Boundedness of the gradient of a solution and Wiener test of order one for the biharmonic equation

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Abstract

The behavior of solutions to the biharmonic equation is well-understood in smooth domains. In the past two decades substantial progress has also been made for the polyhedral domains and domains with Lipschitz boundaries. However, very little is known about higher order elliptic equations in the general setting.

In this paper we introduce new integral identities that allow to investigate the solutions to the biharmonic equation in an *arbitrary* domain. We establish:
(1) boundedness of the gradient of a solution in any three-dimensional domain;
(2) pointwise estimates on the derivatives of the biharmonic Green function;
(3) Wiener-type necessary and sufficient conditions for continuity of the gradient of a solution.

1 Introduction

The maximum principle for harmonic functions is one of the fundamental results in the theory of elliptic equations. It holds in arbitrary domains and guarantees that every solution to the Dirichlet problem for the Laplace equation, with bounded data, is bounded. In 1960 the maximum principle has been extended to higher order elliptic equations on smooth domains ([3]), and later, in the beginning of 90's, to three-dimensional domains diffeomorphic to a polyhedron ([12], [20]) or having a Lipschitz boundary ([25], [26]). In particular, it ensures that in such domains a biharmonic function satisfies the estimate

$$\|\nabla u\|_{L^\infty(\Omega)} \leq C\|\nabla u\|_{L^\infty(\partial\Omega)}. \tag{1.1}$$

Direct analogues of this principle for higher order equations in general domains are unknown (see Problem 4.3, p.275, in J. Nečas's book [23]). Not only the increase of the order leads to the failure of the methods which work for the second order equations, but the properties of the solutions themselves become more involved.

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To be more specific, let $\Omega \subset \mathbb{R}^n$ be a bounded domain and consider the boundary value problem

$$\Delta^2 u = f \text{ in } \Omega, \quad u \in \mathring{W}_2^2(\Omega), \tag{1.2}$$

where the Sobolev space $\mathring{W}_2^2(\Omega)$ is a completion of $C_0^\infty(\Omega)$ in the norm $\|u\|_{\mathring{W}_2^2(\Omega)} = \|\Delta u\|_{L^2(\Omega)}$ and f is a reasonably nice function. Motivated by (1.1), we ask if the gradient of a solution to problem (1.2) is bounded in an arbitrary domain $\Omega \subset \mathbb{R}^n$. It turns out that this property may fail when $n \geq 4$ (see the counterexamples built in [21] and [24]). In dimension three the boundedness of the gradient of a solution has been an open problem.

The absence of any information about the geometry of the domain puts this question beyond the scope of applicability of the previously devised methods – the aforementioned results regarding the maximum principle heavily relied on specific assumptions on Ω. In the present paper we develop a new set of techniques which allows to establish *the boundedness of the gradient of the solution to (1.2) under no restrictions on the underlying domain.* Moreover, we prove the following:

Theorem 1.1 *Let Ω be an arbitrary bounded domain in $\mathbb{R}^3$ and let G be Green's function for the biharmonic equation. Then*

$$|\nabla_x \nabla_y G(x,y)| \leq C|x-y|^{-1}, \qquad x, y \in \Omega, \tag{1.3}$$

$$|\nabla_x G(x,y)| \leq C \quad \textit{and} \quad |\nabla_y G(x,y)| \leq C, \qquad x, y \in \Omega, \tag{1.4}$$

where C is an absolute constant.

The boundedness of the gradient of a solution to the biharmonic equation is a sharp property in the sense that the function u satisfying (1.2) generally does not exhibit more regularity. Indeed, let Ω be the three-dimensional punctured unit ball $B_1 \setminus \{O\}$, where $B_r = \{x \in \mathbb{R}^3 : |x| < r\}$, and consider a function $\eta \in C_0^\infty(B_{1/2})$ such that $\eta = 1$ on $B_{1/4}$. Let

$$u(x) := \eta(x)|x|, \qquad x \in B_1 \setminus \{O\}. \tag{1.5}$$

Obviously, $u \in \mathring{W}_2^2(\Omega)$ and $\Delta^2 u \in C_0^\infty(\Omega)$. While ∇u is bounded, it is not continuous at the origin. Therefore, the *continuity* of the gradient *does not hold* in general and must depend on some delicate properties of the domain.

Even in the case of the Laplacian the issue of continuity is subtle. It has been resolved in 1924, when Wiener gave his famous criterion for the regularity of a boundary point [28]. Needless to say, Wiener's result strongly influenced the development of partial differential equations, the theory of function spaces and probability. Over the years it has been extended to a variety of second order elliptic and parabolic equations ([14], [10], [9], [7], [15], [2], [29], [13], [8]; see also the review papers [18], [1]). However, the case of higher-order operators is far from being well-understood.

Let us recall the original Wiener's criterion. Roughly speaking, it states that a point $O \in \partial\Omega$ is regular (i.e. every solution to the Dirichlet problem for the Laplacian, with continuous data, is continuous at O) if and only if the complement of the domain near the point O, measured in terms of the Wiener (harmonic) capacity, is sufficiently massive. More specifically, the harmonic capacity of a compactum $K \subset \mathbb{R}^n$ can be defined as

$$\operatorname{cap}(K) := \inf\Big\{ \|\nabla u\|^2_{L^2(\mathbb{R}^n)} : \ u \in C_0^\infty(\mathbb{R}^n), \ u = 1 \text{ in a neighborhood of } K \Big\}, \tag{1.6}$$

where $n \geq 3$, and the regularity of the point O is equivalent to the condition

$$\int_0^1 \operatorname{cap}(\overline{B_s} \setminus \Omega) s^{1-n}\, ds = +\infty. \tag{1.7}$$

Recently, some progress has been made in the study of the continuity of solutions for a certain family of higher order elliptic equations in [19] (see also [16], [17]). In particular, these developments extend (1.7) to the context of the biharmonic equation in dimensions 4, 5, 6 and 7, with the potential-theoretic capacity of order four in place of (1.6). In the present paper we pursue a different goal – to obtain an analogue of the Wiener's test governing the *gradient* of the solution.

Turning to this issue, we start with a suitable notion of capacity. Let Π denote the space of functions

$$P(x) = b_0 + b_1 \tfrac{x_1}{|x|} + b_2 \tfrac{x_2}{|x|} + b_3 \tfrac{x_3}{|x|}, \qquad x \in \mathbb{R}^3 \setminus \{O\}, \qquad b_i \in \mathbb{R}, \qquad i = 0,1,2,3, \tag{1.8}$$

and $\Pi_1 := \{P \in \Pi : \ \|P\|_\Pi = 1\}$. Then, given a compactum $K \subset \mathbb{R}^3 \setminus \{0\}$ and $P \in \Pi_1$, let

$$\operatorname{Cap}_P(K) := \inf\Big\{ \|\Delta u\|^2_{L^2(\mathbb{R}^3)} : \ u \in \mathring{W}_2^2(\mathbb{R}^3 \setminus \{0\}), \ u = P \text{ in a neighborhood of } K \Big\}. \tag{1.9}$$

This capacity first appeared in [22], in the upper estimates on $\sup_r (\frac{1}{r^3} \int_{B_r} |\nabla u(x)|^6\, dx)^{1/6}$ for a solution of (1.2).

We say that a point $O \in \partial\Omega$ is 1-regular if for every $f \in C_0^\infty(\Omega)$ the solution u to (1.2) is continuously differentiable at O, i.e. $\nabla u(x) \to 0$ as $x \to O$; and O is 1-irregular otherwise. Our main result concerning 1-regularity is the following.

Theorem 1.2 *Let Ω be an open set in $\mathbb{R}^n$. If for some $a \geq 4$ and some $c > 0$*

$$\int_0^c \inf_{P \in \Pi_1} \operatorname{Cap}_P(\overline{C_{s,as}} \setminus \Omega)\, ds = +\infty, \tag{1.10}$$

then the point O is 1-regular.

Conversely, if the point $O \in \partial\Omega$ is 1-regular then for every $c > 0$ and every $a \geq 8$

$$\inf_{P \in \Pi_1} \int_0^c \operatorname{Cap}_P(\overline{C_{s,as}} \setminus \Omega)\, ds = +\infty. \tag{1.11}$$

Here $C_{s,as}$ is the annulus $\{x \in \mathbb{R}^3 : \ s < |x| < as\}$.

In §9 we further discuss the discrepancy between conditions (1.10) and (1.11) and show by counterexample that (1.10) is not always necessary for 1-regularity.

To the best of our knowledge, Theorem 1.2 is the first Wiener-type result addressing the continuity of the derivatives of a solution. It is accompanied by corresponding estimates, in particular, we prove the following refinement of (1.3):

$$|\nabla_x \nabla_y G(x,y)|$$
$$\leq C \begin{cases} |x-y|^{-1} \exp\Big(-c \int_{c_1|y|}^{c_2|x|} \inf_{P\in\Pi_1} \operatorname{Cap}_P (\overline{C_{s,as}} \setminus \Omega)\, ds \Big), & \text{if} \quad |y| \leq c_0|x|, \\ |x-y|^{-1} \exp\Big(-c \int_{c_1|x|}^{c_2|y|} \inf_{P\in\Pi_1} \operatorname{Cap}_P (\overline{C_{s,as}} \setminus \Omega)\, ds \Big), & \text{if} \quad |x| \leq c_0|y|, \\ |x-y|^{-1}, & \text{if} \quad c_0|y| \leq |x| \leq c_0^{-1}|y|, \end{cases}$$

where $a \geq 4$, c_0, c_1, c_2, c, C are constants independent of Ω.

It has to be noted that Theorem 1.2 brings up a peculiar role of circular cones and planes for 1-regularity of a boundary point. For example, if the complement of Ω is a compactum located on the circular cone (or plane) given by $\{x \in \mathbb{R}^3 \setminus \{0\} : b_0|x| + b_1x_1 + b_2x_2 + b_3x_3 = 0\}$ such that the harmonic capacity $\operatorname{cap}(\mathbb{R}^3 \setminus \Omega) = 0$, then $\operatorname{Cap}_P(\mathbb{R}^3 \setminus \Omega) = 0$ for P associated to the same b_i's. Hence, by Theorem 1.2, the point O is not 1-regular.

