \dot{W}^{1,p}(\mathbb{R}^n) \longrightarrow L^{p,\infty}(\mathbb{R}^n), \quad p = p(n, r) := \max \left\{ 1, \frac{rn}{n+r} \right\},$$

because, given (5.37), the desired estimate (5.35), for the range $p(n, r) < p < 2$, follows by interpolation with (5.36). More precisely, setting $f := L^{-1/2} h$, by (5.36) and the boundedness of S_1 in L^2 , we have, in particular, that

$$(5.38) \quad \|S_1 \sqrt{L} f\|_{L^2(\mathbb{R}^n)} \leq C \|f\|_{\dot{W}^{1,2}(\mathbb{R}^n)}.$$

Thus, interpolating between the latter estimate and (5.37), we obtain

$$(5.39) \quad S_1 \sqrt{L} : \dot{W}^{1,p}(\mathbb{R}^n) \longrightarrow L^p(\mathbb{R}^n) \quad \text{whenever} \quad \max \left\{ 1, \frac{rn}{n+r} \right\} < p < 2,$$

and this is equivalent to (5.35), in the remaining case $p(n, r) < p < 2$.

Hence, it remains only to prove (5.37), i.e., we shall show that

$$(5.40) \quad \left| \left\{ x \in \mathbb{R}^n : S_1 \sqrt{L} f(x) > \alpha \right\} \right| \leq \frac{C}{\alpha^p} \int_{\mathbb{R}^n} |\nabla f(y)|^p dy, \quad \forall \alpha > 0,$$

for p as in (5.37), where by density we may suppose that $f \in C_0^\infty$.

Our proof is based on the use of a “Calderón-Zygmund type” decomposition of Sobolev spaces taken from [7], where it was used to establish an analogue of (5.40), but for \sqrt{L} rather than for $S_1 \sqrt{L}$.

Lemma 5.41. ([7]) *Suppose $n \geq 1$, $1 \leq p < \infty$ and $f \in \dot{W}^{1,p}(\mathbb{R}^n)$. Then for every $\alpha > 0$ there exists a collection of cubes $\{Q_i\}_{i \in \mathbb{Z}}$ with finite overlap, a function g and a family of functions $\{b_i\}_{i \in \mathbb{Z}}$ satisfying*

$$(5.42) \quad \text{supp } b_i \subset Q_i, \quad \|\nabla b_i\|_{L^p(\mathbb{R}^n)} \leq C \alpha |Q_i|^{1/p}, \quad \forall i \in \mathbb{Z},$$

$$(5.43) \quad \|\nabla g\|_{L^p(\mathbb{R}^n)} \leq C \|\nabla f\|_{L^p(\mathbb{R}^n)}, \quad \|\nabla g\|_{L^\infty(\mathbb{R}^n)} \leq C \alpha,$$

such that f can be represented in the form

$$(5.44) \quad f = g + \sum_{i \in \mathbb{Z}} b_i, \quad \text{with} \quad \sum_{i \in \mathbb{Z}} |Q_i| \leq C \alpha^{-p} \|\nabla f\|_{L^p(\mathbb{R}^n)}^p.$$

Returning to (5.40) we can write using the lemma above

$$S_1 \sqrt{L} f(x) \leq S_1 \sqrt{L} g(x) + \left(\iint_{\Gamma(x)} \left| \sum_{i \in \mathbb{Z}} t L e^{-t^2 L} b_i(y) \chi_{(0,t(Q_i))}(t) \right|^2 \frac{dy dt}{t^{n+1}} \right)^{1/2}$$

$$\begin{aligned}
& + \left(\iint_{\Gamma(x)} \left| \sum_{i \in \mathbb{Z}} t L e^{-t^2 L} b_i(y) \chi_{l(Q_i, \infty)}(t) \right|^2 \frac{dy dt}{t^{n+1}} \right)^{1/2} \\
\leq & S_1 \sqrt{L} g(x) + \sum_{i \in \mathbb{Z}} \left(\iint_{\substack{t < l(Q_i) \\ |x-y| < t}} |t L e^{-t^2 L} b_i(y)|^2 \frac{dy dt}{t^{n+1}} \right)^{1/2} \\
& + \left(\iint_{\Gamma(x)} \left| t^2 L e^{-t^2 L} \sum_{i \in \mathbb{Z}} \frac{b_i(y)}{l(Q_i)} \right|^2 \frac{dy dt}{t^{n+1}} \right)^{1/2} \\
(5.45) \quad & =: I_0(x) + I_1(x) + I_2(x),
\end{aligned}$$

for all $x \in \mathbb{R}^n$. Let us assign now

$$A_l := \{x \in \mathbb{R}^n : I_l(x) > \alpha/3\}, \quad l = 0, 1, 2,$$

so that

$$(5.46) \quad \left| \{x \in \mathbb{R}^n : S_1 \sqrt{L} f(x) > \alpha\} \right| \leq |A_0| + |A_1| + |A_2|.$$

Step II. Consider A_0 . By Chebyshev's inequality

$$(5.47) \quad |A_0| \leq \frac{C}{\alpha^2} \int_{\mathbb{R}^n} |S_1 \sqrt{L} g(x)|^2 dx \leq \frac{C}{\alpha^2} \int_{\mathbb{R}^n} |\nabla g(x)|^2 dx$$

where for the last estimate we used boundedness of S_1 in $L^2(\mathbb{R}^n)$ and the Kato square root estimate ([9]). Combining the two statements in (5.43), we obtain that the expression in (5.47) is bounded by $C\alpha^{-p} \|\nabla f\|_{L^p(\mathbb{R}^n)}^p$, as desired.

Step III. The contribution from A_2 can be estimated as follows. By Chebyshev's inequality

$$(5.48) \quad |A_2| \leq \frac{C}{\alpha^r} \int_{\mathbb{R}^n} \left| S \left(\sum_{i \in \mathbb{Z}} \frac{b_i(y)}{l(Q_i)} \right) \right|^r dx,$$

with S as in (1.10). On the other hand, the $L^r - L^2$ off-diagonal estimates for the heat semigroup imply that S is bounded in $L^r(\mathbb{R}^n)$ (see, e.g., [7] for an analogous result in the case of vertical square function and [39]). Therefore, by Hölder's inequality for sequences

$$(5.49) \quad |A_2| \leq \frac{C}{\alpha^r} \left\| \sum_{i \in \mathbb{Z}} \frac{|b_i|}{l(Q_i)} \right\|_{L^r(\mathbb{R}^n)}^r \leq \frac{C}{\alpha^r} \left\| \left(\sum_{i \in \mathbb{Z}} \frac{|b_i|^r}{l(Q_i)^r} \right)^{1/r} \left(\sum_{i \in \mathbb{Z}} \chi_{Q_i} \right)^{1-1/r} \right\|_{L^r(\mathbb{R}^n)}^r.$$

Now we recall that the cubes $\{Q_i\}_{i \in \mathbb{Z}}$ have finite overlap, i.e. there exists some fixed constant C such that $\sum_{i \in \mathbb{Z}} \chi_{Q_i}(x) \leq C$ for all $x \in \mathbb{R}^n$. This implies that

$$(5.50) \quad |A_2| \leq \frac{C}{\alpha^r} \int_{\mathbb{R}^n} \sum_{i \in \mathbb{Z}} \frac{|b_i|^r}{l(Q_i)^r} dx.$$

When $p = \frac{rn}{n+r}$ we deduce from (5.42) and Poincaré's inequality that

$$(5.51) \quad \|b_i\|_{L^r(\mathbb{R}^n)} \leq C \|\nabla b_i\|_{L^p(\mathbb{R}^n)} \leq C\alpha |Q_i|^{1/p} = C\alpha |l(Q_i)|^{1+n/r}.$$

When $p = 1 > \frac{rn}{n+r}$, by Hölder's inequality

$$(5.52) \quad \|b_i\|_{L^r(Q_i)} \leq C|Q_i|^{\frac{1}{r}-\frac{n-1}{n}} \|b_i\|_{L^{\frac{n}{n-1}}(Q_i)} \leq C|Q_i|^{\frac{1}{r}-\frac{n-1}{n}} \|\nabla b_i\|_{L^1(Q_i)} \leq C\alpha|l(Q_i)|^{1+n/r}.$$

Hence, in any case,

$$(5.53) \quad |A_2| \leq C \sum_{i \in \mathbb{Z}} |Q_i| \leq C\alpha^{-p} \|\nabla f\|_{L^p(\mathbb{R}^n)}^p.$$

Step 4. We now proceed to estimate $|A_1|$. For all functions v , define

$$(5.54) \quad T_i v(x) := \left(\iint_{\substack{t < l(Q_i) \\ |x-y| < t}} |tLe^{-t^2L}v(y)|^2 \frac{dydt}{t^{n+1}} \right)^{1/2}, \quad i \in \mathbb{Z}, \quad x \in \mathbb{R}^n.$$

Then

$$(5.55) \quad \begin{aligned} |A_1| &\leq \sum_{i \in \mathbb{Z}} |4Q_i| + \left\{ x \in \mathbb{R}^n \setminus \cup_{i \in \mathbb{Z}} 4Q_i : \left| \sum_{i \in \mathbb{Z}} T_i b_i(x) \right| > \alpha/3 \right\} \\ &\leq \frac{C}{\alpha^p} \|\nabla f\|_{L^p(\mathbb{R}^n)}^p + \frac{C}{\alpha^2} \int_{\mathbb{R}^n} \left| \sum_{i \in \mathbb{Z}} T_i b_i(x) \chi_{\mathbb{R}^n \setminus 4Q_i}(x) \right|^2 dx. \end{aligned}$$

The second term above (referred to as \tilde{T} later on) is bounded by

$$(5.56) \quad \frac{C}{\alpha^2} \left(\sum_{i \in \mathbb{Z}} \int_{\mathbb{R}^n \setminus 4Q_i} T_i b_i(x) u(x) dx \right)^2,$$

for some $u \in L^2(\mathbb{R}^n)$ such that $\|u\|_{L^2(\mathbb{R}^n)} = 1$. Therefore,

$$(5.57) \quad \begin{aligned} \tilde{T} &\leq \frac{C}{\alpha^2} \left(\sum_{i \in \mathbb{Z}} \sum_{j=3}^{\infty} \|T_i b_i\|_{L^2(S_j(Q_i))} \|u\|_{L^2(S_j(Q_i))} \right)^2 \\ &\leq \frac{C}{\alpha^2} \left(\sum_{i \in \mathbb{Z}} \sum_{j=3}^{\infty} \left(\iint_{\substack{t < l(Q_i) \\ (y,t) \in \mathcal{R}(S_j(Q_i))}} |t^2 Le^{-t^2L} b_i(y)|^2 \frac{dydt}{t^3} \right)^{1/2} \|u\|_{L^2(S_j(Q_i))} \right)^2, \end{aligned}$$

where, as before, $\mathcal{R}(S_j(Q_i)) = \cup_{x \in S_j(Q_i)} \Gamma(x)$ stands for the saw-tooth region built on the set $S_j(Q_i)$. Then, using Lemma 2.26 and (5.51)–(5.52), we see that

$$(5.58) \quad \begin{aligned} \tilde{T} &\leq \frac{C}{\alpha^2} \left(\sum_{i \in \mathbb{Z}} \sum_{j=3}^{\infty} \left(\int_0^{l(Q_i)} \|t^2 Le^{-t^2L} b_i\|_{L^2(2^{j+1}Q_i \setminus 2^{j-2}Q_i)}^2 \frac{dt}{t^3} \right)^{1/2} \|u\|_{L^2(S_j(Q_i))} \right)^2 \\ &\leq \frac{C}{\alpha^2} \left(\sum_{i \in \mathbb{Z}} \sum_{j=3}^{\infty} \left(\int_0^{l(Q_i)} e^{-\frac{(2^j l(Q_i))^2}{\alpha^2}} t^{2(\frac{n}{2}-\frac{n}{r}-1)} \|b_i\|_{L^r(Q_i)}^2 \frac{dt}{t} \right)^{1/2} \|u\|_{L^2(S_j(Q_i))} \right)^2 \\ &\leq C \left(\sum_{i \in \mathbb{Z}} \sum_{j=3}^{\infty} 2^{-jN} l(Q_i)^n [\mathcal{M}(|u|^2)(y)]^{1/2} \right)^2 \end{aligned}$$

for any $y \in Q_i$ and any large positive number N . Here \mathcal{M} stands for the Hardy-Littlewood maximal function. Next, one can sum up the expression above in j and integrate in y to obtain

$$(5.59) \quad \begin{aligned} \bar{T} &\leq C \left(\int_{\mathbb{R}^n} \sum_{i \in \mathbb{Z}} \chi_{Q_i}(y) [\mathcal{M}(|u|^2)(y)]^{1/2} dy \right)^2 \\ &\leq C \left(\int_{\bigcup_{i \in \mathbb{Z}} Q_i} [\mathcal{M}(|u|^2)(y)]^{1/2} dy \right)^2, \end{aligned}$$

by the finite overlap property of cubes $\{Q_i\}_{i \in \mathbb{Z}}$. At this point we use Kolmogorov's lemma and weak type $(1, 1)$ boundedness of the Hardy-Littlewood maximal function to control the expression above by

$$(5.60) \quad C \left(\left| \bigcup_{i \in \mathbb{Z}} Q_i \right|^{1/2} \| |u|^2 \|_{L^1(\mathbb{R}^n)}^{1/2} \right)^2 \leq C \sum_{i \in \mathbb{Z}} |Q_i| \leq \frac{C}{\alpha^p} \|\nabla f\|_{L^p(\mathbb{R}^n)}^p,$$

as desired. This concludes the proof of Proposition 5.34, and thus also that of Theorem 5.2. \square

6. SHARP MAXIMAL FUNCTION CHARACTERIZATION.

Recall the sharp maximal function introduced in (1.28). This Section is devoted to the proof of (1.29). More precisely, we define $H_{\sharp, M, L}^p(\mathbb{R}^n)$ to be the completion of the set

$$\mathbb{H}_{\sharp, M, L}^p(\mathbb{R}^n) := \{f \in L^2(\mathbb{R}^n) : \mathcal{M}_M^\sharp f \in L^p(\mathbb{R}^n)\},$$

with respect to the norm

$$\|f\|_{H_{\sharp, M, L}^p(\mathbb{R}^n)} := \|\mathcal{M}_M^\sharp f\|_{L^p(\mathbb{R}^n)}.$$

We have the following:

Theorem 6.1. *Let $2 < p < \infty$ and $M > n/4$. Then $f \in H_L^p(\mathbb{R}^n) = H_{\sharp, M, L}^p(\mathbb{R}^n)$, and*

$$(6.2) \quad \|f\|_{H_L^p(\mathbb{R}^n)} \approx \|\mathcal{M}_M^\sharp f\|_{L^p(\mathbb{R}^n)}, \quad f \in L^2(\mathbb{R}^n).$$

Proof. Recall that we have shown in the proof of Corollary 4.21 that $L^2(\mathbb{R}^n) \cap H_L^p(\mathbb{R}^n)$ is dense in $H_L^p(\mathbb{R}^n)$ when $2 < p < \infty$ (for $p \leq 2$, the analogous density statement holds by definition). Consequently, it suffices to establish (6.2).

Step I. First, we shall establish that for all $M \in \mathbb{N}$

$$(6.3) \quad \mathcal{M}_M^\sharp : L^p(\mathbb{R}^n) \longrightarrow L^p(\mathbb{R}^n), \quad \text{for } 2 < p \leq \infty, \quad M \in \mathbb{N}.$$

Clearly, the latter estimate is an immediate consequence of the pointwise bound $\mathcal{M}_M^\sharp f \leq C \left(\mathcal{M}(|f|^2) \right)^{1/2}$, where \mathcal{M} denotes the Hardy-Littlewood maximal operator. In turn, we establish the pointwise bound as follows:

$$\mathcal{M}_M^\sharp f \leq \sup_{Q \ni x} \sum_{j=0}^{\infty} \left(\frac{1}{|Q|} \int_Q \left| (I - e^{-l(Q)^2 L})^M (f \chi_{S_j(Q)})(y) \right|^2 dy \right)^{1/2}$$

$$\begin{aligned}
&\leq C_M \sup_{Q \ni x} \left(\frac{1}{|Q|} \int_Q |f(y)|^2 dy \right)^{1/2} \\
(6.4) \quad &+ C_M \sup_{Q \ni x} \sup_{1 \leq k \leq M} \sum_{j=1}^{\infty} \left(\frac{1}{|Q|} \int_Q \left| e^{-kl(Q)^2 L} (f \chi_{S_j(Q)})(y) \right|^2 dy \right)^{1/2}.
\end{aligned}$$

Then by the Gaffney estimate (Lemma 2.25), the expression above is controlled by

$$\begin{aligned}
(6.5) \quad &C \left(\mathcal{M}(|f|^2) \right)^{1/2} + C \sup_{Q \ni x} \sup_{1 \leq k \leq M} \sum_{j=1}^{\infty} e^{-\frac{\text{dist}(Q, S_j(Q))^2}{c|Q|^2}} \frac{1}{|Q|^{1/2}} \|f\|_{L^2(S_j(Q))} \\
&\leq C \left(\mathcal{M}(|f|^2) \right)^{1/2}.
\end{aligned}$$

This finishes the proof of (6.3). Since for $2 \leq p < p_+(L)$, the spaces H_L^p coincide with L^p , we also have

$$(6.6) \quad \mathcal{M}_M^\sharp : H_L^p(\mathbb{R}^n) \longrightarrow L^p(\mathbb{R}^n) \quad \text{for } 2 < p < p_+(L), \quad M \in \mathbb{N}.$$

Interpolating (6.6) with the property

$$(6.7) \quad \mathcal{M}_M^\sharp : BMO_L(\mathbb{R}^n) \longrightarrow L^\infty(\mathbb{R}^n), \quad M > n/4,$$

we deduce that

$$(6.8) \quad \mathcal{M}_M^\sharp : H_L^p(\mathbb{R}^n) \longrightarrow L^p(\mathbb{R}^n) \quad \text{for } 2 < p < \infty, \quad M > n/4.$$

Step II. Now we turn to the converse of (6.8). More precisely, let us show that

$$(6.9) \quad \|f\|_{H_L^p(\mathbb{R}^n)} \leq C \|\mathcal{M}_M^\sharp f\|_{L^p(\mathbb{R}^n)},$$

whenever $2 < p < \infty$, $M > n/4$ and $f \in L^2(\mathbb{R}^n)$. Note that for such f , the adapted sharp function $\mathcal{M}_M^\sharp f$ is well-defined.

