Homework due Thursday, April 7

 # 1. Hermite polynomials. Derive the following properties of Hermite polynomials/functions:

Differential equations:

$$L^-(\mathcal{H}_n) = \sqrt{n}\mathcal{H}_{n-1}, \quad L^+(\mathcal{H}_n) = \sqrt{n+1}\mathcal{H}_{n+1}$$

with $L^\pm = \frac{1}{\sqrt{2}}(x \mp D_x)$.

Recursion:

$$\sqrt{n+1}H_{n+1}(x) = \sqrt{2}\,xH_n(x) - \sqrt{n}H_{n-1}(x), \quad n = 1, 2, ...$$

Hermite polynomial vs Hermite function:

$$H_n(-iD_\xi)e^{-\xi^2/2} = i^n\mathcal{H}_n$$

 # 2. Fourier transform. Consider the "square wave" function

$$f(t) = \begin{cases} 1, & \text{when } -2\pi \le t \le 2\pi, \\ 0, & \text{otherwise.} \end{cases}$$

a) Make a prediction (without calculation): how smooth is its Fourier transform $\widehat{f}(\lambda)$ – discontinuous, twice differentiable, infinitely differentiable? Explain your answer.
b) Compute the Fourier transform of f.
c) Find the general formula for the Fourier transform for dilation. That is, compute $\mathcal{F}(u(r\xi))$, where r is a constant and u is an L^2 function. Use it to compute $\mathcal{F}[f(2\pi t)](\lambda)$ for the square wave defined above.
d) Find $\mathcal{F}(f*f)$ (1) directly (computing first the convolution, and then the Furier transform of the result) and (2) using the formula for the Fourier transform of convolution and part b).