Another surprising effect, strikingly different from the classical theory, is that for some domains 1-irregularity turns out to be unstable under affine transformations of coordinates.

In conclusion, we provide some examples further illustrating the geometric nature of conditions (1.10)–(1.11). Among them is the model case when Ω has an inner cusp, i.e. in a neighborhood of the origin $\Omega = \{(r,\theta,\phi) : 0 < r < c,\ h(r) < \theta \leq \pi,\ 0 \leq \phi < 2\pi\}$, where h is a non-decreasing function such that $h(br) \leq h(r)$ for some $b > 1$. For such a domain Theorem 1.2 yields the following criterion:

$$\text{the point } O \text{ is 1-regular} \quad \text{if and only if} \quad \int_0^1 s^{-1} h(s)^2\, ds = +\infty. \tag{1.12}$$

Some other geometrical examples can be found in the body of the paper.

2 Integral identity and global estimate

Let us start with a few remarks about the notation.

Let (r,ω) be spherical coordinates in $\mathbb{R}^3$, i.e. $r = |x| \in (0,\infty)$ and $\omega = x/|x|$ is a point of the unit sphere S^2. Occasionally we will write the spherical coordinates as (r,θ,ϕ), where $\theta \in [0,\pi]$ stands for the colatitude and $\phi \in [0,2\pi)$ is the longitudinal coordinate, i.e.

$$\omega = x/|x| = (\sin\theta\cos\phi,\ \sin\theta\sin\phi,\ \cos\theta). \tag{2.1}$$

Now let $t = \log r^{-1}$. Then by κ and $\varkappa$ we denote the mappings

$$\mathbb{R}^3 \ni x \xrightarrow{\kappa} (r,\phi,\theta) \in [0,\infty) \times [0,2\pi) \times [0,\pi]; \qquad \mathbb{R}^3 \ni x \xrightarrow{\varkappa} (t,\omega) \in \mathbb{R} \times S^2. \tag{2.2}$$

The symbols δ_ω and ∇_ω refer, respectively, to the Laplace-Beltrami operator and the gradient on S^2.

For any domain $\Omega \subset \mathbb{R}^3$ a function $u \in C_0^\infty(\Omega)$ can be extended by zero to $\mathbb{R}^3$ and we will write $u \in C_0^\infty(\mathbb{R}^3)$ whenever convenient. Similarly, the functions in $\mathring{W}_2^2(\Omega)$ will be extended by zero and treated as functions on $\mathbb{R}^3$ without further comments.

By C, c, C_i and c_i, $i \in \mathbb{N}$, we generally denote some constants whose exact values are of no importance. Also, we write $A \approx B$, if $C^{-1} A \leq B \leq C\, A$ for some $C > 0$.

The first result is

Lemma 2.1 *Let Ω be an open set in $\mathbb{R}^3$, $u \in C_0^\infty(\Omega)$ and $v = e^t(u \circ \varkappa^{-1})$. Then*

$$\int_{\mathbb{R}^3} \Delta u(x)\, \Delta\Big(u(x)|x|^{-1}\, \mathcal{G}(\log|x|^{-1})\Big)\, dx$$

$$= \int_{\mathbb{R}} \int_{S^2} \Big[(\delta_\omega v)^2 \mathcal{G} + 2(\partial_t \nabla_\omega v)^2 \mathcal{G} + (\partial_t^2 v)^2 \mathcal{G} - (\nabla_\omega v)^2 \Big(\partial_t^2 \mathcal{G} + \partial_t \mathcal{G} + 2\mathcal{G}\Big)$$

$$-(\partial_t v)^2 \Big(2\partial_t^2 \mathcal{G} + 3\partial_t \mathcal{G} - \mathcal{G}\Big) + \frac{1}{2} v^2 \Big(\partial_t^4 \mathcal{G} + 2\partial_t^3 \mathcal{G} - \partial_t^2 \mathcal{G} - 2\partial_t \mathcal{G}\Big)\Big]\, d\omega dt, \tag{2.3}$$

for every function $\mathcal{G}$ on $\mathbb{R}$ such that both sides of (2.3) *are well-defined.*

Proof. In the system of coordinates (t, ω) the 3-dimensional Laplacian can be written as

$$\Delta = e^{2t} \Lambda(\partial_t, \delta_\omega), \quad \text{where} \quad \Lambda(\partial_t, \delta_\omega) = \partial_t^2 - \partial_t + \delta_\omega. \tag{2.4}$$

Then passing to the coordinates (t, ω), we have

$$\int_{\mathbb{R}^3} \Delta u(x)\, \Delta\Big(u(x)|x|^{-1} \mathcal{G}(\log|x|^{-1})\Big)\, dx = \int_{\mathbb{R}} \int_{S^2} \Lambda(\partial_t - 1, \delta_\omega) v\; \Lambda(\partial_t, \delta_\omega)(v\mathcal{G})\, d\omega dt$$

$$= \int_{\mathbb{R}} \int_{S^2} \Big(\partial_t^2 v - 3\partial_t v + 2v + \delta_\omega v\Big)\ \Big(\partial_t^2 (v\mathcal{G}) - \partial_t (v\mathcal{G}) + \mathcal{G}\, \delta_\omega v\Big)\ d\omega dt$$

$$= \int_{\mathbb{R}} \int_{S^2} \Big(\partial_t^2 v - 3\partial_t v + 2v + \delta_\omega v\Big)$$

$$\times \Big(\mathcal{G}\, \delta_\omega v + \mathcal{G}\, \partial_t^2 v + (2\partial_t \mathcal{G} - \mathcal{G})\, \partial_t v + (\partial_t^2 \mathcal{G} - \partial_t \mathcal{G})\, v\Big)\ d\omega dt$$

$$= \int_{\mathbb{R}} \int_{S^2} \Big(\Big((\delta_\omega v)^2 + 2\, \delta_\omega v \partial_t^2 v + \ (\partial_t^2 v)^2\Big)\, \mathcal{G}$$

$$+ \Big(v\delta_\omega v + v\partial_t^2 v\Big)\ \Big(\partial_t^2 \mathcal{G} - \partial_t \mathcal{G} + 2\mathcal{G}\Big) + \Big(\delta_\omega v \partial_t v + \partial_t^2 v \partial_t v\Big)\, (2\partial_t \mathcal{G} - 4\mathcal{G})$$

$$+(\partial_t v)^2\ (-6\partial_t \mathcal{G} + 3\mathcal{G}) + v\partial_t v\, \Big(-3\partial_t^2 \mathcal{G} + 7\partial_t \mathcal{G} - 2\mathcal{G}\Big) + v^2\, \Big(2\partial_t^2 \mathcal{G} - 2\partial_t \mathcal{G}\Big)\Big)\, d\omega dt. \tag{2.5}$$

This, in turn, is equal to

$$\int_{\mathbb{R}} \int_{S^2} \Big(\mathcal{G}\, (\delta_\omega v)^2 - 2\mathcal{G}\, \delta_\omega \partial_t v\, \partial_t v + \mathcal{G}\, (\partial_t^2 v)^2$$

$$+(\nabla_\omega v)^2 \left(-\partial_t^2\mathcal{G} - (\partial_t^2\mathcal{G} - \partial_t\mathcal{G} + 2\mathcal{G}) + (\partial_t^2\mathcal{G} - 2\partial_t\mathcal{G})\right)$$
$$+(\partial_t v)^2 \left(-(\partial_t^2\mathcal{G} - \partial_t\mathcal{G} + 2\mathcal{G}) + (-\partial_t^2\mathcal{G} + 2\partial_t\mathcal{G}) + (-6\partial_t\mathcal{G} + 3\mathcal{G})\right)$$
$$+v\partial_t v \left(-(\partial_t^3\mathcal{G} - \partial_t^2\mathcal{G} + 2\partial_t\mathcal{G}) + (-3\partial_t^2\mathcal{G} + 7\partial_t\mathcal{G} - 2\mathcal{G})\right)$$
$$+v^2 \left(2\partial_t^2\mathcal{G} - 2\partial_t\mathcal{G}\right)\Big) \, d\omega dt, \tag{2.6}$$

and integrating by parts once again we obtain (2.3). □

In order to single out the term with v^2 in (2.3) we shall need the following auxiliary result.

Lemma 2.2 *Consider the equation*

$$\frac{d^4 g}{dt^4} + 2\frac{d^3 g}{dt^3} - \frac{d^2 g}{dt^2} - 2\frac{dg}{dt} = \delta, \tag{2.7}$$

where δ stands for the Dirac delta function. A unique solution to (2.7) *which is bounded and vanishes at $+\infty$ is given by*

$$g(t) = -\frac{1}{6}\begin{cases} e^t - 3, & t < 0, \\ e^{-2t} - 3\,e^{-t}, & t > 0. \end{cases} \tag{2.8}$$

Proof. Since the equation (2.7) is equivalent to

$$\frac{d}{dt}\left(\frac{d}{dt} + 2\right)\left(\frac{d}{dt} + 1\right)\left(\frac{d}{dt} - 1\right) g = \delta, \tag{2.9}$$

a bounded solution of (2.7) vanishing at $+\infty$ must have the form

$$g(t) = \begin{cases} a\,e^t + b, & t < 0, \\ c\,e^{-2t} + d\,e^{-t}, & t > 0, \end{cases} \tag{2.10}$$

for some constants a, b, c, d. Once this is established, we find the system of coefficients so that $\partial_t^k g$ is continuous for $k = 0, 1, 2$ and $\lim_{t\to 0^+} \partial_t^3 g(t) - \lim_{t\to 0^-} \partial_t^3 g(t) = 1$. □

With Lemma 2.2 at hand, a suitable choice of the function $\mathcal{G}$ yields the positivity of the left-hand side of (2.3), one of the cornerstones of this paper. The details are as follows.

Lemma 2.3 *Let Ω be a bounded domain in $\mathbb{R}^3$, $O \in \mathbb{R}^3 \setminus \Omega$, $u \in C_0^\infty(\Omega)$ and $v = e^t(u \circ \varkappa^{-1})$. Then for every $\xi \in \Omega$ and $\tau = \log|\xi|^{-1}$ we have*

$$\frac{1}{2}\int_{S^{n-1}} v^2(\tau, \omega)\, d\omega \le \int_{\mathbb{R}^n} \Delta u(x)\, \Delta\Big(u(x)|x|^{-1} g(\log(|\xi|/|x|))\Big)\, dx, \tag{2.11}$$

where g is given by (2.8).