Recall the discussion of tent spaces in Section 3. In particular, by (3.20) and Corollary 4.21, we have, for $M > n/4$ and $2 < p < \infty$,

$$(6.10) \quad \|f\|_{H_L^p(\mathbb{R}^n)} \leq C \left\| \sup_{B \ni x} \left(\frac{1}{|B|} \iint_{\tilde{B}} |(t^2 L)^M e^{-t^2 L} f(y)|^2 \frac{dy dt}{t} \right)^{1/2} \right\|_{L^p(\mathbb{R}^n)}.$$

Now in order to conclude (6.9) it remains to show that

$$(6.11) \quad \left\| \sup_{Q \ni x} \left(\frac{1}{|Q|} \int_0^{l(Q)} \int_Q |(t^2 L)^{M+1} e^{-t^2 L} f(x)|^2 \frac{dx dt}{t} \right)^{1/2} \right\|_{L^p(\mathbb{R}^n)} \leq C \|\mathcal{M}_M^\sharp f\|_{L^p(\mathbb{R}^n)},$$

for $2 < p < \infty$. Note that we switched to $M + 1$ on the left-hand side, but by the results of Sections 3 and 4 we can choose $M > n/4$ in (6.10) at our convenience.

Step III. In this part we prove that for every cube $Q \subset \mathbb{R}^n$

$$I_Q := \left(\frac{1}{|Q|} \int_0^{l(Q)} \int_Q |(t^2 L)^{M+1} e^{-t^2 L} f(y)|^2 \frac{dy dt}{t} \right)^{1/2}$$

$$(6.12) \leq C \sum_{j=0}^{\infty} 2^{-jN} \frac{1}{|2^j Q|^{1/2}} \sup_{l(Q) \leq s \leq \sqrt{2}l(Q)} \|(I - e^{-s^2 L})^M f\|_{L^2(2^j Q)}, \quad \forall N \in \mathbb{N}.$$

Following the procedure outlined in (3.39)–(3.44) one can split

$$\begin{aligned} f &= 2^M \left(l(Q)^{-2} \int_{l(Q)}^{\sqrt{2}l(Q)} s (I - e^{-s^2 L})^M ds \right. \\ &\quad \left. + \sum_{k=1}^M C_{k,M} l(Q)^{-2} L^{-1} e^{-kl(Q)^2 L} (I - e^{-l(Q)^2 L}) \sum_{i=0}^{k-1} e^{-il(Q)^2 L} \right)^M f \\ &= C_{1,1} T_{1,1}^{l(Q)} (I - e^{-l(Q)^2 L})^M l(Q)^{-2M} L^{-M} f \\ (6.13) \quad &+ \sum_{i=1}^{(M+1)^{M-1}} C_{i,2} T_{i,2}^{l(Q)} \left(l(Q)^{-2} \int_{l(Q)}^{\sqrt{2}l(Q)} s (I - e^{-s^2 L})^M l(Q)^{-2N_i} L^{-N_i} f ds \right), \end{aligned}$$

where $C_{i,k}$ are some constants, $0 \leq N_i \leq M$, and each $T_{i,k}$ is given by a constant (possibly, zero) plus a linear combination of the terms in the form $e^{-t^2 L}$ with $t \approx l(Q)^2$. In particular, $T_{i,k}$'s are bounded in $L^2(\mathbb{R}^n)$ with the constant independent of $l(Q)$ (see Lemma 2.26) and satisfy Gaffney estimates (2.21) with $t \approx l(Q)^2$.

All the terms on the right-hand side of (6.13) are essentially of the same nature, and will be handled similarly. Let us concentrate on the first one. The corresponding part of I_Q is bounded by

$$(6.14)$$

$$\sum_{j=0}^{\infty} \left(\frac{1}{|Q|} \int_0^{l(Q)} \int_Q \left| \left(\frac{t}{l(Q)} \right)^{2M} t^2 L e^{-t^2 L} T_{1,1}^{l(Q)} [\chi_{S_j(Q)} (I - e^{-l(Q)^2 L})^M f](y) \right|^2 \frac{dy dt}{t} \right)^{1/2}.$$

Since the mapping

$$(6.15) \quad f \mapsto \left(\int_0^{\infty} |t^2 L e^{-t^2 L} f(\cdot)|^2 \frac{dt}{t} \right)^{1/2}, \quad M \in \mathbb{N},$$

is bounded in $L^2(\mathbb{R}^n)$ (see [1]), and the operator $T_{1,1}^{l(Q)}$ is bounded in L^2 , we can write

$$(6.16)$$

$$\begin{aligned} \sum_{j=0,1} \left(\frac{1}{|Q|} \int_0^{l(Q)} \int_Q \left| \left(\frac{t}{l(Q)} \right)^{2M} t^2 L e^{-t^2 L} T_{1,1}^{l(Q)} [\chi_{S_j(Q)} (I - e^{-l(Q)^2 L})^M f](y) \right|^2 \frac{dy dt}{t} \right)^{1/2} \\ \leq C \frac{1}{|Q|^{1/2}} \|(I - e^{-l(Q)^2 L})^M f\|_{L^2(2Q)}. \end{aligned}$$

Furthermore, by Gaffney estimates and Lemma 2.22, when $j \geq 2$ we have

$$\left(\frac{1}{|Q|} \int_0^{l(Q)} \int_Q \left| \left(\frac{t}{l(Q)} \right)^{2M} t^2 L e^{-t^2 L} T_{1,1}^{l(Q)} [\chi_{S_j(Q)} (I - e^{-l(Q)^2 L})^M f](y) \right|^2 \frac{dy dt}{t} \right)^{1/2}$$

$$\begin{aligned}
&\leq \frac{C}{|Q|^{1/2}} \left(\int_0^{l(Q)} e^{-\frac{Q^j l(Q)^2}{ct(Q)^2}} \left(\frac{t}{l(Q)} \right)^{2M} \frac{dt}{t} \right)^{1/2} \| (I - e^{-l(Q)^2 L})^M f \|_{L^2(S_j(Q))} \\
(6.17) \quad &\leq C 2^{-jN} \frac{1}{|2^j Q|^{1/2}} \| (I - e^{-l(Q)^2 L})^M f \|_{L^2(S_j(Q))},
\end{aligned}$$

for any $N \in \mathbb{N}$. Now the combination of (6.16) and (6.17), together with analogous considerations for the remaining terms in (6.13), implies

$$\begin{aligned}
I_Q &\leq C \sum_{j=0}^{\infty} 2^{-jN} \frac{1}{|2^j Q|^{1/2}} \left(\| (I - e^{-l(Q)^2 L})^M f \|_{L^2(S_j(Q))} \right. \\
&\quad \left. + \left\| l(Q)^{-2} \int_{l(Q)}^{\sqrt{2}l(Q)} s (I - e^{-s^2 L})^M f ds \right\|_{L^2(S_j(Q))} \right) \\
&\leq C \sum_{j=0}^{\infty} 2^{-jN} \frac{1}{|2^j Q|^{1/2}} \left(\| (I - e^{-l(Q)^2 L})^M f \|_{L^2(S_j(Q))} \right. \\
&\quad \left. + l(Q)^{-2} \int_{l(Q)}^{\sqrt{2}l(Q)} s \| (I - e^{-s^2 L})^M f \|_{L^2(S_j(Q))} ds \right) \\
(6.18) \quad &\leq C \sum_{j=0}^{\infty} 2^{-jN} \frac{1}{|2^j Q|^{1/2}} \sup_{l(Q) \leq s \leq \sqrt{2}l(Q)} \| (I - e^{-s^2 L})^M f \|_{L^2(S_j(Q))},
\end{aligned}$$

as desired.

Step IV. The next step is to show that

$$\begin{aligned}
&\sup_{Q \ni x} \sum_{j=0}^{\infty} 2^{-jN} \frac{1}{|2^j Q|^{1/2}} \sup_{l(Q) \leq s \leq \sqrt{2}l(Q)} \| (I - e^{-s^2 L})^M f \|_{L^2(S_j(Q))} \\
(6.19) \quad &\leq C \mathcal{M}_2(\mathcal{M}_M^\sharp f)(x), \quad x \in \mathbb{R}^n, \quad M \in \mathbb{N},
\end{aligned}$$

where \mathcal{M}_2 is an L^2 -based version of the Hardy-Littlewood maximal function, i.e.

$$(6.20) \quad \mathcal{M}_2 g(x) = \sup_{Q \ni x} \left(\frac{1}{|Q|} \int_Q |g(y)|^2 dy \right)^{1/2}, \quad x \in \mathbb{R}^n.$$

Clearly,

$$(6.21) \quad \mathcal{M}_2 g(x) = \sup_{Q \ni x} \sup_{j \in \mathbb{N} \cup \{0\}} \left(\frac{1}{|2^j Q|} \int_{2^j Q} |g(y)|^2 dy \right)^{1/2}, \quad x \in \mathbb{R}^n.$$

Hence,

$$\begin{aligned}
&\mathcal{M}_2(\mathcal{M}_M^\sharp f)(x) \\
(6.22) \quad &= \sup_{Q \ni x} \sup_{j \in \mathbb{N} \cup \{0\}} \left(\frac{1}{|2^j Q|} \int_{2^j Q} \sup_{\tilde{Q} \ni y} \frac{1}{|\tilde{Q}|} \int_{\tilde{Q}} |(I - e^{-l(\tilde{Q})^2 L})^M f(z)|^2 dz dy \right)^{1/2}.
\end{aligned}$$

Let us denote by $\{Q_i^j\}_{i=1}^{2^{jn}}$ some partition of $2^j Q$ into subcubes of sidelength $l(Q)$. Then the expression above is further equal to

$$\begin{aligned}
& \sup_{Q \ni x} \sup_{j \in \mathbb{N} \cup \{0\}} \left(\frac{1}{|2^j Q|} \sum_{i=1}^{2^{jn}} \int_{Q_i^j} \sup_{\tilde{Q} \ni y} \frac{1}{|\tilde{Q}|} \int_{\tilde{Q}} |(I - e^{-l(\tilde{Q})^2 L})^M f(z)|^2 dz dy \right)^{1/2} \\
& \geq \sup_{Q \ni x} \sup_{j \in \mathbb{N} \cup \{0\}} \left(\frac{1}{|2^j Q|} \sum_{i=1}^{2^{jn}} \int_{Q_i^j} \frac{1}{|Q_i^j|} \int_{Q_i^j} |(I - e^{-l(Q)^2 L})^M f(z)|^2 dz dy \right)^{1/2} \\
& = \sup_{Q \ni x} \sup_{j \in \mathbb{N} \cup \{0\}} \left(\frac{1}{|2^j Q|} \sum_{i=1}^{2^{jn}} \int_{Q_i^j} |(I - e^{-l(Q)^2 L})^M f(z)|^2 dz \right)^{1/2} \\
(6.23) \quad & = \sup_{Q \ni x} \sup_{j \in \mathbb{N} \cup \{0\}} \left(\frac{1}{|2^j Q|} \int_{2^j Q} |(I - e^{-l(Q)^2 L})^M f(z)|^2 dz \right)^{1/2},
\end{aligned}$$

where we used the fact that $l(Q_i^j) = l(Q)$ for all $i = 1, \dots, 2^{jn}$, $j \in \mathbb{N} \cup \{0\}$, to switch from $e^{-l(Q_i^j)^2 L}$ to $e^{-l(Q)^2 L}$ in the first inequality above. We claim that the expression in the last line of (6.23) controls the left-hand side of (6.19). Indeed,

$$\begin{aligned}
& \sup_{Q \ni x} \sum_{j=0}^{\infty} 2^{-jN} \frac{1}{|2^j Q|^{1/2}} \sup_{l(Q) \leq s \leq \sqrt{2}l(Q)} \|(I - e^{-s^2 L})^M f\|_{L^2(S_j(Q))} \\
(6.24) \quad & \leq C \sup_{Q \ni x} \sum_{j=0}^{\infty} 2^{-jN} \sup_{l(Q) \leq s \leq \sqrt{2}l(Q)} \left(\frac{1}{|2^j Q_s|} \int_{2^j Q_s} |(I - e^{-s^2 L})^M f(z)|^2 dz \right)^{1/2},
\end{aligned}$$

where Q_s is a cube with the same center as Q and sidelength s . Since $s \geq l(Q)$, in particular, $Q_s \supset Q \ni x$. Then the right-hand side of (6.24) is bounded by

$$\begin{aligned}
& C \sup_{Q \ni x} \sup_{j \in \mathbb{N} \cup \{0\}} \left(\frac{1}{|2^j Q|} \int_{2^j Q} |(I - e^{-l(Q)^2 L})^M f(z)|^2 dz \right)^{1/2} \sum_{j=0}^{\infty} 2^{-jN} \\
(6.25) \quad & \leq C \sup_{Q \ni x} \sup_{j \in \mathbb{N} \cup \{0\}} \left(\frac{1}{|2^j Q|} \int_{2^j Q} |(I - e^{-l(Q)^2 L})^M f(z)|^2 dz \right)^{1/2}.
\end{aligned}$$

This finishes the proof of (6.19).

Step IV. Finally, (6.12), (6.19) allow to conclude that

$$\begin{aligned}
(6.26) \quad & \left\| \sup_{Q \ni x} \left(\frac{1}{|Q|} \int_0^{l(Q)} \int_Q |(t^2 L)^{M+1} e^{-t^2 L} f(x)|^2 \frac{dx dt}{t} \right)^{1/2} \right\|_{L^p(\mathbb{R}^n)} \\
& \leq C \|\mathcal{M}_2(\mathcal{M}_M^\sharp f)\|_{L^p(\mathbb{R}^n)},
\end{aligned}$$

for every $0 < p < \infty$. But since the classical Hardy-Littlewood maximal function is bounded in L^p for $1 < p < \infty$, the operator \mathcal{M}_2 is bounded in $L^p(\mathbb{R}^n)$ for

$2 < p < \infty$, and therefore,

$$(6.27) \quad \|\mathcal{M}_2(\mathcal{M}_M^\# f)\|_{L^p(\mathbb{R}^n)} \leq C \|\mathcal{M}_M^\# f\|_{L^p(\mathbb{R}^n)}, \quad 2 < p < \infty.$$

Now the combination of (6.26) and (6.27) yields (6.11) and finishes the proof of the theorem. \square

7. FRACTIONAL POWERS OF THE OPERATOR L .

Recall that for $p_-(L) < p < r < p_+(L)$

$$(7.1) \quad L^{-\alpha} : L^p(\mathbb{R}^n) \longrightarrow L^r(\mathbb{R}^n), \quad \alpha = \frac{1}{2} \left(\frac{n}{p} - \frac{n}{r} \right).$$

This result has been proved in [7]. In this section we aim to prove the generalization of (7.1) to the full scale of H_L^p spaces.

Theorem 7.2. *Let $0 < p < r < \infty$. Then*

$$(7.3) \quad L^{-\alpha} : H_L^p(\mathbb{R}^n) \longrightarrow H_L^r(\mathbb{R}^n), \quad \alpha = \frac{1}{2} \left(\frac{n}{p} - \frac{n}{r} \right),$$

$$(7.4) \quad L^{-\alpha} : H_L^p(\mathbb{R}^n) \longrightarrow BMO_L(\mathbb{R}^n), \quad \alpha = \frac{n}{2p},$$

$$(7.5) \quad L^{-\alpha} : BMO_L(\mathbb{R}^n) \longrightarrow \Lambda_L^{2\alpha}(\mathbb{R}^n), \quad \alpha > 0,$$

$$(7.6) \quad L^{-\alpha} : \Lambda_L^\beta(\mathbb{R}^n) \longrightarrow \Lambda_L^{\beta+2\alpha}(\mathbb{R}^n), \quad \alpha > 0, \quad \beta > 0.$$

Proof. Let us denote $p_n := 2n/(n+2)$ and $p'_n := 2n/(n-2)$. We recall that by [7], we have $p_-(L) < p_n$ and $p_+(L) > p'_n$. We begin by claiming that it is enough to prove (7.3) for

$$(7.7) \quad 0 < p < r \leq 1 \quad \text{such that} \quad \frac{1}{2} \left(\frac{n}{p} - \frac{n}{r} \right) \leq \frac{1}{2} \left(\frac{n}{p_n} - \frac{n}{2} \right) = \frac{1}{2},$$

which, in particular, says that

$$(7.8) \quad 0 < \alpha = \frac{1}{2} \left(\frac{n}{p} - \frac{n}{r} \right) \leq \frac{1}{2}.$$

Indeed, once (7.3) has been proved for this range, by interpolating with (7.1) via Lemma 4.24, we may obtain that (7.3) holds for all

$$(7.9) \quad 0 < p < r < p_+(L) \quad \text{such that} \quad \frac{1}{2} \left(\frac{n}{p} - \frac{n}{r} \right) \leq \frac{1}{2},$$

with α satisfying (7.8). We can then write $L^{-\alpha} = (L^{-\alpha/k})^k$ for k large enough in order to remove restrictions on α and, equivalently, on the difference between p and r , and obtain (7.3) for

$$(7.10) \quad 0 < p < r < p_+(L), \quad \alpha = \frac{1}{2} \left(\frac{n}{p} - \frac{n}{r} \right),$$

without restriction on the size of α . From here the results in (7.3)–(7.6) follow for the full range of indices by duality and another application of the procedure with $L^{-\alpha} = (L^{-\alpha/k})^k$.

Thus, it suffices to establish (7.3) under the restrictions (7.7)–(7.8), and it is to this task that we now turn. We first show that

$$(7.11) \quad \|S(L^{-\alpha}m)\|_{L^r(\mathbb{R}^n)} \leq C, \quad \text{for every } (H_L^p, \varepsilon, M)\text{-molecule } m,$$

where $M > \frac{n}{2}(\frac{1}{p} - \frac{1}{2})$ and $\varepsilon > 0$. To this end, observe that by Hölder's inequality

$$(7.12) \quad \|S(L^{-\alpha}m)\|_{L^r(\mathbb{R}^n)}^r \leq C \sum_{j=0}^{\infty} (2^j l(Q))^{n(1-r/2)} \|S(L^{-\alpha}m)\|_{L^2(S_j(Q))}^r.$$

When $j \leq 10$, we employ boundedness of S in $L^2(\mathbb{R}^n)$ and (7.1) to obtain the estimate

$$(7.13) \quad \|S(L^{-\alpha}m)\|_{L^2(S_j(Q))} \leq \|m\|_{L^q(\mathbb{R}^n)}.$$

Here and throughout the proof q is such that $\alpha = \frac{1}{2}(\frac{n}{q} - \frac{n}{2})$, so that $q \leq 2$ and $q > p_n$ by (7.8). Then by the definition of the molecule the expression above is bounded by $l(Q)^{\frac{n}{2} - \frac{n}{r}}$. Indeed, by Hölder inequality every (H_L^p, ε, M) -molecule satisfies (3.6) for $q \leq 2$. Therefore,

$$(7.14) \quad \|m\|_{L^q(\mathbb{R}^n)} \leq \sum_{j=0}^{\infty} \|m\|_{L^q(S_j(Q))} \leq Cl(Q)^{\frac{n}{q} - \frac{n}{p}} = Cl(Q)^{\frac{n}{2} - \frac{n}{r}},$$

since $\frac{1}{2}(\frac{n}{p} - \frac{n}{r}) = \alpha = \frac{1}{2}(\frac{n}{q} - \frac{n}{2})$.

Turning to the case $j \geq 10$, one can represent the molecule as follows

$$(7.15) \quad \begin{aligned} m &= (I - e^{-l(Q)^2 L})^M m + [I - (I - e^{-l(Q)^2 L})^M] m \\ &= (I - e^{-l(Q)^2 L})^M m + C \sup_{1 \leq k \leq M} \left(\frac{k}{M} l(Q)^2 L e^{-\frac{k}{M} l(Q)^2 L} \right)^M (l(Q)^{-2} L^{-1})^M m. \end{aligned}$$

Starting with the first term above, we write

$$(7.16) \quad \begin{aligned} \|S(L^{-\alpha}(I - e^{-l(Q)^2 L})^M m)\|_{L^2(S_j(Q))} &\leq \|S(L^{-\alpha}(I - e^{-l(Q)^2 L})^M (m\chi_{R_j(Q)}))\|_{L^2(S_j(Q))} \\ &\quad + \|S(L^{-\alpha}(I - e^{-l(Q)^2 L})^M (m\chi_{\mathbb{R}^n \setminus R_j(Q)}))\|_{L^2(S_j(Q))}, \end{aligned}$$

where

$$(7.17) \quad R_j(Q) = 2^{j+2}Q \setminus 2^{j-3}Q.$$

Then

$$(7.18) \quad \begin{aligned} \|S(L^{-\alpha}(I - e^{-l(Q)^2 L})^M (m\chi_{R_j(Q)}))\|_{L^2(\mathbb{R}^n)} &\leq C \|m\|_{L^q(R_j(Q))} \leq C (2^j l(Q))^{\frac{n}{2} - \frac{n}{r}} 2^{-j\varepsilon}. \end{aligned}$$

As for the second part of (7.16), using the notation (3.49), one can write

$$\begin{aligned}
& \|S(L^{-\alpha}(I - e^{-l(Q)^2L})^M(m\chi_{\mathbb{R}^n \setminus R_j(Q)}))\|_{L^2(S_j(Q))} \\
& \leq C \left(\iint_{\mathcal{R}(S_j(Q))} |s^2 L e^{-s^2L} L^{-\alpha}(I - e^{-l(Q)^2L})^M(m\chi_{\mathbb{R}^n \setminus R_j(Q)})(x)|^2 \frac{ds dx}{s} \right)^{1/2} \\
& \leq C \left(\int_{\bar{S}_j(Q)} \int_0^\infty |s^2 L e^{-s^2L} L^{-\alpha}(I - e^{-l(Q)^2L})^M(m\chi_{\mathbb{R}^n \setminus R_j(Q)})(x)|^2 \frac{ds dx}{s} \right)^{1/2} \\
& + C \left(\int_{\mathbb{R}^n \setminus \bar{S}_j(Q)} \int_{c2^j l(Q)}^\infty |s^2 L e^{-s^2L} L^{-\alpha}(I - e^{-l(Q)^2L})^M(m\chi_{\mathbb{R}^n \setminus R_j(Q)})(x)|^2 \frac{ds dx}{s} \right)^{1/2} \\
(7.19) \quad & =: I + II.
\end{aligned}$$

We claim that for arbitrary closed sets $E, F \subset \mathbb{R}^n$

$$(7.20) \quad \left\| \frac{s}{\tau} e^{-sL}(I - e^{-\tau L})g \right\|_{L^2(F)} \leq C e^{-\frac{\text{dist}(E,F)^2}{cs}} \|g\|_{L^2(E)},$$

provided $s \geq \tau$ and $\text{supp } g \subset E$. Indeed,

$$\begin{aligned}
& \left\| \frac{s}{\tau} (e^{-sL} - e^{-(s+\tau)L})g \right\|_{L^2(F)} = \left\| \frac{s}{\tau} \int_0^\tau \partial_r e^{-(s+r)L} g dr \right\|_{L^2(F)} \\
& \leq C \frac{s}{\tau} \int_0^\tau \left\| (s+r)L e^{-(s+r)L} g \right\|_{L^2(F)} \frac{dr}{s+r} \\
(7.21) \quad & \leq C \|g\|_{L^2(E)} \left(\frac{s}{\tau} \int_0^\tau e^{-\frac{\text{dist}(E,F)^2}{c(s+r)}} \frac{dr}{s+r} \right).
\end{aligned}$$

Since $s+r \approx s$ for $s \geq \tau$ and $r \in (0, \tau)$, the expression above does not exceed

$$(7.22) \quad C \|g\|_{L^2(E)} e^{-\frac{\text{dist}(E,F)^2}{cs}} \left(\frac{s}{\tau} \int_0^\tau \frac{dr}{s+r} \right) \leq C e^{-\frac{\text{dist}(E,F)^2}{cs}} \|g\|_{L^2(E)}.$$

Next, recall that

$$(7.23) \quad L^{-\alpha} f = C \int_0^\infty t^{\alpha-1} e^{-tL} f dt.$$

Then, using (7.20) with $\tau = l(Q)^2$ and Lemma 2.26 we obtain the estimate

$$\begin{aligned}
II & \leq C \left(\int_{c[2^j l(Q)]^2}^\infty \left(\int_0^\infty t^{\alpha-1} \|s L e^{-sL} e^{-tL} (I - e^{-l(Q)^2L})^M m\|_{L^2(\mathbb{R}^n)} dt \right)^2 \frac{ds}{s} \right)^{1/2} \\
& \leq C \left(\int_{c[2^j l(Q)]^2}^\infty \left(\int_0^\infty t^{\alpha-1} \left(\frac{l(Q)^2}{s+t} \right)^M \right. \right. \\
& \quad \left. \left. \times \left\| s L e^{-sL} \left(\frac{s+t}{l(Q)^2} \right)^M e^{-(s+t)L} (I - e^{-l(Q)^2L})^M m \right\|_{L^2(\mathbb{R}^n)} dt \right)^2 \frac{ds}{s} \right)^{1/2}
\end{aligned}$$

$$\leq C \left(\int_{c[2^j l(Q)]^2}^{\infty} \left(\int_0^{\infty} t^{\alpha-1} \left(\frac{l(Q)^2}{s+t} \right)^M dt \right)^2 \frac{ds}{s} \right)^{1/2} \|m\|_{L^2(\mathbb{R}^n)}$$

$$(7.24) \quad \leq C(2^j l(Q))^{\frac{n}{2}-\frac{n}{r}} 2^{j(\frac{n}{p}-\frac{n}{2}-2M)} \leq C(2^j l(Q))^{\frac{n}{2}-\frac{n}{r}} 2^{-j\varepsilon},$$

with ε denoting minimum between ε from the definition of the (p, ε, M) molecule and $\frac{n}{p} - \frac{n}{2} - 2M$. We do not distinguish them in the notation as soon as $\varepsilon > 0$.

In order to estimate I , let us denote by S^* the vertical version of square function, i.e.

$$(7.25) \quad S^* f(x) = \left(\int_0^{\infty} |t^2 L e^{-t^2 L} f(x)|^2 \frac{dt}{t} \right)^{1/2}, \quad x \in \mathbb{R}^n,$$

and record the following result:

$$(7.26) \quad \|S^* e^{-tL} (I - e^{-\tau L})^M f\|_{L^2(F)} \leq C \left(\frac{\max\{t, \tau\}}{\text{dist}(E, F)^2} \right)^M \|f\|_{L^2(E)},$$

for arbitrary closed sets $E, F \subset \mathbb{R}^n$, $f \in L^2(E)$ and $t, \tau > 0$. For $t = 0$ this has been established in Theorem 3.2, [39], and the proof of (7.26) follows the same path.

Then

$$(7.27) \quad \begin{aligned} I &\leq C \int_0^{l(Q)^2} t^{\alpha-1} \|S^* e^{-tL} (I - e^{-l(Q)^2 L})^M (m \chi_{\mathbb{R}^n \setminus R_j(Q)})\|_{L^2(\tilde{S}_j(Q))} dt \\ &\quad + C \int_{l(Q)^2}^{\infty} t^{\alpha-1} \left(\int_0^{\infty} \|s L e^{-sL} e^{-tL} (I - e^{-l(Q)^2 L})^M (m \chi_{\mathbb{R}^n \setminus R_j(Q)})\|_{L^2(\tilde{S}_j(Q))}^2 \frac{ds}{s} \right)^{1/2} dt \\ &=: I_1 + I_2 \end{aligned}$$

and by (7.26)

$$(7.28) \quad I_1 \leq C l(Q)^{2\alpha} 2^{-2Mj} \|m\|_{L^2(\mathbb{R}^n)} \leq C(2^j l(Q))^{\frac{n}{2}-\frac{n}{r}} 2^{-j\varepsilon}.$$

Going further,

$$(7.29) \quad \begin{aligned} I_2 &\leq C \int_{l(Q)^2}^{\infty} t^{\alpha-1} \left(\int_0^t \left(\frac{s}{t} \right)^2 \left(\frac{l(Q)^2}{t} \right)^{2M} \right. \\ &\quad \times \left\| t L e^{-tL} e^{-sL} \left(\frac{t}{l(Q)^2} \right)^M (I - e^{-l(Q)^2 L})^M (m \chi_{\mathbb{R}^n \setminus R_j(Q)}) \right\|_{L^2(\tilde{S}_j(Q))}^2 \frac{ds}{s} \right)^{1/2} dt \\ &\quad + C \int_{l(Q)^2}^{\infty} t^{\alpha-1} \left(\int_t^{\infty} \left(\frac{l(Q)^2}{s} \right)^{2M} \right. \\ &\quad \times \left\| s L e^{-sL} e^{-tL} \left(\frac{s}{l(Q)^2} \right)^M (I - e^{-l(Q)^2 L})^M (m \chi_{\mathbb{R}^n \setminus R_j(Q)}) \right\|_{L^2(\tilde{S}_j(Q))}^2 \frac{ds}{s} \right)^{1/2} dt. \end{aligned}$$

According to (7.20), the expression above is bounded by

$$C \|m\|_{L^2(\mathbb{R}^n)} \int_{l(Q)^2}^{\infty} t^{\alpha-1} \left(\frac{l(Q)^2}{t} \right)^M e^{-\frac{(2^j l(Q))^2}{ct}} \left(\int_0^t \left(\frac{s}{t} \right)^2 \frac{ds}{s} \right)^{1/2} dt$$

$$(7.29) \quad + C\|m\|_{L^2(\mathbb{R}^n)} \int_{l(Q)^2}^{\infty} t^{\alpha-1} \left(\int_t^{\infty} \left(\frac{l(Q)^2}{s} \right)^{2M} e^{-\frac{(2^j l(Q))^2}{cs}} \frac{ds}{s} \right)^{1/2} dt.$$

Making the change of variables $t \mapsto r$, $r := -\frac{(2^j l(Q))^2}{ct}$, in the first line above, we control it by

$$(7.30) \quad C(2^j l(Q))^{2\alpha} 2^{-2jM} \|m\|_{L^2(\mathbb{R}^n)} \leq C(2^j l(Q))^{\frac{n}{2} - \frac{n}{r}} 2^{-j\varepsilon}.$$

In order to control the second term in (7.29), let us take some $\delta > 0$ and write

$$(7.31) \quad \begin{aligned} & C\|m\|_{L^2(\mathbb{R}^n)} \int_{l(Q)^2}^{\infty} t^{\alpha-1} \left(\int_t^{\infty} \left(\frac{l(Q)^2}{s} \right)^{2M} e^{-\frac{(2^j l(Q))^2}{cs}} \frac{ds}{s} \right)^{1/2} dt \\ & \leq C\|m\|_{L^2(\mathbb{R}^n)} \int_{l(Q)^2}^{\infty} t^{\alpha-1} \left(\frac{l(Q)^2}{t} \right)^{\alpha+\delta} \left(\int_t^{\infty} \left(\frac{l(Q)^2}{s} \right)^{2M-2\alpha-2\delta} e^{-\frac{(2^j l(Q))^2}{cs}} \frac{ds}{s} \right)^{1/2} dt \\ & \leq C l(Q)^{2\alpha} 2^{-j(2M-2\alpha-2\delta)} \|m\|_{L^2(\mathbb{R}^n)} \leq C(2^j l(Q))^{\frac{n}{2} - \frac{n}{r}} 2^{-j\varepsilon}, \end{aligned}$$

provided $\delta > 0$ is small enough.

All in all, we have the desired control for $S L^{-\alpha}$ acting on the first term in (7.29). The second one can be handled by a similar argument, since $(l(Q)^{-2} L^{-1})^M m$ satisfies the same size conditions as a molecule itself and $(l(Q)^2 L e^{-l(Q)^2 L})^M$ behaves much as $(I - e^{-l(Q)^2 L})^M$. Roughly speaking, these two operators exhibit the same cancellation and decay properties (it can be seen, e.g., from the argument of Theorem 3.52).

This finishes the proof of (7.11), and it remains only to pass to (7.3), under the conditions (7.7)–(7.8). In particular, $\alpha \leq 1/2$, so by (7.1), and the fact that $p_+(L) > 2n/(n-2)$ (cf. [7]), we then have that

$$(7.32) \quad L^{-\alpha} : L^2(\mathbb{R}^n) \rightarrow L^q(\mathbb{R}^n), \quad \frac{1}{q} = \frac{1}{2} - \frac{2\alpha}{n}.$$

Now by density, as usual it is enough to work with $f \in \mathbb{H}_{L, \text{mol}, M}^p(\mathbb{R}^n)$, so that there is an L^2 convergent molecular decomposition $f = \sum \lambda_j m_j$, with $\sum |\lambda_j|^p \lesssim \|f\|_{\mathbb{H}_{L, \text{mol}, M}^p(\mathbb{R}^n)}^p$. Consequently, (7.32) implies that

$$L^{-\alpha} f = \sum \lambda_j L^{-\alpha} m_j \quad \text{in } L^q(\mathbb{R}^n),$$

and therefore also, since $q < p_+(L)$, that

$$S(L^{-\alpha} f) \leq \sum |\lambda_j| S(L^{-\alpha} m_j)$$

(here we have used that by [7], $S : L^q \rightarrow L^q$ whenever $p_-(L) < q < p_+(L)$). It is now immediate that (7.11) implies (7.3), under the conditions (7.7)–(7.8), and as we have observed above, the conclusion of Theorem 7.2 follows. \square

8. FUNCTIONAL CALCULUS AND FRACTIONAL POWERS OF L IN SMOOTHNESS SPACES.8.1. Functional calculus and fractional powers of L in H_L^p - BMO_L - Λ_L^α spaces.

Recall from Section 2.1 that L has a bounded holomorphic functional calculus on L^2 and (2.5) holds. In general, these properties do not extend to all L^p , $1 < p < \infty$. Otherwise, the heat semigroup would be bounded in all L^p , $1 < p < \infty$, as an H^∞ function, which would contradict Proposition 2.10. However, the functional calculus can be extended to a full scale of H_L^p - BMO_L - Λ_L^α spaces.

Lemma 8.1. *The operator L defined in (1.1)–(1.3) has a bounded holomorphic functional calculus in $H_L^p(\mathbb{R}^n)$, $0 < p < \infty$, $BMO_L(\mathbb{R}^n)$ and $\Lambda_L^\alpha(\mathbb{R}^n)$, $\alpha > 0$, in the following sense.*

When $0 < p \leq 2$, for every non-trivial $\psi \in H^\infty(\Sigma_\mu^0)$ the operator $\psi(L)$ originally defined on $L^2(\mathbb{R}^n)$ extends by continuity to a bounded operator on $H_L^p(\mathbb{R}^n)$ satisfying

$$(8.2) \quad \|\psi(L)f\|_{H_L^p(\mathbb{R}^n)} \leq C\|\psi\|_{L^\infty(\Sigma_\mu^0)}\|f\|_{H_L^p(\mathbb{R}^n)} \quad \text{for every } f \in H_L^p(\mathbb{R}^n).$$

For $p > 2$ the operator $\psi(L)$ can be defined on $H_L^p(\mathbb{R}^n)$ by duality:

$$(8.3) \quad \forall f \in H_L^p(\mathbb{R}^n), \quad p > 2, \quad \forall g \in H_{L^*}^{p'}(\mathbb{R}^n) \quad \langle \psi(L)f, g \rangle := \langle f, \psi(L^*)g \rangle,$$

and satisfies (8.2). In the same way $\psi(L)$ can be defined on $BMO_L(\mathbb{R}^n)$ and $\Lambda_L^\alpha(\mathbb{R}^n)$, $\alpha > 0$, and

$$(8.4) \quad \|\psi(L)f\|_{BMO_L(\mathbb{R}^n)} \leq C\|\psi\|_{L^\infty(\Sigma_\mu^0)}\|f\|_{BMO_L(\mathbb{R}^n)} \quad \text{for every } f \in BMO_L(\mathbb{R}^n),$$

$$(8.5) \quad \|\psi(L)f\|_{\Lambda_L^\alpha(\mathbb{R}^n)} \leq C\|\psi\|_{L^\infty(\Sigma_\mu^0)}\|f\|_{\Lambda_L^\alpha(\mathbb{R}^n)} \quad \text{for every } f \in \Lambda_L^\alpha(\mathbb{R}^n), \quad \alpha > 0.$$

Proof. Let $0 < p \leq 2$ and $\beta > \frac{n}{2} \left(\max\{\frac{1}{p}, 1\} - \frac{1}{2} \right)$. Now take $\psi \in \Psi_{\beta,\beta}(\Sigma_\mu^0)$ and build $\tilde{\psi} \in \Psi_{\beta,\beta}(\Sigma_\mu^0)$ using (4.17) so that (4.16) is satisfied. Then for any $g \in H_L^p(\mathbb{R}^n)$

$$(8.6) \quad Q_\psi g \in T^p(\mathbb{R}_+^{n+1}) \quad \text{and} \quad \|Q_\psi g\|_{T^p(\mathbb{R}_+^{n+1})} \leq C\|g\|_{H_L^p(\mathbb{R}^n)}.$$

Furthermore, by Proposition 4.4

$$(8.7) \quad Q_\psi \circ f(L) \circ \pi_{\tilde{\psi}} : T^p(\mathbb{R}_+^{n+1}) \longrightarrow T^p(\mathbb{R}_+^{n+1}),$$

and hence, by (8.6)

$$(8.8) \quad Q_\psi \circ f(L) = Q_\psi \circ f(L) \circ \pi_{\tilde{\psi}} \circ Q_\psi : H_L^p(\mathbb{R}^n) \longrightarrow T^p(\mathbb{R}_+^{n+1}).$$

By virtue of (4.22) the property (8.8) implies that

$$(8.9) \quad f(L) : H_L^p(\mathbb{R}^n) \longrightarrow H_L^p(\mathbb{R}^n),$$

thereby concluding the case $0 < p \leq 2$.