Proof. Representing v as a series of spherical harmonics and noting that the eigenvalues of the Laplace-Beltrami operator on the unit sphere are $k(k+1)$, $k = 0, 1, ...$, we arrive at the inequality

$$\int_{S^2} |\delta_\omega v|^2 \, d\omega \geq 2 \int_{S^2} |\nabla_\omega v|^2 \, d\omega. \tag{2.12}$$

Now, let us take $\mathcal{G}(t) = g(t - \tau)$, $t \in \mathbb{R}$. Since $g \geq 0$, the combination of Lemma 2.2, (2.3) and (2.12) allows one to obtain the estimate

$$\begin{aligned}
&\int_{\mathbb{R}^n} \Delta u(x) \, \Delta\Big(u(x)|x|^{-1} g(\log(|\xi|/|x|))\Big) \, dx \\
&\geq \int_{\mathbb{R}} \int_{S^{n-1}} \Big[-(\nabla_\omega v(t,\omega))^2 \Big(\partial_t^2 g(t-\tau) + \partial_t g(t-\tau)\Big) \\
&-(\partial_t v(t,\omega))^2 \Big(2\partial_t^2 g(t-\tau) + 3\partial_t g(t-\tau) - g(t-\tau)\Big)\Big] \, d\omega dt + \frac{1}{2} \int_{S^{n-1}} v^2(\tau,\omega) \, d\omega.
\end{aligned} \tag{2.13}$$

Thus, the matters are reduced to showing that

$$\partial_t^2 g + \partial_t g \leq 0 \qquad \text{and} \qquad 2\partial_t^2 g + 3\partial_t g - g \leq 0. \tag{2.14}$$

Indeed, we compute

$$\partial_t g(t) = -\frac{1}{6} \begin{cases} e^t, & t < 0, \\ -2e^{-2t} + 3\, e^{-t}, & t > 0, \end{cases} \tag{2.15}$$

and

$$\partial_t^2 g(t) = -\frac{1}{6} \begin{cases} e^t, & t < 0, \\ 4e^{-2t} - 3\, e^{-t}, & t > 0, \end{cases} \tag{2.16}$$

which gives

$$\partial_t^2 g(t) + \partial_t g(t) = -\frac{1}{3} \begin{cases} e^t, & t < 0, \\ e^{-2t}, & t > 0, \end{cases} \tag{2.17}$$

and

$$2\partial_t^2 g(t) + 3\partial_t g(t) - g(t) = -\frac{1}{6} \begin{cases} 4e^t + 3, & t < 0, \\ e^{-2t} + 6e^{-t}, & t > 0. \end{cases} \tag{2.18}$$

Clearly, both functions (2.17), (2.18) are non-positive. The result follows from (2.13). □

3 Local energy and L^2 estimates

This section is devoted to estimates for a solution of the Dirichlet problem near a boundary point, in particular, the proof of Theorem 1.1. To set the stage, let us first record the well-known result following from the energy estimate for solutions of elliptic equations.

Lemma 3.1 *Let Ω be an arbitrary domain in $\mathbb{R}^3$, $Q \in \mathbb{R}^3 \setminus \Omega$ and $R > 0$. Suppose*

$$\Delta^2 u = f \text{ in } \Omega, \quad f \in C_0^\infty(\Omega \setminus B_{4R}(Q)), \quad u \in \mathring{W}_2^2(\Omega). \tag{3.1}$$

Then

$$\int_{B_\rho(Q)\cap\Omega} |\nabla^2 u|^2 \, dx + \frac{1}{\rho^2} \int_{B_\rho(Q)\cap\Omega} |\nabla u|^2 \, dx \leq \frac{C}{\rho^4} \int_{C_{\rho,2\rho}(Q)\cap\Omega} |u|^2 \, dx \tag{3.2}$$

for every $\rho < 2R$.

Here and throughout the paper $B_r(Q)$ and $S_r(Q)$ denote, respectively, the ball and the sphere with radius r centered at Q and $C_{r,R}(Q) = B_R(Q) \setminus \overline{B_r(Q)}$. When the center is at the origin, we write B_r in place of $B_r(O)$, and similarly $S_r := S_r(O)$ and $C_{r,R} := C_{r,R}(O)$. Also, $\nabla^2 u$ stands for a vector of all second derivatives of u.

We omit a standard proof of Lemma 3.1 (see, e.g., [4], [27]) and proceed to estimates for a biharmonic function based upon the results in §2.

Proposition 3.2 *Let Ω be a bounded domain in $\mathbb{R}^3$, $Q \in \mathbb{R}^3 \setminus \Omega$, and $R > 0$. Suppose*

$$\Delta^2 u = f \text{ in } \Omega, \quad f \in C_0^\infty(\Omega \setminus B_{4R}(Q)), \quad u \in \mathring{W}_2^2(\Omega). \tag{3.3}$$

Then

$$\frac{1}{\rho^4} \int_{S_\rho(Q)\cap\Omega} |u(x)|^2 \, d\sigma_x \leq \frac{C}{R^5} \int_{C_{R,4R}(Q)\cap\Omega} |u(x)|^2 \, dx \quad \text{for every} \quad \rho < R, \tag{3.4}$$

where C is an absolute constant.

Proof. For notational convenience we assume that $Q = O$. Let us approximate Ω by a sequence of domains with smooth boundaries $\{\Omega_n\}_{n=1}^\infty$ satisfying

$$\bigcup_{n=1}^\infty \Omega_n = \Omega \quad \text{and} \quad \overline{\Omega}_n \subset \Omega_{n+1} \quad \text{for every} \quad n \in \mathbb{N}. \tag{3.5}$$

Choose $n_0 \in \mathbb{N}$ such that $\operatorname{supp} f \subset \Omega_n$ for every $n \geq n_0$ and denote by u_n a unique solution of the Dirichlet problem

$$\Delta^2 u_n = f \quad \text{in} \quad \Omega_n, \quad u_n \in \mathring{W}_2^2(\Omega_n), \quad n \geq n_0. \tag{3.6}$$

The sequence $\{u_n\}_{n=n_0}^\infty$ converges to u in $\mathring{W}_2^2(\Omega)$ (see, e.g., [23], §6.6).

Next, take some $\eta \in C_0^\infty(B_{2R})$ such that

$$0 \leq \eta \leq 1 \text{ in } B_{2R}, \quad \eta = 1 \text{ in } B_R \quad \text{and} \quad |\nabla^k \eta| \leq C R^{-k}, \quad k \leq 4. \tag{3.7}$$

Also, fix $\tau = \log \rho^{-1}$ and let g be the function defined in (2.8).

Consider the difference

$$\int_{\mathbb{R}^3} \Delta\Big(\eta(x) u_n(x)\Big)\, \Delta\Big(\eta(x) u_n(x) |x|^{-1} g(\log(\rho/|x|))\Big)\, dx$$
$$- \int_{\mathbb{R}^3} \Delta u_n(x)\, \Delta\Big(u_n(x) |x|^{-1} g(\log(\rho/|x|)) \eta^2(x)\Big)\, dx. \tag{3.8}$$

One can view this expression as

$$\int_{\mathbb{R}^3} \Big([\Delta^2, \eta] u_n(x)\Big)\Big(\eta(x) u_n(x) |x|^{-1} g(\log(\rho/|x|))\Big)\, dx, \tag{3.9}$$

where the integral is understood in the sense of pairing between $\mathring{W}_2^2(\Omega_n)$ and its dual. Evidently, the support of the integrand is a subset of $\operatorname{supp} \nabla\eta \subset C_{R,2R}$, and therefore, the difference in (3.8) is bounded by

$$C \sum_{k=0}^{2} \frac{1}{R^{5-2k}} \int_{C_{R,2R}} |\nabla^k u_n(x)|^2\, dx. \tag{3.10}$$

Since u_n is biharmonic in $\Omega_n \cap B_{4R}$ and η is supported in B_{2R}, the second term in (3.8) is equal to zero. Turning to the first term, we shall employ Lemma 2.3 with $u = \eta\, u_n$. The result of the Lemma holds for such a choice of u. This can be seen directly by inspection of the argument or one can approximate each u_n by a sequence of $C_0^\infty(\Omega_n)$ functions in $\mathring{W}_2^2(\Omega_n)$ and then take a limit using that $O \notin \overline{\Omega}_n$. Then (3.8) is bounded from below by

$$\frac{C}{\rho^4} \int_{S_\rho} |\eta(x) u_n(x)|^2\, d\sigma_x. \tag{3.11}$$

Hence, for every $\rho < R$

$$\frac{1}{\rho^4} \int_{S_\rho} |u_n(x)|^2\, d\sigma_x \leq C \sum_{k=0}^{2} \frac{1}{R^{5-2k}} \int_{C_{R,2R}} |\nabla^k u_n(x)|^2\, dx. \tag{3.12}$$

Now the proof can be finished applying Lemma 3.1 and taking the limit as $n \to \infty$. $\square$

Now we show that (3.4) yields a uniform pointwise estimate for ∇u.

Corollary 3.3 *Let Ω be a bounded domain in $\mathbb{R}^3$, $Q \in \mathbb{R}^3 \setminus \Omega$, $R > 0$ and*

$$\Delta^2 u = f \text{ in } \Omega, \quad f \in C_0^\infty(\Omega \setminus B_{4R}(Q)), \quad u \in \mathring{W}_2^2(\Omega). \tag{3.13}$$

Then for every $x \in B_{R/4}(Q) \cap \Omega$

$$|\nabla u(x)|^2 \leq \frac{C}{R^5} \int_{C_{R/4,4R}(Q)\cap\Omega} |u(y)|^2\, dy, \tag{3.14}$$

and

$$|u(x)|^2 \leq C \frac{|x-Q|^2}{R^5} \int_{C_{R/4,4R}(Q)\cap\Omega} |u(y)|^2\, dy. \tag{3.15}$$

In particular, for every bounded domain $\Omega \subset \mathbb{R}^3$ the solution to the boundary value problem (3.13) *satisfies*

$$|\nabla u| \in L^\infty(\Omega). \tag{3.16}$$

Proof. By an interior estimate for solutions of the elliptic equations (see [11], pp. 153-155)

$$|\nabla u(x)|^2 \leq \frac{C}{d(x)^3} \int_{B_{d(x)/2}(x)} |\nabla u(y)|^2 \, dy, \tag{3.17}$$

where $d(x)$ denotes the distance from x to $\partial\Omega$. Let x_0 be a point on the boundary of Ω such that $d(x) = |x - x_0|$. Since $x \in B_{R/4}(Q) \cap \Omega$ and $Q \in \mathbb{R}^3 \setminus \Omega$, we have $x \in B_{R/4}(x_0)$, and therefore

$$\frac{1}{d(x)^3} \int_{B_{d(x)/2}(x)} |\nabla u(y)|^2 \, dy \leq \frac{C}{d(x)^5} \int_{B_{2d(x)}(x_0)} |u(y)|^2 \, dy \leq \frac{C}{R^5} \int_{C_{3R/4,3R}(x_0)} |u(y)|^2 \, dy, \tag{3.18}$$

using Lemma 3.1 for the first estimate and (3.4) for the second one. Indeed, $d(x) \leq R/4$ and therefore, $2d(x) < 3R/4$. On the other hand, u is biharmonic in $B_{4R}(Q) \cap \Omega$ and

$$|Q - x_0| \leq |Q - x| + |x - x_0| \leq R/2. \tag{3.19}$$

Hence, u is biharmonic in $B_{3R}(x_0) \cap \Omega$ and Proposition 3.2 holds with x_0 in place of Q, $3R/4$ in place of R and $\rho = 2d(x)$. Furthermore, (3.19) yields

$$C_{3R/4,3R}(x_0) \subset C_{R/4,4R}(Q), \tag{3.20}$$

and that finishes the argument for (3.14).

To prove (3.15), we start with the estimate

$$|u(x)|^2 \leq \frac{C}{d(x)^3} \int_{B_{d(x)/2}(x)} |u(y)|^2 \, dy, \tag{3.21}$$

and then proceed using (3.4), much as in (3.18)–(3.20). □

Using the Kelvin transform for biharmonic functions, an estimate on a biharmonic function near the origin can be translated into an estimate at infinity. In particular, Proposition 3.2 and Corollary 3.3 lead to the following result.

Proposition 3.4 *Let Ω be a bounded domain in $\mathbb{R}^3$, $Q \in \mathbb{R}^3 \setminus \Omega$, $r > 0$ and assume that*

$$\Delta^2 u = f \text{ in } \Omega, \quad f \in C_0^\infty(B_{r/4}(Q) \cap \Omega), \quad u \in \mathring{W}_2^2(\Omega). \tag{3.22}$$

Then

$$\frac{1}{\rho^2} \int_{S_\rho(Q) \cap \Omega} |u(x)|^2 \, d\sigma_x \leq \frac{C}{r^3} \int_{C_{r/4,r}(Q) \cap \Omega} |u(x)|^2 \, dx, \tag{3.23}$$

for any $\rho > r$.

Furthermore, for any $x \in \Omega \setminus B_{4r}(Q)$

$$|\nabla u(x)|^2 \leq \frac{C}{|x - Q|^2 \, r^3} \int_{C_{r/4,4r}(Q) \cap \Omega} |u(y)|^2 \, dy, \tag{3.24}$$

and

$$|u(x)|^2 \leq \frac{C}{r^3} \int_{C_{r/4,4r}(Q) \cap \Omega} |u(y)|^2 \, dy. \tag{3.25}$$

Proof. As before, it is enough to consider the case $Q = O$. Retain the approximation of Ω with the sequence of smooth domains Ω_n satisfying (3.5) and define u_n according to (3.6). We denote by $\mathcal{I}$ the inversion $x \mapsto y = x/|x|^2$ and by U_n the Kelvin transform of u_n,

$$U_n(y) := |y|\, u_n(y/|y|^2), \quad y \in \mathcal{I}(\Omega_n). \tag{3.26}$$

Then

$$\Delta^2 U_n(y) = |y|^{-7}(\Delta^2 u_n)(y/|y|^2), \tag{3.27}$$

and therefore, U_n is biharmonic in $\mathcal{I}(\Omega_n) \cap B_{4/r}$. Moreover, (3.27) implies that

$$\int_{\mathcal{I}(\Omega_n)} |\Delta U_n(y)|^2\, dy = \int_{\Omega_n} |\Delta u_n(x)|^2\, dx, \tag{3.28}$$

so that

$$U_n \in \mathring{W}_2^2(\mathcal{I}(\Omega_n)) \quad \Longleftrightarrow \quad u_n \in \mathring{W}_2^2(\Omega_n). \tag{3.29}$$

Observe also that Ω_n is a bounded domain with $O \notin \overline{\Omega}_n$, hence, so is $\mathcal{I}(\Omega_n)$ and $O \notin \overline{\mathcal{I}(\Omega_n)}$.

Following Proposition 3.2, we show that

$$\rho^4 \int_{S_{1/\rho}} |U_n(y)|^2\, d\sigma_y \leq C\, r^5 \int_{C_{1/r,4/r}} |U_n(y)|^2\, dy, \tag{3.30}$$

which after the substitution (3.26) and the change of coordinates yields

$$\frac{1}{\rho^2} \int_{S_\rho} |u_n(x)|^2\, d\sigma_x \leq \frac{C}{r^3} \int_{C_{r/4,r}} |u_n(x)|^2\, dx. \tag{3.31}$$

Turning to the pointwise estimates (3.24)–(3.25), let us fix some $x \in \Omega \setminus B_{4r}(Q)$. Observe that

$$|\nabla u_n(x)| \leq C|x|^{-1} \left|(\nabla U_n)(x/|x|^2)\right| + \left|U_n(x/|x|^2)\right|, \tag{3.32}$$

since $u_n(x) = |x|\, U_n(x/|x|^2)$. Therefore, combining (3.32) and Corollary 3.3 applied to the function U_n, we deduce that

$$|\nabla u_n(x)|^2 \leq C\, \frac{r^5}{|x|^2} \int_{C_{1/(4r),4/r}} |U_n(z)|^2\, dz = \frac{C}{|x|^2\, r^3} \int_{C_{r/4,4r}} |u_n(z)|^2\, dz, \tag{3.33}$$

and

$$|u_n(x)|^2 \leq C r^5 \int_{C_{1/(4r),4/r}} |U_n(z)|^2\, dz = \frac{C}{r^3} \int_{C_{r/4,4r}} |u_n(z)|^2\, dz. \tag{3.34}$$

At this point, we can use the limiting procedure to complete the argument. Indeed, since u_n converges to u in $\mathring{W}_2^2(\Omega)$, the integrals in (3.31), (3.33) and (3.34) converge to the corresponding integrals with u_n replaced by u. Turning to $|\nabla u_n(x)|$, we observe that both u_n and u are biharmonic in a neighborhood of x, in particular, for sufficiently small d

$$|\nabla(u_n(x) - u(x))|^2 \leq \frac{C}{d^5} \int_{B_{d/2}(x)} |u_n(z) - u(z)|^2\, dz. \tag{3.35}$$

As $n \to \infty$, the integral on the right-hand side of (3.35) vanishes and therefore, $|\nabla u_n(x)| \to |\nabla u(x)|$. Similar considerations apply to $u_n(x)$. $\square$

4 Estimates for Green's function

Let Ω be a bounded three-dimensional domain. As in the introduction, we denote by $G(x,y)$, $x,y \in \Omega$, Green's function for the biharmonic equation. In other words, for every fixed $y \in \Omega$ the function $G(x,y)$ satisfies

$$\Delta_x^2 G(x,y) = \delta(x-y), \qquad x \in \Omega, \tag{4.1}$$

in the space $\mathring{W}_2^2(\Omega)$. Here and throughout the section Δ_x stands for the Laplacian in x variable, and similarly we use the notation Δ_y, ∇_y, ∇_x for the Laplacian and gradient in y, and gradient in x, respectively. As before, $d(x)$ is the distance from $x \in \Omega$ to $\partial\Omega$.

Proposition 4.1 *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain. Then there exists an absolute constant C such that for every $x,y \in \Omega$*

$$\Big|\nabla_x \nabla_y (G(x,y) - \Gamma(x-y))\Big| \leq \frac{C}{\max\{|x-y|,\, d(x),\, d(y)\}}, \tag{4.2}$$

where $\Gamma(x-y) = \frac{|x-y|}{8\pi}$ is the fundamental solution for the bilaplacian.

Proof. Let us start with some auxiliary calculations. Consider a function η such that

$$\eta \in C_0^\infty(B_{1/2}) \quad \text{and} \quad \eta = 1 \quad \text{in} \quad B_{1/4}, \tag{4.3}$$

and define a vector-valued function $\mathcal{R} = (\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3)$ by

$$\mathcal{R}_j(x,y) := \frac{\partial}{\partial y_j} G(x,y) - \eta\left(\frac{x-y}{d(y)}\right) \frac{\partial}{\partial y_j} \Gamma(x-y), \qquad x,y \in \Omega, \tag{4.4}$$

where $j = 1,2,3$. Also, let us denote

$$f_j(x,y) := \Delta_x^2 \mathcal{R}_j(x,y) = -\left[\Delta_x^2, \eta\left(\frac{x-y}{d(y)}\right)\right] \frac{\partial}{\partial y_j} \Gamma(x-y), \quad j = 1,2,3. \tag{4.5}$$

It is not hard to see that for every j

$$f_j(\cdot,y) \in C_0^\infty(C_{d(y)/4, d(y)/2}(y)) \qquad \text{and} \qquad |f_j(x,y)| \leq C d(y)^{-4}, \quad x,y \in \Omega. \tag{4.6}$$

Then for every fixed $y \in \Omega$ the function $x \mapsto \mathcal{R}_j(x,y)$ is a solution of the boundary value problem

$$\Delta_x^2 \mathcal{R}_j(x,y) = f_j(x,y) \text{ in } \Omega, \quad f_j(\cdot,y) \in C_0^\infty(\Omega), \quad \mathcal{R}_j(\cdot,y) \in \mathring{W}_2^2(\Omega), \tag{4.7}$$

so that

$$\big\|\nabla_x^2 \mathcal{R}_j(\cdot,y)\big\|_{L^2(\Omega)} = \|\mathcal{R}_j(\cdot,y)\|_{\mathring{W}_2^2(\Omega)} \leq C\|f_j(\cdot,y)\|_{W_2^{-2}(\Omega)}, \qquad j = 1,2,3. \tag{4.8}$$