Now the functional calculus of L in H_L^p for $p > 2$, BMO_L and Λ_L^α , $\alpha > 0$, follows from (1.12) and Theorem 3.52. \square

8.2. Classical scales of function spaces measuring smoothness. So far we have worked with a few different scales of function spaces on \mathbb{R}^n : $L^p(\mathbb{R}^n)$, $1 < p \leq \infty$, Hardy spaces $H^p(\mathbb{R}^n)$, $0 < p \leq 1$, homogeneous Sobolev spaces $\dot{W}^{s,p}(\mathbb{R}^n)$, $s \in \mathbb{R}$, $1 < p < \infty$ (cf. (1.32)), and their counterparts for $p \leq 1$ and $s = 1$, namely the regular Hardy spaces $H^{1,p}(\mathbb{R}^n)$ defined in (5.23). All of them belong to (or can be identified with the members of) a more extensive scale of the Triebel-Lizorkin spaces, $\dot{F}_s^{p,q}(\mathbb{R}^n)$, $s \in \mathbb{R}$, $0 < p, q < \infty$.

Let us denote by \mathcal{F} the Fourier transform operator. We fix a Schwartz function φ such that:

- (1) $\text{supp } \mathcal{F}(\varphi) \subseteq \{\xi \in \mathbb{R}^n : \frac{1}{2} \leq |\xi| \leq 2\}$,
- (2) $|\mathcal{F}(\varphi)(\xi)| \geq c > 0$ uniformly for $\frac{2}{3} \leq |\xi| \leq \frac{5}{3}$,
- (3) $\sum_{i \in \mathbb{Z}} |\mathcal{F}(\varphi)(2^i \xi)|^2 = 1$ if $\xi \neq 0$.

Then for $s \in \mathbb{R}$, $0 < p < \infty$ and $0 < q \leq \infty$,

$$(8.10) \quad \dot{F}_s^{p,q}(\mathbb{R}^n) := \left\{ f \in \mathcal{S}'/\mathcal{P} : \|f\|_{\dot{F}_s^{p,q}(\mathbb{R}^n)} := \left\| \left(\sum_{i \in \mathbb{Z}} (2^{is}) |\varphi_i * f|^q \right)^{\frac{1}{q}} \right\|_{L^p} < \infty \right\},$$

where \mathcal{S}'/\mathcal{P} is the space of tempered distributions on \mathbb{R}^n modulo polynomials. We have

$$(8.11) \quad L^p(\mathbb{R}^n) \approx \dot{F}_0^{p,2}(\mathbb{R}^n), \quad 1 < p < \infty,$$

$$(8.12) \quad \dot{W}^{s,p}(\mathbb{R}^n) \approx \dot{F}_s^{p,2}(\mathbb{R}^n), \quad 1 < p < \infty, \quad s \in \mathbb{R},$$

$$(8.13) \quad H^p(\mathbb{R}^n) \approx \dot{F}_0^{p,2}(\mathbb{R}^n), \quad 0 < p \leq 1,$$

$$(8.14) \quad H^{1,p}(\mathbb{R}^n) \approx \dot{F}_1^{p,2}(\mathbb{R}^n), \quad 0 < p \leq 1.$$

The details on the identifications in (8.11) and (8.13) are presented in [32] (Remark 7.8 and Appendix B). The identifications (8.12) and (8.14) will be discussed after Lemma 8.17. We shall use the following basic properties of Triebel-Lizorkin spaces.

Lemma 8.15. *The space*

$$(8.16) \quad \mathcal{Z}(\mathbb{R}^n) := \{\varphi \in \mathcal{S}(\mathbb{R}^n) : (D^\alpha \mathcal{F} \varphi)(0) = 0 \text{ for every multiindex } \alpha\}$$

is a dense subspace of $\dot{F}_s^{p,q}(\mathbb{R}^n)$ for all $s \in \mathbb{R}$, $0 < p, q < \infty$.

Lemma 8.17. *The operator Δ^α , $\alpha \in \mathbb{R}$, is an isomorphism from $\dot{F}_s^{p,q}(\mathbb{R}^n)$ onto $\dot{F}_{s-2\alpha}^{p,q}(\mathbb{R}^n)$, $s \in \mathbb{R}$, $0 < p, q < +\infty$. Also, for any $m \in \mathbb{N}$,*

$$(8.18) \quad \dot{F}_s^{p,q}(\mathbb{R}^n) = \{f \in \mathcal{S}'/\mathcal{P} : D^\alpha f \in \dot{F}_{s-m}^{p,q}(\mathbb{R}^n), \forall \alpha \text{ with } |\alpha| = m\}.$$

Lemma 8.15 is proved in [58], Section 5.1.3, and Lemma 8.17 directly follows from Theorem 5.2.3 in [58]. Note that Lemma 8.17 together with (8.11) and (8.13) implies (8.12) and (8.14).

Finally, we would like to record the following consequence of the Kato estimate.

Lemma 8.19. *Let L be an operator defined by (1.1)–(1.3). Then L^α , $-1/2 \leq \alpha \leq 1/2$, is an isomorphism from $\dot{W}^{s,2}(\mathbb{R}^n)$ onto $\dot{W}^{s-2\alpha,2}(\mathbb{R}^n)$, $-1 \leq s \leq 1$.*

Proof. The Kato estimate (1.4) implies that $L^{1/2}$ maps the Sobolev space $\dot{W}^{1,2}(\mathbb{R}^n)$ isomorphically onto $L^2(\mathbb{R}^n)$. Using this observation and interpolation, one can further show that

$$(8.20) \quad L^\alpha, 0 \leq \alpha \leq 1/2, \text{ is an isomorphism between } \dot{W}^{2\alpha,2}(\mathbb{R}^n) \text{ and } L^2(\mathbb{R}^n),$$

(see, e.g. the proof of Proposition 6.3 in [7] for the details). Now we write $L^\alpha = L^{s/2} \circ L^{-s/2+\alpha}$ and use duality and (8.20) to finish the argument. \square

Interchanging the order in which L^p and ℓ^q norms are taken in (8.10), one would obtain the homogeneous Besov spaces $\dot{B}_s^{p,q}$, $s \in \mathbb{R}$, $0 < p, q \leq \infty$. There are also appropriate versions of (8.10) corresponding to $p = \infty$ or $q = \infty$; see, e.g., [32] for the definitions. Since we aim to concentrate on the properties of the operator L in Sobolev spaces and their counterparts for $p \leq 1$, we do not further elaborate on this point. However, below we will use the notation $\dot{F}_s^{p,2}$ in place of $\dot{W}^{s,p}$ and $H^{s,p}$ for uniformity and to avoid repetition when considering $p > 1$ and $p \leq 1$.

8.3. Weighted tent spaces. Let $s \in \mathbb{R}$, $0 < p, q < \infty$, and consider the spaces

$$(8.21) \quad T_s^{p,q}(\mathbb{R}_+^{n+1}) := \{F : \mathbb{R}_+^{n+1} \rightarrow \mathbb{C}; \|F\|_{T_s^{p,q}(\mathbb{R}_+^{n+1})} := \|\mathcal{A}_s^q F\|_{L^p(\mathbb{R}^n)} < \infty\},$$

where

$$(8.22) \quad \mathcal{A}_s^q F(x) := \left(\iint_{\Gamma(x)} |F(y,t)|^q \frac{dy dt}{t^{sq+n+1}} \right)^{1/q}, \quad x \in \mathbb{R}^n.$$

When $s = 0$, these are the classical tent spaces we discussed in Section 4. They were first introduced and studied in [21]. In particular, the authors established the complex interpolation of tent spaces for $s = 0$ and $p, q \geq 1$ (when the underlying spaces are Banach). Later on the complex interpolation of tent spaces was proved for $0 < p, q < \infty$ and $s = 0$ in [14], [19] (see also [3], [4], [15]). We stated a partial case of this result in (3.22). However, for the applications we have in mind we need to show that the tent spaces interpolate in s , p and q for the full range of indices.

Lemma 8.23. *For all $s_0, s_1 \in \mathbb{R}$, $0 < p_0, p_1 < \infty$, $0 < q_0, q_1 < \infty$,*

$$(8.24) \quad \left[T_{s_0}^{p_0,q_0}(\mathbb{R}_+^{n+1}), T_{s_1}^{p_1,q_1}(\mathbb{R}_+^{n+1}) \right]_\theta = T_s^{p,q}(\mathbb{R}_+^{n+1}), \quad 0 < \theta < 1,$$

where $s = (1 - \theta)s_0 + \theta s_1$, $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ and $\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$.

Proof. As we already mentioned (see the discussion preceding Lemma 4.24), extension of the complex interpolation method to quasi-Banach spaces is not straight forward, and over the years several approaches to this issue have been developed. Here we continue to follow the method of complex interpolation of analytically convex spaces which have been employed in the classical Hardy-Sobolev-Besov-Triebel-Lizorkin scales in [44], [51], [45], and for the tent spaces with $s = 0$ in [19].

According to Theorem 7.9 in [45] (see also [44]), we have

$$(8.25) \quad T_s^{p,q}(\mathbb{R}_+^{n+1}) = \left(T_{s_0}^{p_0,q_0}(\mathbb{R}_+^{n+1}) \right)^{(1-\theta)} \left(T_{s_1}^{p_1,q_1}(\mathbb{R}_+^{n+1}) \right)^\theta,$$

provided that $T_{s_i}^{p_i,q_i}(\mathbb{R}_+^{n+1})$, $i = 0, 1$, are analytically convex and separable. Here the space on the right-hand side of (8.25) is interpreted as a set of functions F :

$\mathbb{R}_+^{n+1} \rightarrow \mathbb{C}$ such that $|F| \leq |G|^{1-\theta}|H|^\theta$ for some $G \in T_{s_0}^{p_0, q_0}$ and $H \in T_{s_1}^{p_1, q_1}$, equipped with the natural infimum norm.

The fact that the tent spaces are separable is fairly obvious (note that $p, q < \infty$). Furthermore, any tent space is a quasi-Banach lattice (a quasi-Banach space with a partial order), and a quasi-Banach lattice X is analytically convex if it is lattice r -convex, i.e.

$$(8.26) \quad \left\| \left(\sum_{j=1}^m |f_j|^r \right)^{1/r} \right\|_X \leq \left(\sum_{j=1}^m \|f_j\|_X^r \right)^{1/r}$$

for any finite family $\{f_j\}_{1 \leq j \leq m} \subset X$ (see Theorem 7.8 in [45]). The elements of $T_s^{p, q}$ satisfy (8.26) with $r = \min\{p, q\}$ by Minkowski inequality. Hence, the spaces (8.21) are analytically convex and (8.25) applies.

Now recall the factorization results from [19] for the tent spaces without weight: (8.27)

$$T_0^{p, q}(\mathbb{R}_+^{n+1}) = T_0^{p_0, q_0}(\mathbb{R}_+^{n+1}) \cdot T_0^{p_1, q_1}(\mathbb{R}_+^{n+1}), \quad \frac{1}{p} = \frac{1}{p_0} + \frac{1}{p_1}, \quad \frac{1}{q} = \frac{1}{q_0} + \frac{1}{q_1},$$

where $0 < p, q \leq \infty$ and the product in (8.27) is interpreted similarly to (8.25). Since for all functions $F, G : \mathbb{R}_+^{n+1} \rightarrow \mathbb{C}$ and $s \in \mathbb{R}$ we have $\frac{F}{t^{s_0}} \frac{G}{t^{s_1}} = \frac{FG}{t^{s_0+s_1}}$, the formula (8.27) entails

$$(8.28) \quad T_s^{p, q}(\mathbb{R}_+^{n+1}) = T_{s_0}^{p_0, q_0}(\mathbb{R}_+^{n+1}) \cdot T_{s_1}^{p_1, q_1}(\mathbb{R}_+^{n+1}),$$

with $s = s_0 + s_1$, $\frac{1}{p} = \frac{1}{p_0} + \frac{1}{p_1}$ and $\frac{1}{q} = \frac{1}{q_0} + \frac{1}{q_1}$. Furthermore, it can be checked directly that $(T_s^{p, q})^r = T_{sr}^{p/r, q/r}$, so that (8.28) implies

$$(8.29) \quad (T_{s_0}^{p_0, q_0})^{(1-\theta)} (T_{s_1}^{p_1, q_1})^\theta = T_{s_0(1-\theta)}^{\frac{p_0}{(1-\theta)}, \frac{q_0}{(1-\theta)}} \cdot T_{s_1\theta}^{\frac{p_1}{\theta}, \frac{q_1}{\theta}} = T_s^{p, q}(\mathbb{R}_+^{n+1}),$$

for $s = (1-\theta)s_0 + \theta s_1$, $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ and $\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$. Together with (8.25) this finishes the proof. \square

8.4. Hardy-Sobolev spaces associated to L : general theory. Let us now define a smooth version of the Hardy spaces $H_L^{s, p}(\mathbb{R}^n)$, $0 \leq s \leq 1$, $0 < p \leq 2$, as a completion of $L^{-s/2}(L^2 \cap H_L^p)$ in the norm

$$(8.30) \quad \|f\|_{H_L^{s, p}(\mathbb{R}^n)} := \|S L^{s/2} f\|_{L^p(\mathbb{R}^n)} = \|L^{s/2} f\|_{H_L^p(\mathbb{R}^n)}.$$

Recall that $L^{-s/2}$ is an isomorphism of L^2 onto the space $\dot{W}^{s, 2}$, hence, $L^{-s/2}(L^2 \cap H_L^p)$ is a subspace of $\dot{W}^{s, 2}$, in particular, $L^{s/2} f$ is well-defined for every $f \in L^{-s/2}(L^2 \cap H_L^p)$. Moreover, it follows that

$$(8.31) \quad \dot{W}^{s, 2}(\mathbb{R}^n) \cap H_L^{s, p}(\mathbb{R}^n) \text{ is dense in } H_L^{s, p}(\mathbb{R}^n), \quad \text{for all } 0 \leq s \leq 1, 0 < p \leq 2.$$

Lemma 8.32. *The operator L^α , $-1/2 \leq \alpha \leq 1/2$, is an isomorphism of $H_L^{s, p}$ onto $H_L^{s-2\alpha, p}$ provided $0 \leq s - 2\alpha \leq 1$, $0 \leq s \leq 1$ and $0 < p \leq 2$.*

Proof. This result is a direct consequence of the definitions. Indeed, by definition $L^2 \cap H_L^p$ is dense in H_L^p and

$$(8.33) \quad \|L^{-\alpha} f\|_{H_L^{2\alpha,p}} = \|S L^\alpha L^{-\alpha} f\|_{L^p} = \|f\|_{H_L^p}, \quad \forall f \in L^2 \cap H_L^p, \quad 0 \leq \alpha \leq 1/2.$$

Hence, the operator $L^{-\alpha}$ extends by continuity to $L^{-\alpha} : H_L^p \rightarrow H_L^{2\alpha,p}$ and its range is closed in $H_L^{2\alpha,p}$. On the other hand, its range contains $L^{-\alpha}(L^2 \cap H_L^p)$, a dense subset of $H_L^{2\alpha,p}$, and therefore, the range of $L^{-\alpha}$ in $H_L^{2\alpha,p}$ actually coincides with $H_L^{2\alpha,p}$. Then $L^{-\alpha}$ is an isomorphism of H_L^p onto $H_L^{2\alpha,p}$, $0 \leq \alpha \leq 1/2$. Using this fact and writing $L^\alpha = L^{s/2} \circ L^{\alpha-s/2}$ we finish the proof of the Lemma. \square

Clearly, $H_L^{s,p}$ are analogues of the Sobolev spaces adapted to the elliptic operator L . In particular, Lemmas 8.32, 8.17 and the remark after (1.15) show that

$$(8.34) \quad H_\Delta^{s,p}(\mathbb{R}^n) \approx \dot{W}^{s,p}(\mathbb{R}^n), \quad 0 \leq s \leq 1, \quad 1 < p \leq 2.$$

As their counterparts for $L = \Delta$, the spaces $H_L^{s,p}(\mathbb{R}^n)$ are amenable to complex interpolation, satisfy natural duality properties, admit some version of the molecular decomposition etc. If necessary, the scale of $H_L^{s,p}$ spaces can be extended to the full range of p and s analogously to the Triebel-Lizorkin spaces. We do not pursue this subject in the present paper, and only mention the results which are important for the applications we have in mind.

Lemma 8.35. *The operator L has bounded holomorphic functional calculus in $H_L^{s,p}(\mathbb{R}^n)$ for all $0 \leq s \leq 1$ and $0 < p \leq 2$, in the sense that for every $\varphi \in H^\infty(\Sigma_\mu^0)$*

$$(8.36) \quad \varphi(L) : H_L^{s,p}(\mathbb{R}^n) \longrightarrow H_L^{s,p}(\mathbb{R}^n),$$

with the norm bounded by $\|\varphi\|_{L^\infty(\Sigma_\mu^0)}$.