Here $W^2_{-2}(\Omega)$ stands for the Banach space dual of $\mathring{W}^2_2(\Omega)$, i.e.

$$\|f_j(\cdot,y)\|_{W^2_{-2}(\Omega)} = \sup_{v:\,\|v\|_{\mathring{W}^2_2(\Omega)}=1} \int_\Omega f_j(x,y)v(x)\,dx. \tag{4.9}$$

Recall that by Hardy's inequality

$$\left\|\frac{v}{|\cdot - Q|^2}\right\|_{L^2(\Omega)} \le C\left\|\nabla^2 v\right\|_{L^2(\Omega)} \quad \text{for every} \quad v\in \mathring{W}^2_2(\Omega), \quad Q\in\partial\Omega. \tag{4.10}$$

Then for some $y_0\in\partial\Omega$ such that $|y-y_0| = d(y)$

$$\begin{aligned}\int_\Omega f_j(x,y)v(x)\,dx &\le C\left\|\frac{v}{|\cdot - y_0|^2}\right\|_{L^2(\Omega)} \left\|f_j(\cdot,y)|\cdot - y_0|^2\right\|_{L^2(\Omega)}\\ &\le Cd(y)^2\left\|\nabla^2 v\right\|_{L^2(\Omega)} \|f_j(\cdot,y)\|_{L^2(C_{d(y)/4,d(y)/2}(y))},\end{aligned} \tag{4.11}$$

and therefore, by (4.6)

$$\left\|\nabla^2_x \mathcal{R}(\cdot,y)\right\|_{L^2(\Omega)} \le Cd(y)^{-1/2}. \tag{4.12}$$

Turning to (4.2), let us first consider the case $|x-y|\ge Nd(y)$ for some large N to be specified later. As before, we denote by y_0 some point on the boundary such that $|y-y_0| = d(y)$. Then by (4.6) the function $x\mapsto \mathcal{R}(x,y)$ is biharmonic in $\Omega\setminus B_{3d(y)/2}(y_0)$. Hence, by Proposition 3.4 with $r=6d(y)$

$$|\nabla_x \mathcal{R}(x,y)|^2 \le \frac{C}{|x-y_0|^2\,d(y)^3}\int_{C_{3d(y)/2,24d(y)}(y_0)} |\mathcal{R}(z,y)|^2\,dz, \tag{4.13}$$

provided $|x-y|\ge 4r+d(y)$, i.e $N\ge 25$. The right-hand side of (4.13) is bounded by

$$\frac{Cd(y)}{|x-y_0|^2}\int_{C_{3d(y)/2,24d(y)}(y_0)} \frac{|\mathcal{R}(z,y)|^2}{|z-y_0|^4}\,dz \le \frac{C\,d(y)}{|x-y_0|^2}\int_\Omega |\nabla_z^2\mathcal{R}(z,y)|^2\,dz \le \frac{C}{|x-y|^2}, \tag{4.14}$$

by Hardy's inequality and (4.12).

Now one can directly check that

$$|\nabla_x\nabla_y\Gamma(x,y)| \le \frac{C}{|x-y|} \qquad \text{for all} \qquad x,y\in\Omega, \tag{4.15}$$

and combine it with (4.13)–(4.14) to deduce that

$$\Big|\nabla_x\nabla_y(G(x,y)-\Gamma(x-y))\Big| \le \frac{C}{|x-y|} \qquad \text{whenever} \qquad |x-y|\ge Nd(y). \tag{4.16}$$

We claim that this settles the case

$$|x-y| \ge N\min\{d(y),d(x)\}. \tag{4.17}$$

Indeed, if $d(y) \leq d(x)$, (4.16) gives the desired result and if $d(y) \geq d(x)$ and $|x-y| \geq Nd(x)$, we employ the version of (4.16) with $d(x)$ in place of $d(y)$ which follows from the symmetry of Green's function and the fundamental solution in x and y variables.

Next, assume that $|x-y| \leq N^{-1}d(y)$. For such x we have $\eta(\frac{x-y}{d(y)}) = 1$ and therefore

$$\frac{\partial}{\partial y_j}\left(G(x,y) - \Gamma(x-y)\right) = \mathcal{R}_j(x,y). \tag{4.18}$$

By the interior estimates for solutions of elliptic equations

$$|\nabla_x \mathcal{R}(x,y)|^2 \leq \frac{C}{d(y)^5}\int_{B_{d(y)/8}(x)} |\mathcal{R}(z,y)|^2\, dz, \tag{4.19}$$

since the function $\mathcal{R}$ is biharmonic in $B_{d(y)/8}(x) \subset B_{d(y)/4}(y)$. Now we bound the expression above by

$$\frac{C}{d(y)}\int_{B_{d(y)/4}(y)} \frac{|\mathcal{R}(z,y)|^2}{|z-y_0|^4}\, dz \leq \frac{C}{d(y)}\left\|\nabla_x^2 \mathcal{R}(\cdot,y)\right\|^2_{L^2(\Omega)} \leq \frac{C}{d(y)^2}. \tag{4.20}$$

When $|x-y| \leq N^{-1}d(y)$, we have

$$(N-1)\, d(y) \leq Nd(x) \leq (N+1)\, d(y), \tag{4.21}$$

i.e. $d(y) \approx d(x)$, and therefore (4.19)–(4.20) give the desired result. By symmetry, one can handle the case $|x-y| \leq N^{-1}d(x)$ and hence all $x, y \in \Omega$ such that

$$|x-y| \leq N^{-1} \max\{d(x),\, d(y)\}. \tag{4.22}$$

Finally, it remains to consider the situation when

$$|x-y| \approx d(x) \approx d(y), \tag{4.23}$$

or more precisely, when

$$N^{-1}\, d(x) \leq |x-y| \leq Nd(x) \quad \text{and} \quad N^{-1}\, d(y) \leq |x-y| \leq Nd(y). \tag{4.24}$$

In this case we use the biharmonicity of $x \mapsto G(x,y)$ in $B_{d(x)/(2N)}(x)$. By the interior estimates, with $x_0 \in \partial\Omega$ such that $|x-x_0| = d(x)$, we have

$$\begin{aligned}
|\nabla_x \nabla_y G(x,y)|^2 &\leq \frac{C}{d(x)^5}\int_{B_{d(x)/(2N)}(x)} |\nabla_y G(z,y)|^2\, dz \\
&\leq \frac{C}{d(x)^5}\int_{B_{d(x)/(2N)}(x)} |\nabla_y \Gamma(z-y)|^2\, dz + \frac{C}{d(x)}\int_{B_{2d(x)}(x_0)} \frac{|\mathcal{R}(z,y)|^2}{|z-x_0|^4}\, dz \\
&\leq \frac{C}{d(x)^5}\int_{B_{d(x)/(2N)}(x)} |\nabla_y \Gamma(z-y)|^2\, dz + \frac{C}{d(x)}\int_{\Omega} |\nabla_z^2 \mathcal{R}(z,y)|^2\, dz \\
&\leq \frac{C}{d(x)^2} + \frac{C}{d(x)d(y)},
\end{aligned} \tag{4.25}$$

invoking Hardy's inequality and (4.12). In view of (4.23) this finishes the argument. □

Proof of Theorem 1.1. The estimate (1.3) follows directly from (4.2). The second inequality in (1.4) can be proved closely following the above argument, via an analogue of (4.2). The first inequality in (1.4) is based on the second one and the symmetry of Green's function. □

Green's function estimates proved in this section allow to investigate the solutions of the Dirichlet problem (1.2) for a wide class of data. For example, consider the boundary value problem

$$\Delta^2 u = \operatorname{div} f + h, \quad u \in \mathring{W}_2^2(\Omega), \tag{4.26}$$

where $f = (f_1, f_2, f_3)$ is some vector valued function and $h \in L^1(\Omega)$. Then the solution satisfies the estimate

$$|\nabla u(x)| \leq C \int_\Omega \frac{|f(y)|}{|x-y|}\, dy + C \int_\Omega |h(y)|\, dy, \qquad x \in \Omega. \tag{4.27}$$

Indeed, the integral representation formula

$$u(x) = \int_\Omega G(x,y) \Big(\operatorname{div} f(y) + h(y)\Big)\, dy, \qquad x \in \Omega, \tag{4.28}$$

follows directly from the definition of Green's function. It implies that

$$\begin{aligned} \nabla u(x) &= \nabla_x \int_\Omega G(x,y) \Big(\operatorname{div} f(y) + h(y)\Big)\, dy \\ &= -\int_\Omega \nabla_x (\nabla_y G(x,y) \cdot f(y))\, dy + \int_\Omega \nabla_x G(x,y) h(y)\, dy, \end{aligned} \tag{4.29}$$

and Theorem 1.1 leads to (4.27).

One can further observe that by the mapping properties of the Riesz potential the estimate (4.27) entails that

$$\|\nabla u\|_{L^\infty(\Omega)} \leq C \|f\|_{L^{3/2,1}(\Omega)} + C \|h\|_{L^1(\Omega)}, \tag{4.30}$$

where $L^{3/2,1}(\Omega)$ is a Lorentz space. Consequently,

$$\|\nabla u\|_{L^\infty(\Omega)} \leq C \|f\|_{L^p(\Omega)} + C \|h\|_{L^1(\Omega)}, \quad p > 3/2, \tag{4.31}$$

whenever $f \in L^p(\Omega)$ for some $p > 3/2$.

5 The capacity Cap_P

This section is devoted to basic properties of the capacity Cap_P. A part of the results presented here and in §9 have been obtained in [22]. For the convenience of the reader we present a self-contained discussion.

To begin, we introduce a capacity of a compactum K relative to some open set $\Omega \subset \mathbb{R}^3 \setminus \{O\}$, $K \subset \Omega$. To this end, recall that Π is the space of functions (1.8) equipped with some norm. For example, we can take

$$\|P\|_\Pi = \sqrt{b_0^2 + b_1^2 + b_2^2 + b_3^2}, \tag{5.1}$$

and $\Pi_1 := \{P \in \Pi : \|P\|_\Pi = 1\}$. A different norm in the space Π would yield an equivalent relative capacity.

Now fix some $P \in \Pi_1$. Then

$$\mathrm{Cap}_P\,(K, \Omega) := \inf \left\{ \int_\Omega (\Delta u(x))^2\, dx : \ u \in \mathring{W}_2^2(\Omega),\ u = P \text{ in a neighborhood of } K \right\}, \tag{5.2}$$

and

$$\mathrm{Cap}\,(K, \Omega) := \inf_{P \in \Pi_1} \mathrm{Cap}_P\,(K, \Omega). \tag{5.3}$$

Observe that in the introduction, for the sake of brevity, we dropped the reference to Ω. There we had $\Omega = \mathbb{R}^3 \setminus \{0\}$.

It follows directly from the definition that the capacity Cap_P is monotone in the sense that for every $P \in \Pi_1$

$$K_1 \subseteq K_2 \subset \Omega \qquad \Longrightarrow \qquad \mathrm{Cap}_P\,(K_1, \Omega) \leq \mathrm{Cap}_P\,(K_2, \Omega), \tag{5.4}$$

$$K \subset \Omega_1 \subseteq \Omega_2 \qquad \Longrightarrow \qquad \mathrm{Cap}_P\,(K, \Omega_1) \geq \mathrm{Cap}_P\,(K, \Omega_2), \tag{5.5}$$

and analogous statements hold for Cap in place of Cap_P.

We shall be concerned mostly with the case when a compactum is contained in some annulus centered at the origin for the reasons that will become apparent in the sequel. In such a case, it will be convenient to work with an equivalent definition of capacity by means of the form

$$\Psi[u; \Omega] = \int_{\tilde{\varkappa}(\Omega)} \left((\partial_r^2 v)^2 + 2r^{-2} (\partial_r v)^2 + 2r^{-2} |\partial_r \nabla_\omega v|^2 + r^{-4} (\delta_\omega v)^2 + 2r^{-4} v \delta_\omega v \right) r^2\, d\omega dr, \tag{5.6}$$

where (r, ω) are the spherical coordinates in the three dimensional space, $\tilde{\varkappa}$ is the mapping

$$\mathbb{R}^3 \ni x \stackrel{\tilde{\varkappa}}{\longrightarrow} (r, \omega) \in [0, \infty) \times S^2, \tag{5.7}$$

and $v = u \circ \tilde{\varkappa}^{-1}$.

Lemma 5.1 *For every r, R such that $0 < r < R < \infty$ and every function $u \in W_2^2(C_{r,R})$*

$$\Psi[u; C_{r,R}] = \int_{C_{r,R}} \Bigg[(\Delta u)^2 - \frac{2}{|x|^4} \Big(x_i \frac{\partial}{\partial x_i} - 1 \Big) \left(\Big(x_j \frac{\partial u}{\partial x_j} + u \Big) \Big(|x|^2 \Delta u - x_i \frac{\partial}{\partial x_i} \Big(x_j \frac{\partial u}{\partial x_j} \Big) - u \Big) + u^2 \Big) \right) \Bigg]\, dx, \tag{5.8}$$

where, as customary, we imply summation on repeated indices. Furthermore, for every open set Ω *in* $\mathbb{R}^3 \setminus \{0\}$ *and every* $u \in \mathring{W}_2^2(\Omega)$

$$\Psi[u;\Omega] = \int_\Omega (\Delta u(x))^2\, dx. \tag{5.9}$$

The formulas (5.8)–(5.9) can be checked directly using the representation of the Laplacian in spherical coordinates

$$\Delta u = r^{-2}\left(\partial_r(r^2\partial_r) + \delta_\omega\right). \tag{5.10}$$

They give rise to an alternative definition of the biharmonic capacity. Indeed, if K is a compact subset of $\Omega \subset \mathbb{R}^3 \setminus \{0\}$, then for every $P \in \Pi_1$

$$\mathrm{Cap}_P\,(K,\Omega) = \inf\{\Psi[u;\Omega]:\ u \in \mathring{W}_2^2(\Omega),\ u = P \text{ in a neighborhood of } K\} \tag{5.11}$$

and an analogous equality holds for Cap in place of Cap_P.

Lemma 5.2 *Suppose* K *is a compactum in* $\overline{C_{s,as}}$ *for some* $s > 0$, $a > 1$*. Then for every* $P \in \Pi_1$

$$\mathrm{Cap}_P(K, \mathbb{R}^3 \setminus \{O\}) \approx \mathrm{Cap}_P(K, C_{s/2,2as}) \qquad \text{and} \qquad \mathrm{Cap}_P(K, C_{s/2,2as}) \leq Cs^{-1}, \tag{5.12}$$

with the constants independent of s*.*

Proof. The inequality

$$\mathrm{Cap}_P(K, \mathbb{R}^3 \setminus \{O\}) \leq \mathrm{Cap}_P(K, C_{s/2,2as}) \tag{5.13}$$

is a consequence of the monotonicity property (5.5). As for the opposite inequality, we take $u \in \mathring{W}_2^2(\mathbb{R}^3 \setminus \{O\})$ such that $u = P$ in a neighborhood of K and

$$\mathrm{Cap}_P(K, \mathbb{R}^3 \setminus \{O\}) + \varepsilon > \int_{\mathbb{R}^3} |\Delta u(x)|^2\, dx = \Psi[u; \mathbb{R}^3 \setminus \{O\}]. \tag{5.14}$$

Consider now the cut-off function

$$\zeta \in C_0^\infty(1/2, 2a),\ \zeta = 1 \text{ on } [3/4, 3a/2], \tag{5.15}$$

and let $w(x) := \zeta(|x|/s)u(x)$, $x \in \mathbb{R}^3$. Then

$$w \in \mathring{W}_2^2(C_{s/2,2as}) \quad \text{and} \quad w = P \text{ in a neighborhood of } K. \tag{5.16}$$

Hence,

$$\mathrm{Cap}_P(K, C_{s/2,2as}) \leq \Psi[w; C_{s/2,2as}] \tag{5.17}$$

and

$$\begin{aligned}\Psi[w, C_{s/2,2as}] = &\int_{s/2}^{2as}\int_{S^2}\Big((\partial_r^2(\zeta(r/s)v))^2 + 2r^{-2}(\partial_r(\zeta(r/s)v))^2 \\ &+2r^{-2}|\partial_r(\zeta(r/s)\nabla_\omega v)|^2 + r^{-4}\zeta^2(r/s)(\delta_\omega v)^2 + 2r^{-4}\zeta^2(r/s)v\delta_\omega v\Big)\, r^2\, d\omega dr \\ &\leq C\Psi[v, C_{s/2,2as}],\end{aligned} \tag{5.18}$$

using the properties of ζ and the one dimensional Hardy's inequality in the r variable. This finishes the proof of the first assertion in (5.12).

As for the second one, observe first that if $v(x) = u(sx)$, $x \in \mathbb{R}^3$, the functions u and v belong to $\mathring{W}_2^2(\mathbb{R}^3 \setminus \{O\})$ simultaneously, and $u = P$ in a neighborhood of K if and only if $v = P$ in a neighborhood of $s^{-1}K := \{x \in \mathbb{R}^3 : sx \in K\}$. Also,

$$\int_{\mathbb{R}^3} |\Delta v(x)|^2\, dx = \int_{\mathbb{R}^3} |\Delta_x u(sx)|^2\, dx = s \int_{\mathbb{R}^3} |\Delta_y u(y)|^2\, dy, \tag{5.19}$$

so that

$$s\mathrm{Cap}_P(K, \mathbb{R}^3 \setminus \{O\}) = \mathrm{Cap}_P(s^{-1}K, \mathbb{R}^3 \setminus \{O\}). \tag{5.20}$$

However, $s^{-1}K \subset \overline{C_{1,a}}$ and therefore by (5.12) the right-hand side of (5.20) is controlled by $\mathrm{Cap}_P(\overline{C_{1,a}}, \mathbb{R}^3 \setminus \{O\})$, uniformly in s. □

Lemma 5.3 *Assume that for some $s > 0$, $a > 1$ the function $u \in L^2(C_{s,as})$ is such that $\Psi[u; C_{s,as}] < \infty$. Then there exists $\mathcal{P} = \mathcal{P}(u, s, a) \in \Pi$ with the property*

$$\|u - \mathcal{P}\|^2_{L^2(C_{s,as})} \leq Cs^4 \Psi[u; C_{s,as}]. \tag{5.21}$$

Proof. Let us start with the expansion of u by means of spherical harmonics:

$$u = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} u_l^m(r) Y_l^m(\omega), \tag{5.22}$$

where Y_l^m are the spherical harmonics of degree $l \in \mathbb{N}$ and order $m \in \mathbb{Z}$. By Poincaré's inequality, for $l = 0, 1$, and the corresponding m there exist constants σ_l^m (depending on u) such that

$$\int_s^{as} |u_l^m(r) - \sigma_l^m|^2\, dr \leq Cs^2 \int_s^{as} |\partial_r u_l^m(r)|^2\, dr. \tag{5.23}$$

Let

$$\mathcal{P}(x) := \sigma_0^0 + \sigma_1^1 \frac{x_1}{|x|} + \sigma_1^{-1} \frac{x_2}{|x|} + \sigma_1^0 \frac{x_3}{|x|}, \qquad x \in \mathbb{R}^3 \setminus \{O\}. \tag{5.24}$$

Then (5.23) yields (5.21). □

Proposition 5.4 *([22]) Suppose $s > 0$, $a \geq 2$ and K is a compact subset of $\overline{C_{s,as}}$. Then for every $u \in L^2(C_{s,as})$ such that $\Psi[u; C_{s,as}] < \infty$ and $u = 0$ in a neighborhood of K*

$$\frac{1}{s^3} \int_{C_{s,as}} |u(x)|^2\, dx \leq \frac{C}{\mathrm{Cap}\,(K, \mathbb{R}^3 \setminus \{O\})}\ \Psi[u; C_{s,as}]. \tag{5.25}$$

Proof. For the purposes of this argument let us take $\|P\|_\Pi := \|P\|_{L^2(C_{1,a})}$ and let $\Pi_1 := \{P \in \Pi : \|P\|_\Pi = 1\}$ with such a norm. This is an equivalent norm in the space Π and hence it yields the capacity equivalent to the one defined in (5.1)–(5.2). We claim that for every $P \in \Pi_1$

$$\mathrm{Cap}_P(K, C_{s/2,2as}) \leq Cs^{-4} \|P - u\|^2_{L^2(C_{s,as})} + C\Psi[u; C_{s,as}]. \tag{5.26}$$

To prove this, let us denote by $V_2^2(C_{s,as})$ a collection of functions on $C_{s,as}$ such that

$$\|u\|_{V_2^2(C_{s,as})} := \left(\frac{1}{s^4} \int_{C_{s,as}} |u(x)|^2\, dx + \Psi[u; C_{s,as}] \right)^{1/2}, \tag{5.27}$$

is finite. One can construct an extension operator

$$\mathrm{Ex} : V_2^2(C_{s,as}) \to V_2^2(C_{s/2,2as}) \tag{5.28}$$

with the operator norm independent of s satisfying the properties

$$\mathrm{Ex}\, u = u \text{ in } C_{s,as}, \qquad \mathrm{Ex}\, P = P \text{ for every } P \in \Pi_1, \tag{5.29}$$

and such that if $u = 0$ in some neighborhood of K intersected with $\overline{C_{s,as}}$ then $\mathrm{Ex}\, u$ vanishes in a neighborhood of K contained in $C_{s/2,2as}$. For example, one can start with the corresponding one-dimensional extension operator and then use the expansion (5.22) to define Ex.