Moreover, for every $\varphi \in \Psi'(\Sigma_\mu^0)$ and for all $0 \leq \alpha, \beta \leq 1$ and $0 < p \leq q \leq 2$

$$(8.37) \quad \varphi(L) : H_L^{\alpha,p}(\mathbb{R}^n) \longrightarrow H_L^{\beta,q}(\mathbb{R}^n),$$

and

$$(8.38) \quad \|\varphi(L)f\|_{H_L^{\beta,q}(\mathbb{R}^n)} \leq C \left\| z^{\frac{\alpha-\beta}{2} + \frac{1}{2}(\frac{n}{p} - \frac{n}{q})} \varphi \right\|_{L^\infty(\Sigma_\mu^0)} \|f\|_{H_L^{\alpha,p}(\mathbb{R}^n)},$$

whenever the L^∞ norm on the right-hand side is finite.

Proof. The Lemma follows directly from Lemmas 8.32 and 8.1 as soon as we observe that

$$(8.39) \quad \varphi(L) = \left(L^{\frac{\alpha-\beta}{2} + \frac{1}{2}(\frac{n}{p} - \frac{n}{q})} \varphi(L) \right) L^{-\frac{\alpha-\beta}{2} - \frac{1}{2}(\frac{n}{p} - \frac{n}{q})},$$

and by our assumptions the function $z \mapsto z^{\frac{\alpha-\beta}{2} + \frac{1}{2}(\frac{n}{p} - \frac{n}{q})} \varphi(z)$ belongs to $H^\infty(\Sigma_\mu^0)$. \square

Lemma 8.40. *For all $0 \leq s_0, s_1 \leq 1$ and $0 < p_0, p_1 \leq 2$*

$$(8.41) \quad \left[H_L^{s_0,p_0}(\mathbb{R}^n), H_L^{s_1,p_1}(\mathbb{R}^n) \right]_\theta = H_L^{s,p}(\mathbb{R}^n), \quad 0 < \theta < 1,$$

where $s = (1 - \theta)s_0 + \theta s_1$ and $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$.

Proof. Similarly to the case $s_0 = s_1 = 0$ we prove (8.41) via the reduction to the interpolation of tent spaces, this time, using the weighted tent spaces discussed in Section 8.3. Recall the operators Q_ψ and π_ψ introduced in (4.1) and (4.2), respectively. Let $\mu \in (\omega, \pi/2)$. Using Proposition 4.13, Lemma 8.32 and the fact that multiplication by t^{-s} is an isomorphism from $T_s^{p,2}(\mathbb{R}_+^{n+1})$ onto $T^p(\mathbb{R}_+^{n+1})$, we can verify that

$$(8.42) \quad Q_\psi : H_L^{s,p}(\mathbb{R}^n) \longrightarrow T_s^{p,2}(\mathbb{R}_+^{n+1}), \quad \text{and} \quad \pi_{\tilde{\psi}} : T_s^{p,2}(\mathbb{R}_+^{n+1}) \longrightarrow H_L^{s,p}(\mathbb{R}^n),$$

for $\psi \in \Psi_{\alpha,\beta}(\Sigma_\mu^0)$ and $\tilde{\psi} \in \Psi_{\beta,\alpha}(\Sigma_\mu^0)$, where $\alpha > \frac{s}{2}$ and $\beta > \frac{n}{2} \left(\max\{\frac{1}{p}, 1\} - \frac{1}{2} \right) - \frac{s}{2}$.

Now for any given (s_0, p_0) and (s_1, p_1) , $s_0 \leq s_1$, we choose $\psi \in \Psi_{\alpha,\beta}(\Sigma_\mu^0)$ and $\tilde{\psi} \in \Psi_{\beta,\alpha}(\Sigma_\mu^0)$, where $\alpha > \frac{s_1}{2}$ and $\beta > \frac{n}{2} \left(\max\{\frac{1}{p}, 1\} - \frac{1}{2} \right) - \frac{s_1}{2}$. Then the corresponding Q_ψ and $\pi_{\tilde{\psi}}$ satisfy (8.42) for all $s_0 \leq s \leq s_1$ and all p between p_0 and p_1 . The rest of the proof follows the same lines as the proof of Lemma 4.24. We omit the remaining details, except to note that by Lemma 8.32 and Theorem 7.2, the $H_L^{s,p}$ spaces under consideration embed into H_L^p or Λ_L^α spaces falling under the scope of Proposition 10.1 below, and thus may all be embedded into a common ambient Banach space. \square

8.5. Hardy-Sobolev spaces associated to L : identifications with classical scales.

Proposition 8.43. *For every p such that $\frac{p-(L)n}{n+p-(L)} < p \leq 2$*

$$(8.44) \quad H_L^{1,p}(\mathbb{R}^n) \approx \dot{F}_1^{p,2}(\mathbb{R}^n).$$

Proof. Step I. First, we would like to show that

$$(8.45) \quad H_L^{1,p}(\mathbb{R}^n) \hookrightarrow \dot{F}_1^{p,2}(\mathbb{R}^n), \quad \text{for} \quad \frac{n}{n+1} < p \leq 2.$$

By Propositions 5.6, 5.32 and (8.11), (8.13) we have

$$(8.46) \quad \nabla L^{-1/2} : H_L^p(\mathbb{R}^n) \longrightarrow \dot{F}_0^{p,2}(\mathbb{R}^n), \quad \text{if} \quad \frac{n}{n+1} < p < 2 + \varepsilon(L).$$

On the other hand, according to Lemma 8.32 the operator $L^{1/2}$ is an isomorphism of $H_L^{1,p}$ onto H_L^p for $0 < p \leq 2$. Hence, $L^{1/2}g \in H_L^p$ for every $g \in H_L^{1,p}$, and

$$(8.47) \quad \|\nabla g\|_{\dot{F}_0^{p,2}(\mathbb{R}^n)} = \|\nabla L^{-1/2} L^{1/2} g\|_{\dot{F}_0^{p,2}(\mathbb{R}^n)} \leq C \|L^{1/2} g\|_{H_L^p(\mathbb{R}^n)} \leq C \|g\|_{H_L^{1,p}(\mathbb{R}^n)}, \quad \forall g \in H_L^{1,p},$$

if $\frac{n}{n+1} < p < 2 + \varepsilon(L)$. This gives the desired norm estimate (see Lemma 8.17). It remains to show that the elements of $H_L^{1,p}(\mathbb{R}^n)$ can be seen as tempered distributions modulo polynomials.

Indeed, by (8.31) for every $g \in H_L^{1,p}$ there is a sequence $\{g_n\}_{n=1}^\infty \subset \dot{W}^{1,2} \cap H_L^{1,p}$ converging to g in $H_L^{1,p}$ norm. Then

$$(8.48) \quad \{g_n\}_{n=1}^\infty \subset \dot{W}^{1,2} \approx \dot{F}_1^{2,2} \subset \mathcal{S}'/\mathcal{P},$$

in particular, g_n , $n = 1, 2, \dots$, are tempered distributions modulo polynomials. Also, $\{g_n\}_{n=1}^\infty$ is a Cauchy sequence in $H_L^{1,p}$ norm. Hence,

$$(8.49) \quad \{\nabla g_n\}_{n=1}^\infty \text{ is Cauchy in } \dot{F}_0^{p,2} \text{ norm}$$

by (8.47). Combining (8.48), (8.49) and Lemma 8.17, we conclude that $\{g_n\}_{n=1}^\infty \subset \dot{F}_1^{p,2}$ and $\{g_n\}_{n=1}^\infty$ is Cauchy in $\dot{F}_1^{p,2}$. Now g can be identified with the limit of $\{g_n\}$ in $\dot{F}_1^{p,2}$.

Step II. Now we concentrate on the inverse inclusion, and show that

$$(8.50) \quad H_L^{1,p}(\mathbb{R}^n) \hookrightarrow \dot{F}_1^{p,2}(\mathbb{R}^n), \quad \text{for } \frac{p-(L)n}{n+p-(L)} < p \leq 2.$$

It follows from (5.22), (5.39), (8.12) and (8.14) that

$$(8.51) \quad S_1 \sqrt{L} : \dot{F}_1^{p,2}(\mathbb{R}^n) \longrightarrow L^p(\mathbb{R}^n), \quad \text{for } \frac{p-(L)n}{n+p-(L)} < p \leq 2.$$

Combining this with (5.21) we have

$$(8.52) \quad \|f\|_{H_L^{1,p}(\mathbb{R}^n)} \leq C \|f\|_{\dot{F}_1^{p,2}(\mathbb{R}^n)}, \quad \forall f \in \dot{F}_1^{p,2}(\mathbb{R}^n), \quad \frac{p-(L)n}{n+p-(L)} < p \leq 2,$$

and it remains to show that f actually belongs to $H_L^{1,p}(\mathbb{R}^n)$, i.e., that it can be approximated by the elements of $L^{-1/2}(L^2 \cap H_L^p)$.

According to Lemma 8.15, $\mathcal{Z}(\mathbb{R}^n)$ is a dense subset of $\dot{F}_1^{p,2}(\mathbb{R}^n)$. Then every f in (8.52) can be approximated in $\dot{F}_1^{p,2}(\mathbb{R}^n)$ norm by a sequence $\{f_n\}_{n=1}^\infty \subset \mathcal{Z}(\mathbb{R}^n)$. The operator \sqrt{L} maps $\dot{W}^{1,2} \approx \dot{F}_1^{2,2}$ to $L^2(\mathbb{R}^n)$ and $\mathcal{Z}(\mathbb{R}^n)$ is a subset of $\dot{F}_1^{2,2}$. Hence,

$$(8.53) \quad \sqrt{L}f_n \in L^2(\mathbb{R}^n), \quad n = 1, 2, \dots$$

Since, in addition, $\|\sqrt{L}f_n\|_{H_L^p(\mathbb{R}^n)}$ is finite for every $n = 1, 2, \dots$ by (8.51), we can conclude that $\{\sqrt{L}f_n\}_{n=1}^\infty \subset L^2 \cap H_L^p$ and therefore, $\{f_n\}_{n=1}^\infty \subset L^{-1/2}(L^2 \cap H_L^p)$.

By our assumptions $\{f_n\}_{n=1}^\infty$ is Cauchy in $\dot{F}_1^{p,2}(\mathbb{R}^n)$ norm. Then by (8.51), it is also Cauchy in $H_L^{1,p}(\mathbb{R}^n)$ norm and belongs to $L^{-1/2}(L^2 \cap H_L^p)$. Now we identify its limit in $H_L^{1,p}(\mathbb{R}^n)$ with $f \in \dot{F}_1^{p,2}(\mathbb{R}^n)$, and derive (8.50) with the appropriate norm estimate. \square

8.6. Functional calculus and fractional powers of L in Sobolev and regular Hardy spaces. In this section we restrict ourselves to the case $n \geq 3$. One can derive analogues of all the results below for $n = 2$ following the same arguments. We will not state them for the sake of brevity.

Theorem 8.54. *Let L be an elliptic operator satisfying (1.1)–(1.3), and let $p(L)$ and $\varepsilon(L)$ retain the same significance as before. Assume that $-1 \leq s \leq 1$ and $0 < p < \infty$ are such that either of the conditions (1) or (2) below is satisfied*

$$(1) \quad -1 \leq s \leq 0 \quad \text{and}$$

$$(8.55) \quad \max \left\{ 0, \frac{1}{n} s + 1 - \frac{1}{p_-(L^*)} \right\} < \frac{1}{p} < \left(\frac{1}{2+\varepsilon(L^*)} - 1 + \frac{1}{p_-(L)} \right) s + \frac{1}{p_-(L)},$$

$$(8.56)$$

$$(2) \quad 0 \leq s \leq 1 \quad \text{and} \quad \left(\frac{1}{2+\varepsilon(L)} - 1 + \frac{1}{p_-(L^*)} \right) s + 1 - \frac{1}{p_-(L^*)} < \frac{1}{p} < \frac{1}{n} s + \frac{1}{p_-(L)}.$$

Then L has bounded holomorphic functional calculus in $\dot{F}_s^{p,2}(\mathbb{R}^n)$, in the sense that for every $\varphi \in H^\infty(\Sigma_\mu^0)$

$$(8.57) \quad \varphi(L) : \dot{F}_s^{p,2}(\mathbb{R}^n) \longrightarrow \dot{F}_s^{p,2}(\mathbb{R}^n),$$

with the norm bounded by $\|\varphi\|_{L^\infty(\Sigma_\mu^0)}$.

Moreover, for every $\varphi \in \Psi'(\Sigma_\mu^0)$

$$(8.58) \quad \varphi(L) : \dot{F}_\alpha^{p,2}(\mathbb{R}^n) \longrightarrow \dot{F}_\beta^{q,2}(\mathbb{R}^n),$$

and

$$(8.59) \quad \|\varphi(L)f\|_{\dot{F}_\beta^{q,2}(\mathbb{R}^n)} \leq C \left\| z^{\frac{\alpha-\beta}{2} + \frac{1}{2}(\frac{n}{p} - \frac{n}{q})} \varphi \right\|_{L^\infty(\Sigma_\mu^0)} \|f\|_{\dot{F}_\alpha^{p,2}(\mathbb{R}^n)},$$

whenever the L^∞ norm on the right-hand side is finite, $p \leq q$, and the pairs $(\alpha, 1/p)$, $(\beta, 1/q)$ satisfy (8.55) or (8.56), i.e., at least one of the conditions (8.55), (8.56) holds with $s = \alpha$ and at least one of the conditions (8.55), (8.56) holds with $s = \beta$, $p = q$.

The range of s and p satisfying either (8.55) or (8.56) can be identified with a polygon on the $(s, 1/p)$ plane. The shape of such a polygon depends on whether $\frac{n+p_-(L^*)}{np_-(L^*)} < 1$ (in which case we will denote the corresponding polygon by $\mathcal{R}_1(L)$) or $\frac{n+p_-(L^*)}{np_-(L^*)} \geq 1$ (then the polygon will be denoted by $\mathcal{R}_2(L)$).

First, assume that $\frac{n+p_-(L^*)}{np_-(L^*)} < 1$. The region $\mathcal{R}_1(L)$ consists of the open polygon with vertices

$$(8.60) \quad \begin{aligned} B_L &= \left(-1, 1 - \frac{1}{2+\varepsilon(L^*)}\right), & E_L &= \left(0, \frac{1}{p_-(L)}\right), & C_L &= \left(1, \frac{n+p_-(L)}{np_-(L)}\right), \\ A_L &= \left(-1, 1 - \frac{n+p_-(L^*)}{np_-(L^*)}\right), & F_L &= \left(0, 1 - \frac{1}{p_-(L^*)}\right), & D_L &= \left(1, \frac{1}{2+\varepsilon(L)}\right), \end{aligned}$$

together with the sides $A_L B_L$ and $C_L D_L$. It is shown on Figure 3.

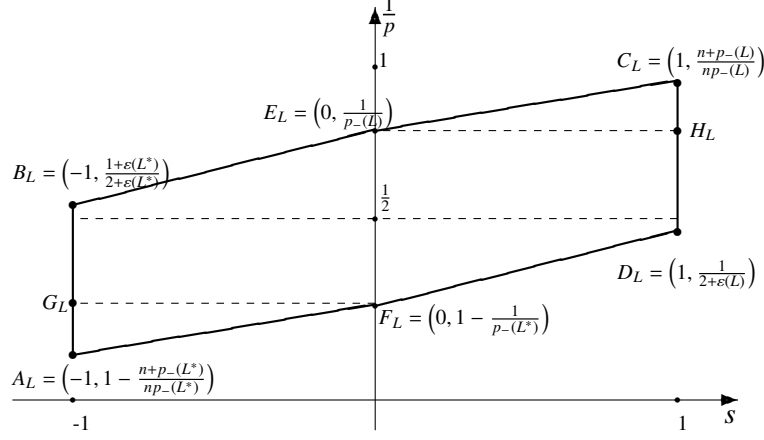


Figure 3 – the region $\mathcal{R}_1(L)$.

For the case $\frac{n+p_-(L^*)}{np_-(L^*)} \geq 1$ we define the second region, $\mathcal{R}_2(L)$, as an open polygon with the vertices

$$(8.61) \quad \begin{aligned} B_L &= \left(-1, \frac{1+\varepsilon(L^*)}{2+\varepsilon(L^*)}\right), & E_L &= \left(0, \frac{1}{p_-(L)}\right), & C_L &= \left(1, \frac{n+p_-(L)}{np_-(L)}\right), \\ \tilde{A}_L &= (-1, 0), & \tilde{F}_L &= \left(0, \frac{n}{p_-(L^*)} - n\right), \\ F_L &= \left(0, 1 - \frac{1}{p_-(L^*)}\right), & D_L &= \left(1, \frac{1}{2+\varepsilon(L)}\right), \end{aligned}$$

together with the sides $\tilde{A}_L B_L$ and $C_L D_L$. Its picture is a modified version of Figure 3, much as Figure 2 is a modification of Figure 1.

Proof of Theorem 8.54. Let us introduce auxiliary points $O = (0, 0)$, $B = (-1, 1/2)$ and $D = (1, 1/2)$. As we already mentioned, the statement of the theorem was proved for all p, q which in addition to the aforementioned restrictions satisfy $p, q > p_-(L)$ (see [7]). Thus, the interior of the polygon $G_L B_L E_L H_L D_L F_L$ is already covered (i.e. the statement of the Theorem holds with \mathcal{R}_1 substituted by $G_L B_L E_L H_L D_L F_L$). The same argument applies to the segments $G_L B_L$ and $H_L D_L$.

Next, (1.15), (8.44) and Lemma 8.40 together with the well-known results on the complex interpolation of Triebel-Lizorkin spaces lead to the conclusion that $H_L^{s,p}(\mathbb{R}^n) \approx \dot{F}_s^{p,2}(\mathbb{R}^n)$ whenever $(s, 1/p)$ belongs to $OE_L C_L D$ or the segment $C_L D$. Then, by Lemma 8.35, the statement of the theorem holds in $OE_L C_L D$ and on the segment $C_L D$.

Combining these observations, we recover the result on the entire \mathcal{R}_1 or \mathcal{R}_2 using duality and interpolation. \square

Remark. When $\frac{n+p_-(L^*)}{np_-(L^*)} \geq 1$, then Theorem 8.54 can be complemented by the corresponding results for $p = \infty$. Specifically, consider the spaces $\dot{F}_s^{\infty,\infty}$. For $-1 \leq s < 0$ they can be seen, e.g., as the dual spaces for $\dot{F}_1^{p,2}(\mathbb{R}^n)$ with $p = \frac{n}{n+s+1}$.