Having this at hand, we define $w(x) := \zeta(|x|/s)(P(x) - \mathrm{Ex}\, u(x))$, $x \in C_{s/2,2as}$, where ζ is a function introduced in (5.15). Then w satisfies (5.16) and therefore $\mathrm{Cap}_P(K, C_{s/2,2as})$ is controlled by

$$\begin{aligned} \Psi[w; C_{s/2,2as}] &\leq \Psi[P - \mathrm{Ex}\, u; C_{s/2,2as}] = \Psi[\mathrm{Ex}\,(P - u); C_{s/2,2as}] \\ &\leq C s^{-4} \|P - u\|^2_{L^2(C_{s,as})} + C\Psi[P - u; C_{s,as}], \end{aligned} \tag{5.30}$$

where the first inequality is proved analogously to (5.18) and the second one follows from the mapping properties of Ex. Using that $\delta_\omega \omega_i = -2\omega_i$, $i = 1, 2, 3$, one can directly check that

$$\Psi[P - u; C_{s,as}] = \Psi[u; C_{s,as}], \tag{5.31}$$

and obtain (5.26).

The next step is to pass from (5.26) to (5.25). Without loss of generality we may assume that $\|u\|_{L^2(C_{s,as})} = s^{3/2}$. Then the desired result reads as

$$\inf_{P \in \Pi_1} \mathrm{Cap}_P(K, C_{s/2,2as}) \leq \Psi[u; C_{s,as}]. \tag{5.32}$$

Let $\mathcal{P} = \mathcal{P}(u, s, a)$ be a function in Π satisfying (5.21), and denote by C_0 the constant C in (5.21). First of all, the case

$$\Psi[u; C_{s,as}] \geq 1/(4C_0 s) \tag{5.33}$$

is trivial, since Lemma 5.2 guarantees that the right-hand side of (5.33) is bounded from below by the capacity of K, modulo a multiplicative constant.

On the other hand,

$$\Psi[u; C_{s,as}] \leq 1/(4C_0 s) \qquad \Longrightarrow \qquad 2\|u - \mathcal{P}\|_{L^2(C_{s,as})} \leq s^{3/2} = \|u\|_{L^2(C_{s,as})}, \tag{5.34}$$

by (5.21) and the normalization of u. This, in turn, implies that

$$\frac{s^{3/2}}{2} \leq \|\mathcal{P}\|_{L^2(C_{s,as})} \leq \frac{3s^{3/2}}{2}. \tag{5.35}$$

Finally, we choose

$$P := \frac{\mathcal{P}}{\|\mathcal{P}\|_{L^2(C_{1,a})}} = s^{3/2}\,\frac{\mathcal{P}}{\|\mathcal{P}\|_{L^2(C_{s,as})}}. \tag{5.36}$$

Then

$$\begin{aligned}\|P-\mathcal{P}\|_{L^2(C_{s,as})} &= \left|s^{3/2} - \|\mathcal{P}\|_{L^2(C_{s,as})}\right| \\ &= \left|\|u\|_{L^2(C_{s,as})} - \|\mathcal{P}\|_{L^2(C_{s,as})}\right| \leq \|u-\mathcal{P}\|_{L^2(C_{s,as})}.\end{aligned} \tag{5.37}$$

Hence,

$$||u-P|| \leq ||u-\mathcal{P}|| + ||\mathcal{P}-P|| \leq 2||u-\mathcal{P}||, \tag{5.38}$$

so that

$$\|u-P\|^2_{L^2(C_{s,as})} \leq 16\|u-\mathcal{P}\|^2_{L^2(C_{s,as})} \leq 16C_0 s^4 \Psi[u; C_{s,as}], \tag{5.39}$$

by (5.21). Combining (5.39) with (5.26), we complete the argument. □

6 1-regularity of a boundary point

Let Ω be a domain in $\mathbb{R}^3$ and consider the boundary value problem

$$\Delta^2 u = f \text{ in } \Omega, \quad f \in C_0^\infty(\Omega), \quad u \in \mathring{W}_2^2(\Omega). \tag{6.1}$$

We say that the point $Q \in \partial\Omega$ is *1-regular* (with respect to Ω) if for every $f \in C_0^\infty(\Omega)$ the gradient of the solution to (6.1) is continuous, i.e.

$$\nabla u(x) \to 0 \text{ as } x \to Q,\ x \in \Omega. \tag{6.2}$$

Otherwise, $Q \in \partial\Omega$ is called 1-irregular.

Observe that in the case $Q = O$ this definition coincides with the one given in the introduction.

In this section we would like to show that 1-regularity is a local property. In particular, while most of the statements in Sections 1–5 were confined to the case of a bounded domain, the proposition below will allow us to study 1-regularity with respect to any open set in $\mathbb{R}^3$.

Proposition 6.1 *Let Ω be a bounded domain in $\mathbb{R}^3$ and the point $Q \in \partial\Omega$ be 1-regular with respect to Ω. If Ω' is another domain with the property that $B_r(Q) \cap \Omega = B_r(Q) \cap \Omega'$ for some $r > 0$ then Q is 1-regular with respect to Ω'.*

The proof of the proposition rests on the ideas from [19]. It starts with the following result.

Lemma 6.2 *Let Ω be a bounded domain in $\mathbb{R}^3$ and the point $Q \in \partial\Omega$ be 1-regular with respect to Ω. Then*

$$\nabla u(x) \to 0 \text{ as } x \to Q,\ x \in \Omega, \tag{6.3}$$

for every $u \in \mathring{W}_2^2(\Omega)$ satisfying

$$\Delta^2 u = \sum_{\alpha:\,|\alpha|\leq 2} \partial^\alpha f_\alpha \text{ in } \Omega, \quad f_\alpha \in L^2(\Omega) \cap C^\infty(\Omega), \quad f_\alpha = 0 \text{ in a neighborhood of } Q. \tag{6.4}$$

Proof. Take some $\eta \in C_0^\infty(\Omega)$ and let v be the solution of the Dirichlet problem

$$\Delta^2 v = \sum_{\alpha\colon |\alpha|\le 2} \partial^\alpha(\eta f_\alpha) \text{ in } \Omega, \quad v \in \mathring{W}_2^2(\Omega), \tag{6.5}$$

and $w := u - v \in W_2^2(\Omega)$. Since the point Q is 1-regular, the function v automatically satisfies (6.3) and it remains to consider w.

Since $f_\alpha = 0$ in a neighborhood of Q, the function w is biharmonic in some neighborhood of Q and, therefore, for some $R > 0$ depending on the $\operatorname{supp} f_\alpha$, we have

$$|\nabla w(x)| \le \frac{C}{d(x)^3} \int_{B_{d(x)/2}(x)} |\nabla w(y)|^2\,dy \le \frac{C}{R^5} \int_{C_{R/4,4R}(Q)} |w(y)|^2\,dy, \quad \forall\, x \in B_{R/4}(Q), \tag{6.6}$$

analogously to (3.17)–(3.18). On the other hand, according to Lemma 2.3 the last expression in (6.6) does not exceed

$$\begin{aligned} &C \sup_{\xi \in C_{R/4,4R}(Q)\cap\Omega} \int_{\mathbb{R}^n} \Delta w(y)\,\Delta\left(\frac{w(y)}{|x-Q|}\, g\left(\log\frac{|\xi-Q|}{|y-Q|}\right)\right) dy \\ &\le C \sup_{\xi \in C_{R/4,4R}(Q)\cap\Omega} \sum_{\alpha\colon |\alpha|\le 2} \int_{\mathbb{R}^n} (1-\eta(y)) f_\alpha(y)\,(-\partial_y)^\alpha \left(\frac{w(y)}{|y-Q|}\, g\left(\log\frac{|\xi-Q|}{|y-Q|}\right)\right) dy, \end{aligned} \tag{6.7}$$

where g is given by (2.8). When x approaches Q, the support of $1-\eta$ can be chosen arbitrarily small. Hence, the integral on the right-hand side of (6.7) shrinks. Then (6.6)–(6.7) imply that $|\nabla w(x)| \to 0$ when $x \to Q$. □

Proof of Proposition 6.1. Consider a solution of the Dirichlet problem

$$\Delta^2 u = f \text{ in } \Omega', \quad f \in C_0^\infty(\Omega'), \quad u \in \mathring{W}_2^2(\Omega'), \tag{6.8}$$

and take some cut-off function $\eta \in C_0^\infty(B_r(Q))$ equal to 1 on $B_{r/2}(Q)$. Then $\eta u \in \mathring{W}_2^2(\Omega)$ and

$$\Delta^2(\eta u) = \eta f + [\Delta^2, \eta]u. \tag{6.9}$$

Since $\eta f \in C_0^\infty(\Omega)$,

$$[\Delta^2,\eta] : \mathring{W}_2^2(\Omega) \longrightarrow (\mathring{W}_2^2(\Omega))^* = W_{-2}^2(\Omega) \quad \text{and} \quad \operatorname{supp}([\Delta^2,\eta]u) \subset C_{r/2,r}(Q)\cap\Omega, \tag{6.10}$$

one can write

$$\Delta^2(\eta u) = \sum_{\alpha\colon |\alpha|\le 2} \partial^\alpha f_\alpha, \quad \text{for some} \quad f_\alpha \in L^2(\Omega)\cap C^\infty(\Omega), \tag{6.11}$$

with $f_\alpha = 0$ in a neighborhood of Q given by the intersection of $B_{r/2}(Q)$ and the complement to $\operatorname{supp} f$. Then, by Lemma 6.2, the gradient of ηu (and therefore, the gradient of u) vanishes as $x \to Q$. □

7 Sufficient condition for 1-regularity

The following proposition provides the first part of Theorem 1.2, i.e. sufficiency of condition (1.10) for 1-regularity of a boundary point.