Then for every $\varphi \in H^\infty(\Sigma_\mu^0)$

$$(8.62) \quad \varphi(L) : \dot{F}_s^{\infty, \infty}(\mathbb{R}^n) \longrightarrow \dot{F}_s^{\infty, \infty}(\mathbb{R}^n),$$

whenever $-1 \leq \alpha < n\left(\frac{1}{p-(L^*)} - 1\right)$. In the same way the spaces $\dot{F}_\alpha^{\infty, \infty}$ can be incorporated in (8.58)–(8.59), that is, we can say that (8.58)–(8.59) hold whenever $p \leq q$ and $(\alpha, 1/p), (\beta, 1/q)$ belong to $\widetilde{\mathcal{R}}_2 = \mathcal{R}_2 \cup \widetilde{A}_L \widetilde{F}_L$, where the segment $\widetilde{A}_L \widetilde{F}_L$ corresponds to the classes $\dot{F}_s^{\infty, \infty}$.

The Theorem 8.54 and sharpness results in Section 2.2 lead to the complete description of all functions spaces on Hardy-Sobolev-Triebel-Lizorkin scale where one can develop functional calculus for an arbitrary elliptic operator satisfying (1.1)–(1.3).

Corollary 8.63. *Let L be an elliptic operator satisfying (1.1)–(1.3), and assume that $s \in \mathbb{R}$, $p \in (0, \infty)$ are such that*

$$(8.64) \quad -1 \leq s \leq 1 \quad \text{and} \quad \max\left\{0, \frac{1}{n}s + \frac{n-2}{2n}\right\} \leq \frac{1}{p} \leq \frac{1}{n}s + \frac{n+2}{2n}.$$

Then L has a bounded holomorphic functional calculus in $\dot{F}_s^{p,2}(\mathbb{R}^n)$, in the sense that

$$(8.65) \quad \varphi(L) : \dot{F}_s^{p,2}(\mathbb{R}^n) \longrightarrow \dot{F}_s^{p,2}(\mathbb{R}^n), \quad \text{for every } \varphi \in H^\infty(\Sigma_\mu^0),$$

with the norm bounded by $\|\varphi\|_{L^\infty(\Sigma_\mu^0)}$.

More generally, if $0 < p \leq q < \infty$ and the pairs α, p and β, q satisfy (8.64), i.e.

$$(8.66) \quad -1 \leq \alpha \leq 1 \quad \text{and} \quad \max\left\{0, \frac{1}{n}\alpha + \frac{n-2}{2n}\right\} \leq \frac{1}{p} \leq \frac{1}{n}\alpha + \frac{n+2}{2n},$$

$$(8.67) \quad -1 \leq \beta \leq 1 \quad \text{and} \quad \max\left\{0, \frac{1}{n}\beta + \frac{n-2}{2n}\right\} \leq \frac{1}{q} \leq \frac{1}{n}\beta + \frac{n+2}{2n},$$

then

$$(8.68) \quad \varphi(L) : \dot{F}_\alpha^{p,2}(\mathbb{R}^n) \longrightarrow \dot{F}_\beta^{q,2}(\mathbb{R}^n),$$

with

$$(8.69) \quad \|\varphi(L)f\|_{\dot{F}_\beta^{q,2}(\mathbb{R}^n)} \leq C \left\| \left| z \right|^{\frac{\alpha-\beta}{2} + \frac{1}{2}\left(\frac{n}{p} - \frac{n}{q}\right)} \varphi \right\|_{L^\infty(\Sigma_\mu^0)} \|f\|_{\dot{F}_\alpha^{p,2}(\mathbb{R}^n)},$$

for every $\varphi \in \Psi'(\Sigma_\mu^0)$ such that the L^∞ norm on the right-hand side of (8.69) is finite.

These result are sharp for all $n \geq 3$. For every $-1 \leq s \leq 1$, $0 < p < \infty$ not satisfying (8.64) there exists an elliptic operator L such that the heat semigroup is not bounded in $\dot{F}_s^{p,2}(\mathbb{R}^n)$ and hence, the property (8.65) does not hold. Similarly, (8.68), (8.69) need not hold if α, p or β, q do not satisfy (8.66)–(8.67).

The Corollary 8.63 extends to the case $p = \infty$ in the vein of remark after the proof of Theorem 8.54.

As we mentioned in the introduction, the range of indices s and p satisfying (8.64) can be described as a region on $(s, 1/p)$ plane.

Assume first that $n \geq 4$. We denote by \mathcal{R}_1 a *closed* polygon on $(s, 1/p)$ plane with vertices at

$$(8.70) \quad \begin{aligned} A &= \left(-1, \frac{n-4}{2n}\right), & B &= \left(-1, \frac{1}{2}\right), \\ C &= \left(1, \frac{n+4}{2n}\right), & D &= \left(1, \frac{1}{2}\right). \end{aligned}$$

On an $(s, 1/p)$ plane the Region \mathcal{R}_1 is shown on Figure 1.

Now let $n \leq 4$, and let \mathcal{R}_2 be a *closed* polygon on $(s, 1/p)$ plane with vertices at

$$(8.71) \quad \begin{aligned} \tilde{A} &= (-1, 0), & B &= \left(-1, \frac{1}{2}\right), \\ C &= \left(1, \frac{n+4}{2n}\right), & D &= \left(1, \frac{1}{2}\right), \\ \tilde{F} &= \left(\frac{2-n}{2}, 0\right). \end{aligned}$$

The region \mathcal{R}_2 is depicted on Figure 2.

Observe that for $n = 4$ we have $\mathcal{R}_1 = \mathcal{R}_2$, and the corresponding picture can be seen as an extreme case of \mathcal{R}_1 (with $A = (-1, 0)$ and $C = (1, 1)$) or an extreme case of \mathcal{R}_2 (with $\tilde{A} = \tilde{F} = (-1, 0)$).

In general, as dimension decreases, the slope of the line BC becomes larger, while B is fixed and C moves up along the line $\{s = 1\}$. When $n = 4$, $C = (1, 1)$ and for $n \leq 3$ the point C corresponds to $p < 1$. Strictly speaking, the Figure 2 shows \mathcal{R}_2 for $n = 3$, and as we mentioned above, $n = 4$ is its extreme case.

All in all, $s \in [-1, 1]$ and $p \in (0, \infty]$ satisfy (8.64) if and only if the point $(s, 1/p)$ belongs to \mathcal{R}_1 ($n \geq 4$) or to \mathcal{R}_2 ($n \leq 4$). As before, the segment $\tilde{A}\tilde{F}$ corresponds to the spaces $\dot{F}_s^{\infty, \infty}$.

Proof of Corollary 8.63. The Corollary follows from Theorem 8.54 and the fact that $p_-(L) < \frac{2n}{n+2}$ for every elliptic operator L . The sharpness is a consequence of Proposition 2.10. Indeed, if $n \geq 4$ and for some point $(s_0, 1/p_0) \notin \mathcal{R}_1$ the heat semigroup e^{-tL} , $t > 0$, is bounded in $\dot{F}_{s_0}^{p_0, 2}(\mathbb{R}^n)$ for all elliptic operators L , then by interpolation the heat semigroup is bounded in all $\dot{F}_s^{p, 2}(\mathbb{R}^n)$ with $(s, 1/p)$ in the linear span of $(s_0, 1/p_0)$ and \mathcal{R}_1 . In particular, there exists $p \notin \left[\frac{2n}{n+2}, \frac{2n}{n-2}\right]$ such that the heat semigroup is bounded in L^p for all L , which contradicts Proposition 2.10. Similarly, when $n = 3$, we discover such a contradiction starting with any $(s_0, 1/p_0) \notin \mathcal{R}_2$. \square

9. APPENDIX 1: RELATIONSHIPS BETWEEN H_L^p AND CLASSICAL H^p

In this Appendix, we establish (1.15) - (1.17). We note that the containments in (1.16) (resp. (1.17)) are strict if $1 < p_-(L)$ (resp. $p_+(L) < \infty$). For example, see item (vi) in Proposition 9.1 below, and its proof.

We recall that classical $H^p(\mathbb{R}^n) = L^p(\mathbb{R}^n)$, if $1 < p < \infty$, that $(p_-(L), p_+(L))$ is the interior of the interval of L^p boundedness of the heat semigroup e^{-tL} , and that $p_-(L) < 2n/(n+2)$ and $p_+(L) > 2n/(n-2)$, if $n > 2$. For $\alpha > 0$, we let $\Lambda^\alpha(\mathbb{R}^n)$ denote the classical homogeneous ‘‘Lip $_\alpha$ ’’ spaces (cf. (9.6) below), and in the case $\alpha = 0$, we let $\Lambda^0(\mathbb{R}^n)$, $\Lambda_L^0(\mathbb{R}^n)$ denote, respectively, the classical and L -adapted

BMO spaces $BMO(\mathbb{R}^n)$ and $BMO_L(\mathbb{R}^n)$. We define null spaces

$$\mathcal{N}_p(L) := \{f \in L^p(\mathbb{R}^n) \cap W_{loc}^{1,2} : Lf = 0\}, \quad p_+(L) \leq p < \infty,$$

and

$$\mathcal{N}_\alpha(L) := \{\varphi \in \Lambda^\alpha(\mathbb{R}^n) \cap W_{loc}^{1,2} : L\varphi = 0\}, \quad 0 \leq \alpha.$$

Proposition 9.1. *We have the following containments and continuous embeddings:*

$$(i) \quad L^2(\mathbb{R}^n) \cap H_L^p(\mathbb{R}^n) \subset L^2(\mathbb{R}^n) \cap H^p(\mathbb{R}^n), \quad n/(n+1) < p \leq 1, \text{ and}$$

$$(9.2) \quad \|f\|_{H^p(\mathbb{R}^n)} \lesssim \|f\|_{H_L^p(\mathbb{R}^n)}, \quad f \in L^2(\mathbb{R}^n) \cap H_L^p(\mathbb{R}^n).$$

$$(ii) \quad L^2(\mathbb{R}^n) \cap H_L^p(\mathbb{R}^n) \subset L^2(\mathbb{R}^n) \cap L^p(\mathbb{R}^n), \quad 1 < p \leq p_-(L), \text{ and}$$

$$(9.3) \quad \|f\|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_{H_L^p(\mathbb{R}^n)}, \quad f \in L^2(\mathbb{R}^n) \cap H_L^p(\mathbb{R}^n).$$

$$(iii) \quad L^p(\mathbb{R}^n)/\mathcal{N}_p(L) \hookrightarrow H_L^p(\mathbb{R}^n), \quad p_+(L) \leq p < \infty, \text{ and}$$

$$(9.4) \quad \|f\|_{H_L^p(\mathbb{R}^n)} \leq C\|f\|_{L^p(\mathbb{R}^n)}, \quad p_+(L) \leq p < \infty.$$

$$(iv) \quad \Lambda^\alpha(\mathbb{R}^n)/\mathcal{N}_\alpha(L) \hookrightarrow \Lambda_L^\alpha(\mathbb{R}^n), \quad 0 \leq \alpha < 1,^{12} \text{ and}$$

$$(9.5) \quad \|\varphi\|_{\Lambda_L^\alpha(\mathbb{R}^n)} \leq C\|\varphi\|_{\Lambda^\alpha(\mathbb{R}^n)}, \quad 0 \leq \alpha < 1.$$

Moreover,

$$(v) \quad H_L^p(\mathbb{R}^n) = L^p(\mathbb{R}^n), \quad p_-(L) < p < p_+(L).$$

$$(vi) \quad H_L^p(\mathbb{R}^n) \neq L^p(\mathbb{R}^n), \quad 1 < p \leq p_-(L) \text{ or } p_+(L) \leq p < \infty.$$

Finally, for each $p > 2n/(n-2)$, $n \geq 3$ (resp., for each $\alpha \in [0, 1)$), there is an operator L and a non-trivial $u \in L^p(\mathbb{R}^n)$ (resp., $u \in \Lambda^\alpha(\mathbb{R}^n)$) such that $Lu = 0$ weakly in \mathbb{R}^n . Thus, for each such p or α , there is an operator L for which the corresponding null space $\mathcal{N}_p(L)$ or $\mathcal{N}_\alpha(L)$ is non-trivial.

Proof. We carry out the proof in the following order: (iv), (v), (iii), (i), (ii), (vi) and then conclude by presenting examples of non-trivial global null solutions.

Proof of (iv). Fix $\varphi \in \Lambda^\alpha$, $0 \leq \alpha < 1$. By definition, for $n/(n+1) < p \leq 1$ (as is the case if $0 \leq \alpha = n(p^{-1} - 1) < 1$), an $H_L^p(\mathbb{R}^n)$ -molecule is, in particular, a classical $H^p(\mathbb{R}^n)$ -molecule (since the operator L kills constants). Consequently, by the classical duality results [31, 28] we have that $\varphi \in \mathbf{M}_{\alpha, L^*}^{M, *}$, the ambient space in which Λ_L^α is defined (cf. (1.25) and the related discussion, bearing in mind that in our present context, the roles of L and L^* have been reversed). Also, $\|\varphi\|_{\Lambda_L^\alpha} = 0$ for $\varphi \in \mathcal{N}_\alpha(L)$, by definition of the Λ_L^α norm (cf. (1.26), but with L in place of L^*). Thus, to prove (iv), it suffices to show that φ satisfies the norm estimate (9.5). To this end, we fix a cube $Q \subset \mathbb{R}^n$, and use the fact that $e^{-tL}1 = 1$ to write

$$\begin{aligned} & \frac{1}{|Q|^{\alpha/n}} \left(\frac{1}{|Q|} \int_Q \left| (I - e^{-t(Q)^2 L})^M \varphi(x) \right|^2 dx \right)^{1/2} \\ &= \frac{1}{|Q|^{\alpha/n}} \left(\frac{1}{|Q|} \int_Q \left| (I - e^{-t(Q)^2 L})^M (\varphi - \varphi_Q)(x) \right|^2 dx \right)^{1/2}, \end{aligned}$$

¹²In the presence of pointwise heat kernel bounds, the case $\alpha = 0$ of (iv) was previously obtained in [25].

where $\varphi_Q := \int_Q \varphi$. It is then a routine matter to verify that this last expression is bounded uniformly in Q by either $\|\varphi\|_{BMO}$ (if $\alpha = 0$), or by

$$(9.6) \quad \|\varphi\|_{\Lambda^\alpha(\mathbb{R}^n)} := \sup_{x \neq y} \frac{|\varphi(x) - \varphi(y)|}{|x - y|^\alpha}$$

(if $0 < \alpha < 1$), using a dyadic annular decomposition plus the Gaffney estimates, much as in the proof of (6.3). We omit the details.

Proof of (v). Recall that $L^2(\mathbb{R}^n) \cap H_L^p(\mathbb{R}^n)$ is dense in $H_L^p(\mathbb{R}^n)$ (by definition, if $0 < p \leq 2$, and as proved in Corollary 4.21, if $2 < p < \infty$). Of course, $L^2(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$. Therefore, it is enough to show that $L^2(\mathbb{R}^n) \cap L^p(\mathbb{R}^n) = L^2(\mathbb{R}^n) \cap H_L^p(\mathbb{R}^n)$, with equivalence of norms.

One direction is easy: fix $f \in L^2(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$, $p_-(L) < p < p_+(L)$. By Corollary 4.21, for appropriate ψ we have that

$$\|f\|_{H_L^p(\mathbb{R}^n)} \approx \left\| \left(\int \int_{\Gamma(\cdot)} |\psi(t^2 L)f(y)|^2 \frac{dy dt}{t^{n+1}} \right)^{1/2} \right\|_{L^p(\mathbb{R}^n)} \leq C \|f\|_{L^p(\mathbb{R}^n)},$$

where the last step follows by the same argument used in [7] to treat the special case $\psi(z) = ze^{-z}$.

Conversely, suppose that $f \in L^2(\mathbb{R}^n) \cap H_L^p(\mathbb{R}^n)$, and let $g \in L^2(\mathbb{R}^n) \cap L^{p'}(\mathbb{R}^n)$, with $\|g\|_{L^{p'}(\mathbb{R}^n)} = 1$. By the Calderón reproducing formula (4.16), for appropriate $\psi, \tilde{\psi}$ we have that

$$\begin{aligned} \left| \int_{\mathbb{R}^n} f \bar{g} \right| &= \left| \int_{\mathbb{R}^n} \pi_{\tilde{\psi}, L} \circ Q_{\psi, L} f \bar{g} \right| \\ &\leq \|Q_{\psi, L} f\|_{T^p(\mathbb{R}_+^{n+1})} \|Q_{\tilde{\psi}, L^*} g\|_{T^{p'}(\mathbb{R}_+^{n+1})} \leq C \|f\|_{H_L^p(\mathbb{R}^n)} \|g\|_{L^{p'}(\mathbb{R}^n)}, \end{aligned}$$

where in the last step we have used (4.14) and the square function bounds of [7]. The latter are applicable to the adjoint operator L^* in $L^{p'}(\mathbb{R}^n)$ since $p_+(L^*) = (p_-(L))'$. Taking the supremum over all such g , we obtain that

$$\|f\|_{L^p(\mathbb{R}^n)} \leq C \|f\|_{H_L^p(\mathbb{R}^n)},$$

as desired.

Proof of (iii). We interpolate the inclusion map between $p = 2$ and $p = \infty$ (i.e., $\alpha = 0$ in (iv)), to obtain (9.4). In turn, Theorem 6.1 implies that $\|f\|_{H_L^p(\mathbb{R}^n)} = 0$ for $f \in \mathcal{N}_p(L)$, whence (iii) follows.

Proof of (i). We suppose that $n/(n+1) < p \leq 1$. As noted above, an $H_L^p(\mathbb{R}^n)$ -molecule is also a classical $H^p(\mathbb{R}^n)$ -molecule, if $n/(n+1) < p \leq 1$. Consequently, by (3.7) and the molecular decomposition of classical H^p spaces, we have that $L^2(\mathbb{R}^n) \cap H_L^p(\mathbb{R}^n) \subset L^2(\mathbb{R}^n) \cap H^p(\mathbb{R}^n)$ and (9.2) holds.

Remark: by the density of $L^2(\mathbb{R}^n) \cap H_L^p(\mathbb{R}^n)$ in $H_L^p(\mathbb{R}^n)$, one may now extend the identity map by continuity to produce an “embedding” $\mathcal{J} : H_L^p(\mathbb{R}^n) \rightarrow H^p(\mathbb{R}^n)$, which equals the identity on $L^2(\mathbb{R}^n) \cap H_L^p(\mathbb{R}^n)$. It remains an open question to determine whether, in general, this embedding is necessarily 1-1.

We further remark that, in the case $p = 1$, the containment $L^2 \cap H_L^1 \subset L^2 \cap H^1$ amounts to saying that, for $f \in L^2 \cap H_L^1$, the limits of the molecular decomposition $f = \sum \lambda_j m_j$, in H_L^1 , H^1 and L^1 , are all the same. It is not known whether the same can be said for an *arbitrary* element of H_L^1 , except in the special case that the kernel of the heat semigroup e^{-tL} enjoys a *pointwise* Gaussian upper bound. In that case, it is a routine matter to verify that one has the 1-1 embedding $H_L^1 \hookrightarrow H^1$.

Proof of (ii). Let $f \in L^2(\mathbb{R}^n) \cap H_L^p(\mathbb{R}^n)$, $1 < p \leq p_-(L)$, and let $g \in L^2(\mathbb{R}^n) \cap L^{p'}(\mathbb{R}^n)$, so that in particular, by (iii) above, we have that $g \in H_{L^*}^{p'}(\mathbb{R}^n)$ (here we are using that $(p_-(L))' = p_+(L^*)$). Then for such f, g , we have that

$$\left| \int_{\mathbb{R}^n} f \bar{g} \right| = |\langle f, g \rangle| \leq \|f\|_{H_L^p(\mathbb{R}^n)} \|g\|_{H_{L^*}^{p'}(\mathbb{R}^n)} \lesssim \|f\|_{H_L^p(\mathbb{R}^n)} \|g\|_{L^{p'}(\mathbb{R}^n)},$$

where $\langle \cdot, \cdot \rangle$ denotes the $H_L^p(\mathbb{R}^n) - H_{L^*}^{p'}(\mathbb{R}^n)$ duality pairing, and where in the last step we have used the L^* version of (9.4). Taking a supremum over all g as above, with $\|g\|_{L^{p'}(\mathbb{R}^n)} = 1$, we obtain that $f \in L^p$ and satisfies (9.3).

Proof of (vi). By duality, it suffices to treat the case $1 < p \leq p_-(L)$, since $p_+(L) = (p_-(L^*))'$. Moreover, it is enough to treat the case $p = p_-(L)$: indeed, if (vi) holds in that case, then it must also hold for $1 < p < p_-(L)$, or else we would reach a contradiction by interpolating with the case $p = 2$.

We therefore suppose that $p = p_-(L) > 1$. We recall that by [7], the Riesz transform $\nabla L^{-1/2}$ fails to be bounded on L^p , if $p = p_-(L)$ (cf. (1.5)). Thus, by Proposition 5.32, we must have that $H_L^p(\mathbb{R}^n)$ cannot equal $L^p(\mathbb{R}^n)$ if $p = p_-(L)$.

To conclude the proof of the proposition, it remains to construct examples to show that the null spaces $\mathcal{N}_\alpha(L)$, $0 \leq \alpha < 1$ and $\mathcal{N}_p(L)$, $2n/(n-2) < p < \infty$, may be non-trivial. To this end, we recall the examples of Frehse [33], discussed above in Section 2, namely that for each $q < n/2$ and $\lambda > 0$, there exists $L := -\nabla \cdot A \nabla$, with A complex elliptic, $L^\infty(\mathbb{R}^n)$ and $C^\infty(\mathbb{R}^n \setminus \{0\})$, for which the $W_{loc}^{1,2}$ function

$$(9.7) \quad u(x) := \frac{x_1}{|x|^q} e^{i\lambda \ln|x|}$$

is a global weak solution of the equation $Lu = 0$ in \mathbb{R}^n . Taking $\alpha = 1 - q$, we then have that u in (9.7) belongs to $\Lambda_\alpha(\mathbb{R}^n)$ if $0 < q \leq 1$; in fact, if $q = 1$ we even have the stronger statement that $u \in L^\infty(\mathbb{R}^n)$. Thus, $u \in \mathcal{N}_\alpha(L)$.

To exhibit an L for which $\mathcal{N}_p(L)$ is non-trivial is a bit more delicate, although matters will still depend on the construction in [33]. Fix now $p > 2n/(n-2)$ and choose $q < n/2$ such that $p(q-1) > n$. We observe that for such p, q , the solution u in (9.7) belongs to L^p “at infinity”, i.e., in the complement of any ball centered at the origin. However, u is not in L^p in any neighborhood of the origin, so we shall have to work a little harder to produce a null solution that belongs globally to L^p .

Let $L := -\nabla \cdot A \nabla$ be the complex elliptic matrix constructed in [33], for which u in (9.7) is a global weak solution in \mathbb{R}^n (the matrix A is given explicitly in (2.12) above). We note that A is smooth away from the origin, and that $|\nabla A(x)| \leq C$ if, say, $|x| > 1/4$. Fix a smooth cut-off function $\eta \in C_0^\infty(|x| \leq 3/8)$, with $0 \leq \eta \leq 1$, and $\eta(x) \equiv 1$ if $|x| \leq 1/4$. Let $\mathbf{1}$ denote the $n \times n$ identity matrix, and define an

auxiliary matrix

$$A_1 := \eta \mathbf{1} + (1 - \eta)A.$$

Then $A_1 \in C^\infty(\mathbb{R}^n)$ is complex elliptic (in the sense of (1.2)), with $\|\nabla A_1\|_{L^\infty(\mathbb{R}^n)} \leq C$. Set $L_1 := -\nabla \cdot A_1 \nabla$.

Next, we smoothly truncate u away from 0. Let $0 \leq \Phi \in C^\infty(\mathbb{R}^n)$, with $\Phi(x) \equiv 1$ if $|x| \geq 1$, and $\Phi(x) \equiv 0$ if $|x| \leq 1/2$, and define

$$w := u \Phi.$$

We observe that

$$L_1 w = L w = -\operatorname{div}(u A \nabla \Phi) - A \nabla u \cdot \nabla \Phi =: f \in C_0^\infty\left(\frac{1}{2} \leq |x| \leq 1\right).$$

We now fix $r := 2n/(n-2)$ and $r' = 2n/(n+2)$. Recall that by [7], we have that

$$(9.8) \quad L_1^{-1} : L^{r'}(\mathbb{R}^n) \rightarrow \dot{W}^{1,2}(\mathbb{R}^n) \cap L^r(\mathbb{R}^n).$$

Thus,

$$w_1 := L_1^{-1} f \in \dot{W}^{1,2}(\mathbb{R}^n) \cap L^r(\mathbb{R}^n).$$

On the other hand, since $q < n/2$, the solution u in (9.7), and hence also w , do **not** belong to $L^r(\mathbb{R}^n)$, nor to $\dot{W}^{1,2}(\mathbb{R}^n)$ (this is related to the failure of semigroup bounds for L_1 in L^p , when $p > n/(q-1)$). Consequently, $v := w - w_1$ is non-trivial, and solves $L_1 v = 0$, globally in \mathbb{R}^n in the weak sense.

It therefore remains only to show that $v \in L^p(\mathbb{R}^n)$ (in spite of the failure of functional calculus for L_1 in L^p), where we recall that $p > 2n/(n-2)$ was fixed above. We begin with the following

Lemma 9.9. *Let $r = 2n/(n-2)$. Suppose that $A \in C^1(\mathbb{R}^n)$ is complex elliptic (in the sense of (1.2)), and that $\|\nabla A\|_{L^\infty(\mathbb{R}^n)} \leq C_0$. Set $L := -\nabla \cdot A \nabla$, and suppose that $v \in W_{loc}^{1,2}$ is a global weak solution of $Lv = 0$. Then there are constants C_1 and κ , depending only on n , C_0 and ellipticity, such that for every **unit** cube $Q \subset \mathbb{R}^n$, we have that*

$$(9.10) \quad \|v\|_{L^\infty(Q)} \leq C_1 \left(\int_{\kappa Q} |v|^r \right)^{1/r},$$

where κQ denotes the concentric dilate of the unit cube Q , with side length κ .

Let us momentarily take the lemma for granted, and conclude the proof of Proposition 9.1. We apply Lemma 9.9 to the operator L_1 and to the solution $v = w - w_1$ constructed above. We recall that $w \in L^p(\mathbb{R}^n)$, $w_1 \in L^r(\mathbb{R}^n)$, with $p > r := 2n/(n-2)$. Let $\{Q_j\}$ be an enumeration of the dyadic grid of unit cubes in \mathbb{R}^n , and we observe that for κ as in the lemma,

$$\sum a_j := \sum \int_{\kappa Q_j} |w_1|^r \approx \int_{\mathbb{R}^n} |w_1|^r < \infty,$$

since the dilated cubes κQ_j have bounded overlaps. We now consider

$$\int_{\mathbb{R}^n} |v|^p = \sum \int_{Q_j} |v|^p$$

$$\begin{aligned} &\lesssim \sum \left(\int_{\kappa Q_j} |v|^r \right)^{p/r} \lesssim \sum \left(\int_{\kappa Q_j} |w|^r \right)^{p/r} + \sum (a_j)^{p/r} \\ &=: \sum_1 + \sum_2, \end{aligned}$$

where in the first inequality we have used (9.10). By Hölder's inequality, we have

$$\sum_1 \lesssim \sum \int_{\kappa Q_j} |w|^p \lesssim \int_{\mathbb{R}^n} |w|^p < \infty.$$

Moreover,

$$\sum_2 \leq \left(\sum a_j \right)^{p/r} < \infty,$$

since $p > r$. This concludes the proof of Proposition 9.1, modulo the proof of Lemma 9.9. \square

Proof of Lemma 9.9. The inequality (9.10) is a variant of standard classical estimates, and can be extracted, for example, from results proved in the monograph of Giaquinta [35]. For the reader's convenience, we provide a proof here using a well known perturbation argument, plus an iteration scheme.

For the moment, we fix an arbitrary (i.e., not necessarily unit) cube Q , of side length $\ell(Q)$, and a point $x_0 \in Q$, and define a constant coefficient complex elliptic operator $L_0 := -\nabla \cdot A_0 \nabla$, where $A_0 := A(x_0)$. By standard results for constant coefficient operators, we have that Γ_0 , the fundamental solution for L_0 , belongs to $C^\infty(\mathbb{R}^n \setminus \{0\})$ and satisfies

$$(9.11) \quad |\Gamma_0(x)| \lesssim |x|^{2-n}, \quad |\nabla \Gamma_0(x)| \lesssim |x|^{1-n}, \quad |\nabla^2 \Gamma_0(x)| \lesssim |x|^{-n},$$

where the implicit constants depend only upon ellipticity and dimension.

Let ϕ_Q be a smooth non-negative cut-off function supported in $3Q$, with $\phi_Q \equiv 1$ on $2Q$, and satisfying $\|\nabla \phi_Q\|_\infty \lesssim \ell(Q)^{-1}$, $\|\nabla^2 \phi_Q\|_\infty \lesssim \ell(Q)^{-2}$. We now write

$$\begin{aligned} v(x_0) &= v(x_0) \phi_Q(x_0) = \int \overline{\nabla_y \Gamma_0(x_0 - y)} \cdot A_0 \nabla (v(y) \phi_Q(y)) dy \\ &= \int \overline{\nabla_y \Gamma_0(x_0 - y)} \cdot A_0 \nabla v(y) \phi_Q(y) dy + \int \overline{\nabla_y \Gamma_0(x_0 - y)} \cdot A_0 \nabla \phi_Q(y) v(y) dy \\ &= \int \overline{\nabla_y (\Gamma_0(x_0 - y) \phi_Q(y))} \cdot (A_0 - A(y)) \nabla v(y) dy - \int \overline{\Gamma_0(x_0 - y)} \nabla \phi_Q(y) \cdot A_0 \nabla v(y) dy \\ &\quad + \int \overline{\nabla_y \Gamma_0(x_0 - y)} \cdot A_0 \nabla \phi_Q(y) v(y) dy =: I + II + III, \end{aligned}$$

where we have used in term I that $Lv = 0$.

By (9.11) and the definition of ϕ_Q , we have that

$$|III| \lesssim \frac{1}{|Q|} \int_{3Q \setminus 2Q} |v|.$$

The same bound holds for II , as may be seen by integrating by parts to move the gradient away from v . Similarly, integrating by parts in term I yields the estimate

$$\begin{aligned} |I| &\lesssim \int |\nabla\Gamma_0| |\nabla\phi_Q| |v| + \int |\Gamma_0| |\nabla^2\phi_Q| |v| \\ &+ \|\nabla A\|_\infty \int_{3Q} |\nabla^2\Gamma_0(x_0 - y)| |x_0 - y| |v(y)| dy + \|\nabla A\|_\infty \int |\Gamma_0| |\nabla\phi_Q| |v| \\ &+ \|\nabla A\|_\infty \int_{3Q} |\nabla\Gamma_0(x_0 - y)| |v(y)| dy =: I_1 + I_2 + I_3 + I_4 + I_5. \end{aligned}$$

The terms I_1, I_2 satisfy the same bound as do II and III . For the remaining terms, we have

$$|I_3 + I_4 + I_5| \lesssim \int_{3Q} |x_0 - y|^{1-n} |v(y)| dy =: I_Q v(x_0).$$

Combining our estimates, we obtain

$$(9.12) \quad |v(x)| \lesssim \frac{1}{|Q|} \int_{3Q} |v| + I_Q v(x), \quad \forall x \in Q.$$

By Hölder's inequality, we have

$$I_Q v(x) \lesssim \ell(Q) \left(\frac{1}{|Q|} \int_{3Q} |v|^t \right)^{1/t},$$

for any $t > n$, and each $x \in Q$, so that also

$$(9.13) \quad |v(x)| \lesssim \frac{1}{|Q|} \int_{3Q} |v| + \ell(Q) \left(\frac{1}{|Q|} \int_{3Q} |v|^t \right)^{1/t}, \quad \forall x \in Q.$$

Iterating, we obtain for $x \in Q$,

$$\begin{aligned} |v(x)| &\lesssim \frac{1}{|Q|} \int_{3Q} |v| + \ell(Q) \left(\frac{1}{|Q|} \int_{3Q} |v|^t \right)^{1/t} \\ &\lesssim \frac{1}{|Q|} \int_{3Q} |v| + \frac{\ell(Q)}{|Q|} \int_{9Q} |v| + \ell(Q) \left(\frac{1}{|Q|} \int_{3Q} |I_{3Q} v|^t \right)^{1/t} \\ &\lesssim \frac{1}{|Q|} \int_{3Q} |v| + \frac{\ell(Q)}{|Q|} \int_{9Q} |v| + (\ell(Q))^2 \left(\frac{1}{|Q|} \int_{9Q} |v|^s \right)^{1/s}, \end{aligned}$$

where in the last step $1/t = 1/s - 1/n$ and we have used the fractional integral theorem. Iterating further, and taking Q to be a unit cube, we obtain the conclusion of the lemma. \square

10. APPENDIX 2: EMBEDDING OF $H_L^p(\mathbb{R}^n)$ SPACES INTO AN AMBIENT BANACH SPACE

We shall continue to use the notational convention that $\Lambda_L^0(\mathbb{R}^n) := BMO_L(\mathbb{R}^n)$. In this appendix, we prove the following:

Proposition 10.1. *Let $0 < p_0 < 1$, and $0 \leq \alpha_0 < \infty$. Then there exists a Banach space $\mathcal{B} = \mathcal{B}(p_0, \alpha_0)$ such that the spaces $H_L^p(\mathbb{R}^n)$, $p_0 \leq p < \infty$, and $\Lambda_L^\alpha(\mathbb{R}^n)$, $0 \leq \alpha \leq \alpha_0$, are all continuously embedded into \mathcal{B} .*

Proof. We shall realize the space \mathcal{B} as the dual of an appropriate normed space $\mathbf{M}_0 = \mathbf{M}_0(p_0, \alpha_0)$, which in turn will be a subspace of the intersection of $\mathbf{M}_{\alpha_0, L^*}^{\varepsilon_0, M}$ (cf. Section 1) and $\mathcal{D}((L^*)^M)$ (the domain of $(L^*)^M$ in $L^2(\mathbb{R}^n)$), where $\varepsilon_0 > 0$ and

$$(10.2) \quad M > \max\left(\frac{1}{2}(\alpha_0 + n/2), \frac{n}{2}\left(\frac{1}{p_0} - \frac{1}{2}\right)\right).$$

More precisely, for such ε_0 and M fixed, we define $\mathbf{M}_0 = \mathbf{M}_0(p_0, \alpha_0)$ as the collection of all $\varphi \in L^2(\mathbb{R}^n)$ such that φ belongs to the $\mathcal{R}((L^*)^k)$, the range of $(L^*)^k$ in $L^2(\mathbb{R}^n)$, and also to $\mathcal{D}((L^*)^k)$, for each $k = 0, 1, \dots, M$, and satisfies

$$(10.3) \quad \|\varphi\|_{\mathbf{M}_0} := \sup_{j \geq 0} 2^{j(n/2 + \alpha_0 + \varepsilon_0)} \sum_{k=-M}^M \|(L^*)^k \mu\|_{L^2(S_j(Q_0))} < \infty,$$

where Q_0 is the unit cube centered at 0 and $S_j(Q_0)$, $j \in \mathbb{N}$, are the corresponding dyadic annuli (see (3.2)). We note that $\|\cdot\|_{\mathbf{M}_0}$ clearly defines a norm. We observe also that it is easy to construct elements of \mathbf{M}_0 : just set $\varphi = (L^*)^M e^{-L^*} f$, where $f \in L^2$ with support in Q_0 . The bound $\|\varphi\|_{\mathbf{M}_0} \leq C\|f\|_{L^2(Q_0)}$ follows immediately from Gaffney estimates.

We now set $\mathcal{B} := \mathbf{M}'_0$, the dual space of \mathbf{M}_0 , and we consider first the embedding $\Lambda_L^\alpha(\mathbb{R}^n) \hookrightarrow \mathcal{B}$, for $0 \leq \alpha \leq \alpha_0$. Suppose that $\varphi \in \mathbf{M}_0$, with $\|\varphi\|_{\mathbf{M}_0} = 1$. Then φ is an $(H_L^p, (\alpha_0 - \alpha) + \varepsilon_0, M)$ -molecule adapted to Q_0 (cf. (3.3)), up to multiplication by some harmless constant C , with $\alpha = n(1/p - 1)$, for every p such that $n/(n + \alpha_0) \leq p \leq 1$. Thus, by Lemma 3.37, for every $g \in \Lambda_L^\alpha(\mathbb{R}^n)$, $0 \leq \alpha \leq \alpha_0$, we have

$$|\langle \varphi, g \rangle| \leq C\|g\|_{\Lambda_L^\alpha(\mathbb{R}^n)} = C\|\varphi\|_{\mathbf{M}_0}\|g\|_{\Lambda_L^\alpha(\mathbb{R}^n)},$$

whence it follows that $\Lambda_L^\alpha(\mathbb{R}^n) \hookrightarrow \mathcal{B}$.

Next, we consider the embedding $H_L^p(\mathbb{R}^n) \hookrightarrow \mathcal{B}$, $p_0 \leq p \leq 1$. Since $\mathbf{M}_0 \subset L^2(\mathbb{R}^n)$, by (3.7) and Definition 3.4, it is enough to show that, given $\varepsilon > 0$,

$$(10.4) \quad \left| \int_{\mathbb{R}^n} \varphi(x) m(x) dx \right| \leq C\|\varphi\|_{\mathbf{M}_0},$$

for every (H_L^p, ε, M) -molecule m . We fix such a molecule m , associated to a cube Q . It is clear from the definitions (cf. (10.3) and (3.3)) that for $k = 0, 1, \dots, M$,

$$(10.5) \quad \|(L^*)^k \varphi\|_{L^2(\mathbb{R}^n)} \leq C\|\varphi\|_{\mathbf{M}_0} \quad \text{and} \quad \|((\ell(Q))^2 L)^{-k} m\|_{L^2(\mathbb{R}^n)} \leq C\ell(Q)^{n/2 - n/p},$$

Thus, for $\ell(Q) \geq 1$, the bound (10.4) follows immediately from Schwarz's inequality and (10.5) with $k = 0$. On the other hand, if $\ell(Q) < 1$, we have

$$\begin{aligned} \left| \int_{\mathbb{R}^n} \varphi(x) m(x) dx \right| &= \ell(Q)^{2M} \left| \int_{\mathbb{R}^n} (L^*)^M \varphi(x) \left((\ell(Q))^2 L\right)^{-M} m(x) dx \right| \\ &\leq C\|\varphi\|_{\mathbf{M}_0} \ell(Q)^{2M + n/2 - n/p} \end{aligned}$$

by (10.5) with $k = M$. Since $p \geq p_0$, for M as in (10.2), we obtain (10.4).

Finally, we suppose that $1 < p < \infty$, and let $f \in L^2(\mathbb{R}^n) \cap H_L^p(\mathbb{R}^n)$. Setting $\psi(\zeta) := \zeta^M e^{-\zeta}$, by the Calderón reproducing formula (4.16) and duality, we have

$$\begin{aligned} \left| \int_{\mathbb{R}^n} \varphi(x) f(x) dx \right| &\leq C \|Q_{\psi, L^*} f\|_{T^p(\mathbb{R}_+^{n+1})} \|Q_{\psi, L^*} \varphi\|_{T^{p'}(\mathbb{R}_+^{n+1})} \\ &\leq C \|f\|_{H_L^p(\mathbb{R}^n)} \|Q_{\psi, L^*} \varphi\|_{T^{p'}(\mathbb{R}_+^{n+1})}. \end{aligned}$$

It is therefore enough to show that, for $\|\varphi\|_{\mathbf{M}_0} = 1$,

$$(10.6) \quad \|Q_{\psi, L^*} \varphi\|_{T^{p'}(\mathbb{R}_+^{n+1})} \equiv \|\mathcal{A}(Q_{\psi, L^*} \varphi)\|_{L^{p'}(\mathbb{R}^n)} \leq C, \quad 1 < p' < \infty,$$

where we remind the reader that the ‘‘area integral’’ \mathcal{A} is defined in (3.16). We first note that (10.6) with $p' = 2$ follows immediately by standard quadratic estimates and the case $k = 0$ of (10.5). Moreover, $\|\varphi\|_{H_L^1(\mathbb{R}^n)} \leq C$ (indeed, as mentioned above, φ is an $(H_L^1, \alpha_0 + \varepsilon_0, M)$ -molecule adapted to Q_0 , up to multiplication by a harmless constant), so that by Proposition 4.13, we have

$$\|Q_{\psi, L^*} \varphi\|_{T^1(\mathbb{R}_+^{n+1})} = \|\mathcal{A}(Q_{\psi, L^*} \varphi)\|_{L^1(\mathbb{R}^n)} \leq C.$$

Combining the latter bound with that for $p' = 2$, we obtain immediately (10.6) in the case $1 < p' < 2$.

Similarly, to handle the case $2 < p' < \infty$, it is enough to show that $\mathcal{A}(Q_{\psi, L^*} \varphi) \in L^\infty(\mathbb{R}^n)$. To this end, we write

$$\begin{aligned} (\mathcal{A}(Q_{\psi, L^*} \varphi)(x))^2 &:= \iint_{|x-y|<t} |(t^2 L^*)^M e^{-t^2 L^*} \varphi(y)|^2 \frac{dy dt}{t^{n+1}} \\ &\leq \int_0^1 \int_{\mathbb{R}^n} |t^{2M} e^{-t^2 L^*} (L^*)^M \varphi(y)|^2 \frac{dy dt}{t^{n+1}} + \int_1^\infty \int_{\mathbb{R}^n} |(t^2 L^*)^M e^{-t^2 L^*} \varphi(y)|^2 \frac{dy dt}{t^{n+1}} \\ &\leq \int_0^1 t^{4M-n-1} dt + \int_1^\infty t^{-n-1} dt \leq C, \end{aligned}$$

where in the next-to-last inequality we have used (10.5) with $k = M$ in the first term and with $k = 0$ in the second, along with L^2 boundedness of $(t^2 L^*)^k e^{-t^2 L^*}$ for every non-negative integer k , and in the very last step we have used that $M > n/4$, by (10.2) and the fact that $p_0 \leq 1$. \square

REFERENCES

- [1] D. Albrecht, X. Duong, A. McIntosh, *Operator theory and harmonic analysis*, Instructional Workshop on Analysis and Geometry, Part III (Canberra, 1995), 77–136, Proc. Centre Math. Appl. Austral. Nat. Univ., 34, Austral. Nat. Univ., Canberra, 13, 48
- [2] P. Auscher, X.T. Duong and A. McIntosh, Boundedness of Banach space valued singular integral operators and Hardy spaces. Unpublished preprint (2005). 3
- [3] J. Alvarez, M. Milman, *Spaces of Carleson measures: duality and interpolation*, Ark. Mat. 25 (1987), no. 2, 155–174. 58
- [4] J. Alvarez, M. Milman, *Interpolation of tent spaces and applications*, Function spaces and applications (Lund, 1986), 11–21, Lecture Notes in Math., 1302, Springer, Berlin, 1988. 58

- [5] P. Auscher, *Some questions on elliptic operators*, (English summary) Heat kernels and analysis on manifolds, graphs, and metric spaces (Paris, 2002), 1–10, Contemp. Math., 338, Amer. Math. Soc., Providence, RI, 2003. [4](#)
- [6] P. Auscher, *On L^p -estimates for square roots of second order elliptic operators on \mathbb{R}^n* , Publ. Mat. 48 (2004), 159–186. [4](#)
- [7] P. Auscher, *On necessary and sufficient conditions for L^p -estimates of Riesz transforms associated with elliptic operators on \mathbb{R}^n and related estimates*, preprint. [4](#), [8](#), [10](#), [12](#), [14](#), [15](#), [16](#), [17](#), [42](#), [43](#), [44](#), [51](#), [55](#), [58](#), [64](#), [68](#), [69](#), [70](#)
- [8] P. Auscher, T. Coulhon, Ph. Tchamitchian, *Absence de principe du maximum pour certaines équations paraboliques complexes*, Coll. Math. 171 (1996), 8795. [4](#)
- [9] P. Auscher, S. Hofmann, M. Lacey, A. McIntosh, Ph. Tchamitchian, *The solution of the Kato square root problem for second order elliptic operators on \mathbb{R}^n* , Ann. of Math. (2) 156 (2002), no. 2, 633–654. [3](#), [40](#), [41](#), [44](#)
- [10] P. Auscher, A. McIntosh, E. Russ, *Hardy spaces of differential forms on Riemannian manifolds*, Journal of Geometric Analysis, 18 (2008), 192–248. [3](#), [5](#), [9](#), [12](#), [13](#), [23](#), [26](#), [31](#), [32](#), [33](#), [35](#)
- [11] P. Auscher, E. Russ, *Hardy spaces and divergence operators on strongly Lipschitz domains of \mathbb{R}^n* , J. Funct. Anal. 201 (2003), no. 1, 148–184. [12](#)
- [12] P. Auscher, Ph. Tchamitchian, *Calcul fonctionnel précisé pour des opérateurs elliptiques complexes en dimension un (et applications à certaines équations elliptiques complexes en dimension deux)*, Ann. Inst. Fourier 45 (1995), 721–778. [4](#)
- [13] P. Auscher, Ph. Tchamitchian, *Square root problem for divergence operators and related topics*, volume 249 of Astérisque, Soc. Math. France, 1998. [4](#), [5](#), [14](#), [15](#), [16](#)
- [14] A. Bernal, *Some results on complex interpolation of T_q^p spaces*, Interpolation spaces and related topics (Haifa, 1990), 1–10, Israel Math. Conf. Proc., 5, Bar-Ilan Univ., Ramat Gan, 1992. [58](#)
- [15] A. Bernal, J. Cerda, *Complex interpolation of quasi-Banach spaces with an A -convex containing space*, Ark. Mat. 29 (1991), no. 2, 183–201. [58](#)
- [16] S. Blunck, P. Kunstmann, *Weak-type (p, p) estimates for Riesz transforms*, Math. Z. 247 no. 1 (2004), 137–148. [4](#)
- [17] S. Blunck, P. Kunstmann, *Calderón-Zygmund theory for non-integral operators and the H^∞ functional calculus*, Rev. Mat. Iberoamericana 19 (2003), no. 3, 919–942. [4](#), [17](#)
- [18] A. P. Calderón and A. Torchinsky, *Parabolic maximal functions associated with a distribution. II*, Adv. in Math., 24 (1977), 101–171. [36](#)
- [19] W. S. Cohn, I. E. Verbitsky, *Factorization of tent spaces and Hankel operators*, J. Funct. Anal. 175 (2000), no. 2, 308–329. [36](#), [37](#), [58](#), [59](#)
- [20] R. Coifman, *A real variable characterization of H^p* , Studia Math., **51**(1974), 269–274. [6](#)
- [21] R.R. Coifman, Y. Meyer, E.M. Stein, *Some new function spaces and their applications to harmonic analysis*, J. Funct. Anal. 62 (1985), no. 2, 304–335. [23](#), [24](#), [25](#), [31](#), [33](#), [36](#), [58](#)
- [22] R.R. Coifman, G. Weiss, *Extensions of Hardy spaces and their use in analysis*, Bull. Amer. Math. Soc. 83 (1977), no. 4, 569–645. [19](#), [38](#)
- [23] M. Cwikel, M. Milman and Y. Sagher, *Complex interpolation of some quasi-Banach spaces*, J. Funct. Anal., 65 (1986), 339–347. [36](#)
- [24] E.B. Davies, *Limits on L^p regularity of selfadjoint elliptic operators*, J. Differential Equations 135, no. 1 (1997), 83–102. [4](#)
- [25] X.T. Duong and L.X. Yan, *Duality of Hardy and BMO spaces associated with operators with heat kernel bounds*, J. Amer. Math. Soc. 18 (2005), 943–973. [3](#), [5](#), [9](#), [12](#), [23](#), [29](#), [67](#)
- [26] X.T. Duong and L.X. Yan, *New function spaces of BMO type, the John-Nirenberg inequality, interpolation, and applications*, Comm. Pure Appl. Math. 58 (2005), no. 10, 1375–1420. [3](#), [5](#), [9](#), [12](#)
- [27] X.T. Duong and L.X. Yan, *Old and new Morrey spaces with heat kernel bounds*, J. Fourier Anal. Appl. 13 (2007), 87–111. [9](#), [12](#)

- [28] P. Duren, B. Romberg and A. Shields, *Linear functionals on H^p spaces with $0 < p < 1$* , J. Reine Angew. Math. 238 (1969), 32-60. [8](#), [29](#), [67](#)
- [29] J. Dziubanski and M. Preisner, Zienkiewicz, *The Riesz transform characterization of Hardy spaces associated with Schrödinger operators with compactly supported potentials*, preprint. [7](#)
- [30] J. Dziubanski and J. Zienkiewicz, *Hardy spaces associated with some Schrödinger operators*, Studia Math. 126 (1997), 149-160. [7](#)
- [31] C. Fefferman, E.M. Stein, *H^p spaces of several variables*, Acta Math. 129 (1972), no. 3-4, 137-193. [3](#), [6](#), [7](#), [8](#), [9](#), [10](#), [29](#), [67](#)
- [32] M. Frazier and B. Jawerth, *A discrete transform and decompositions of distribution spaces*, J. Funct. Anal., 93 (1990), no. 1, 34-170. [40](#), [57](#), [58](#)
- [33] J. Frehse, *An irregular complex valued solution to a scalar uniformly elliptic equation*, preprint. [4](#), [14](#), [69](#)
- [34] J. Garcia-Cuerva, J.L. Rubio de Francia, *Weighted norm inequalities and related topics*, North-Holland Mathematics Studies, 116. North-Holland Publishing Co., Amsterdam, 1985. [38](#)
- [35] M. Giaquinta, *Multiple Integrals in the Calculus of Variations and Nonlinear Elliptic Systems*, Annals of Math. Studies 105, Princeton Univ. Press, Princeton, NJ, 1983. [71](#)
- [36] M. Gomez and M. Milman, *Complex interpolation of H^p -spaces on product domains*, Ann. Math. Pura Appl., 155 (1989), 103-115. [36](#)
- [37] S. Hofmann, G. Lu, D. Mitrea, M. Mitrea and L. Yan, *Hardy spaces associated to non-negative self-adjoint operators satisfying Davies-Gaffney estimates*, preprint. [3](#), [20](#), [23](#), [26](#)
- [38] S. Hofmann, J.M. Martell, *L^p bounds for Riesz transforms and square roots associated with second order elliptic operators*, Publ. Mat. 47 (2003), no. 2, 497-515. [4](#), [16](#), [39](#)
- [39] S. Hofmann, S. Mayboroda, *Hardy and BMO spaces associated to divergence form elliptic operators*, Math. Ann. [3](#), [5](#), [6](#), [9](#), [12](#), [13](#), [20](#), [23](#), [26](#), [28](#), [29](#), [30](#), [31](#), [35](#), [37](#), [39](#), [44](#), [54](#)
- [40] S. Hofmann, S. Mayboroda, *Correction to "Hardy and BMO spaces associated to divergence form elliptic operators"*, preprint. [20](#), [23](#), [26](#)
- [41] T. Hytonen, J. Van Neerven, P. Portal, *Conical square function estimates in UMD Banach spaces and application to H^∞ functional calculi*, J. Anal. Math. 106 (2008), 317-351. [35](#)
- [42] J. Janson and P.W. Jones, *Interpolation between H^p spaces: The complex method*, J. Funct. Anal., 48 (1982), 58-80. [36](#)
- [43] R. Jiang and D. Yang, *New Orlicz-Hardy spaces associated with divergence form elliptic operators*, preprint. [12](#), [13](#), [23](#), [26](#), [30](#), [31](#)
- [44] N. Kalton and M. Mitrea, *Stability of Fredholm properties on interpolation scales of quasi-Banach spaces and applications*, Transactions of the Amer. Math. Soc., 350, (1998), 3837-3901. [36](#), [58](#)
- [45] N. Kalton, S. Mayboroda, M. Mitrea *Interpolation of Hardy-Sobolev-Besov-Triebel-Lizorkin Spaces and applications to problems in partial differential equations*, preprint. [36](#), [37](#), [58](#), [59](#)
- [46] R.H. Latter, *A decomposition of $H^p(\mathbb{R}^n)$ in terms of atoms*, Studia Math., **62** (1977), 92-102. [6](#)
- [47] J. M. Martell, *Sharp maximal functions associated with approximations of the identity in spaces of homogeneous type and applications*, Studia Math. 161 (2004), no. 2, 113-145. [9](#), [10](#)
- [48] S. Mayboroda, *The connections between Dirichlet, Regularity and Neumann problems for second order elliptic operators with complex bounded measurable coefficients*, preprint. [8](#)
- [49] V. G. Maz'ya, S. A. Nazarov and B. A. Plamenevskii, *Absence of a De Giorgi-type theorem for strongly elliptic equations with complex coefficients*, Boundary value problems of mathematical physics and related questions in the theory of functions, 14. Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) 115 (1982), 156-168, 309. [4](#), [14](#)
- [50] A. McIntosh, *Operators which have an H^∞ functional calculus*, Miniconference on Operator Theory and Partial Differential Equations, Macquarie University, 1986. Proceedings of the Centre for Mathematical Analysis, Australian National University, 14 (1986) 210-231. [13](#), [14](#), [27](#), [31](#), [32](#)

- [51] O. Mendez, M. Mitrea, *The Banach envelopes of Besov and Triebel-Lizorkin spaces and applications to partial differential equations*, J. Fourier Anal. Appl. 6 (2000), no. 5, 503–531. 40, 58
- [52] N. Meyers, *Mean oscillation over cubes and Hölder continuity*, Proc. Amer. Math. Soc. 15 (1964), 717–721. 9
- [53] I. Sneiberg, *Spectral properties of linear operators in interpolation families of Banach spaces*, Mat. Issled. 9 (1974), 214–229.
- [54] E.M. Stein, *Singular integrals and differentiability properties of functions*, Princeton Mathematical Series, No. 30 Princeton University Press, Princeton, N.J. 1970.
- [55] E. M. Stein *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*, Princeton Mathematical Series, 43. Princeton University Press, Princeton, NJ, 1993. 7
- [56] E.M. Stein, G. Weiss, *On the theory of harmonic functions of several variables. I. The theory of H^p -spaces*, Acta Math. 103 (1960), 25–62. 3, 7
- [57] M. Taibleson, and G. Weiss, *The molecular characterization of certain Hardy spaces. Representation theorems for Hardy spaces*, pp. 67–149, Astérisque, 77, Soc. Math. France, Paris, 1980. 19, 38
- [58] H. Triebel, *Theory of Function Spaces*, Birkhäuser, Berlin, 1983. 57
- [59] T. Wolff, *A note on interpolation spaces*, Proc. Conf. Harmonic Analysis, Minneapolis, Lecture Notes in Math. Vol. 908, pp.199–204, Springer-Verlag, New York/Berlin, 1982. 37
- [60] L. X. Yan, *Classes of Hardy spaces associated with operators, duality theorem and applications*, preprint. 12