Proposition 7.1 *Let Ω be a bounded domain in $\mathbb{R}^3$, $O \in \mathbb{R}^3 \setminus \Omega$, $R > 0$ and*

$$\Delta^2 u = f \text{ in } \Omega, \quad f \in C_0^\infty(\Omega \setminus B_{4R}), \quad u \in \mathring{W}_2^2(\Omega). \tag{7.1}$$

Fix some $a \geq 4$. Then for every $x \in B_{R/a^4} \cap \Omega$

$$\begin{aligned} |\nabla u(x)|^2 + \frac{|u(x)|^2}{|x|^2} &\leq \frac{C}{R^5} \int_{C_{R,4R} \cap \Omega} |u(y)|^2 \, dy \\ &\quad \times \exp\left(- c \int_{a^2|x|}^{R/a^2} \operatorname{Cap}\left(\overline{C_{s,as}} \setminus \Omega, \mathbb{R}^3 \setminus \{O\}\right) ds \right). \end{aligned} \tag{7.2}$$

In particular, when O is a boundary point of Ω,

$$\text{if } \int_0^{R/a^2} \operatorname{Cap}\left(\overline{C_{s,as}} \setminus \Omega, \mathbb{R}^3 \setminus \{O\}\right) ds = +\infty \text{ then } O \text{ is 1-regular.} \tag{7.3}$$

Proof. Fix $s \leq R/a^2$ and let us introduce some extra notation. First,

$$\gamma(s) := \operatorname{Cap}\left(\overline{C_{s,a^2 s}} \setminus \Omega, \mathbb{R}^3 \setminus \{O\}\right). \tag{7.4}$$

Further, let $Q_\tau[u; \Omega]$, $\tau \in \mathbb{R}$, be the quadratic form

$$\begin{aligned} Q_\tau[u;\Omega] := \int_{\varkappa(\Omega)} \Big[&(\delta_\omega v)^2 g(t-\tau) + 2(\partial_t \nabla_\omega v)^2 g(t-\tau) + (\partial_t^2 v)^2 g(t-\tau) \\ &- (\nabla_\omega v)^2 \Big(\partial_t^2 g(t-\tau) + \partial_t g(t-\tau) + 2 g(t-\tau) \Big) \\ &- (\partial_t v)^2 \Big(2\partial_t^2 g(t-\tau) + 3\partial_t g(t-\tau) - g(t-\tau) \Big) \Big] \, d\omega dt, \end{aligned} \tag{7.5}$$

where $v = e^t (u \circ \varkappa^{-1})$, g is defined by (2.8) and $\varkappa$ is the change of coordinates (2.2). Throughout this proof $\tau = \log s^{-1}$.

Now take $\eta \in C_0^\infty(B_{2s})$ such that

$$0 \leq \eta \leq 1 \text{ in } B_{2s}, \quad \eta = 1 \text{ in } B_s \quad \text{and} \quad |\nabla^k \eta| \leq C/|x|^k, \quad k \leq 4. \tag{7.6}$$

Following the argument in (3.8)–(3.10) and the discussion after (3.10), and then passing to the limit as $n \to \infty$, we have

$$\begin{aligned} Q_\tau[u; B_s] &\leq Q_\tau[\eta u; \Omega] \leq \int_{\mathbb{R}^3} \Delta\Big(\eta(x) u(x)\Big) \Delta\Big(\eta(x) u(x) |x|^{-1} g(\log(s/|x|))\Big) dx \\ &\leq C \sum_{k=0}^{2} \frac{1}{s^{5-2k}} \int_{C_{s,2s}} |\nabla^k u(x)|^2 \, dx \leq \frac{C}{s^5} \int_{C_{s,4s}} |u(x)|^2 \, dx. \end{aligned} \tag{7.7}$$

Denote

$$\varphi(s) := \sup_{|x|\leq s}\left(|\nabla u(x)|^2 + \frac{|u(x)|^2}{|x|^2}\right) + Q_\tau[u; B_s], \qquad \tau = \log s^{-1}, \qquad s \leq \frac{R}{a^2}. \tag{7.8}$$

Then combining (7.7) with Corollary 3.3 and Proposition 3.2,

$$\varphi(s) \leq \frac{C}{s^5}\int_{C_{s,16s}} |u(x)|^2\,dx \leq \frac{C}{s^5}\int_{C_{s,a^2 s}} |u(x)|^2\,dx. \tag{7.9}$$

For $\gamma(s) > 0$ the expression on the right-hand side of (7.9) does not exceed

$$\frac{C}{s^3}\int_{C_{s,a^2 s}} \frac{|u(x)|^2}{|x|^2}\,dx \leq \frac{C}{\gamma(s)}\ \Psi\left[\frac{u}{|x|}; C_{s,a^2 s}\right] \leq \frac{C}{s\gamma(s)}\ Q_\tau[u; C_{s,a^2 s}], \tag{7.10}$$

where we used Proposition 5.4 for the first inequality. The second one can be proved directly using that $e^{-\tau} = s$ and calculations from the proof of Lemma 2.3. All in all,

$$\varphi(s) \leq \frac{C}{s\gamma(s)}\ Q_\tau[u; C_{s,a^2 s}] \leq \frac{C}{s\gamma(s)}\left(\varphi(a^2 s) - \varphi(s)\right), \tag{7.11}$$

which, in turn, implies that

$$\varphi(s) \leq \frac{1}{1 + C^{-1}\, s\gamma(s)}\,\varphi(a^2 s) \leq \exp\left(-cs\gamma(s)\right)\,\varphi(a^2 s), \tag{7.12}$$

since $s\gamma(s)$ is bounded by (5.12). In particular, employing (7.12) for $s = a^{-2j}r$, $r \leq R$, $j \in \mathbb{N}$, one can conclude that

$$\varphi(a^{-2l}r) \leq \exp\left(-c\sum_{j=1}^{l} a^{-2j}r\,\gamma(a^{-2j}r)\right)\varphi(r), \tag{7.13}$$

for all $l \in \mathbb{N}$.

Let us choose $l \in \mathbb{N}$ so that

$$a^{-2l-4}R \leq |x| \leq a^{-2l-2}R. \tag{7.14}$$

Next, observe that by (5.4)

$$\begin{aligned}
&\int_{a^2|x|}^{R/a^2} \operatorname{Cap}\left(\overline{C_{s,as}} \setminus \Omega, \mathbb{R}^3 \setminus \{O\}\right) ds \\
&\leq \sum_{j=1}^{l}\sum_{k=1}^{2}\int_{a^{-2j+k-3}\,R}^{a^{-2j+k-2}\,R} \operatorname{Cap}\left(\overline{C_{s,as}} \setminus \Omega, \mathbb{R}^3 \setminus \{O\}\right) ds \\
&\leq C\sum_{j=1}^{l}\sum_{k=1}^{2} a^{-2j+k-3}\,R\operatorname{Cap}\left(\overline{C_{a^{-2j+k-3}\,R,a^{-2j+k-1}\,R}} \setminus \Omega, \mathbb{R}^3 \setminus \{O\}\right) \\
&\leq C\sum_{j=1}^{l}\sum_{k=1}^{2} a^{-2j}r_k\,\gamma(a^{-2j}r_k),
\end{aligned} \tag{7.15}$$

where $r_k = a^{k-3} R$, $k = 1, 2$. Using (7.13) with $r = r_k$, we deduce that for every $x \in B_{R/a^4} \cap \Omega$ and l defined by (7.14)

$$|\nabla u(x)|^2 + \frac{|u(x)|^2}{|x|^2} \leq \varphi(a^{-2l} r_k) \leq \exp\left(-c \sum_{j=1}^{l} a^{-2j} r_k \, \gamma(a^{-2j} r_k)\right) \varphi(R/a), \quad k = 1, 2, \tag{7.16}$$

which implies

$$\begin{aligned} |\nabla u(x)|^2 + \frac{|u(x)|^2}{|x|^2} &\leq \exp\left(-c \sum_{j=1}^{l} \sum_{k=1}^{l} a^{-2j} r_k \, \gamma(a^{-2j} r_k)\right) \varphi(R/a) \\ &\leq \exp\left(-c \int_{a^2|x|}^{R/a^2} \operatorname{Cap}\,(\overline{C_{s,as}} \setminus \Omega, \mathbb{R}^3 \setminus \{O\})\, ds\right) \varphi(R/a), \end{aligned} \tag{7.17}$$

by (7.15).

Finally, analogously to (7.7)–(7.9)

$$\begin{aligned} \varphi(R/a) \quad &\leq \quad \sup_{|x| \leq R/a} \left(|\nabla u(x)|^2 + \frac{|u(x)|^2}{|x|^2}\right) + C \sum_{k=0}^{2} \int_{C_{R/a,2R/a}} \frac{|\nabla^k u(x)|^2}{|x|^{5-2k}}\, dy \\ &\leq \quad \frac{C}{R^5} \int_{C_{R/a,16R/a}} |u(y)|^2\, dy \leq \frac{C}{R^5} \int_{C_{R,4R}} |u(y)|^2\, dy, \end{aligned} \tag{7.18}$$

using Proposition 3.2 for the last inequality. Combining (7.17) and (7.18), we finish the proof of (7.2). The statement (7.3) is a direct consequence of (7.2) and the definition of the 1-regularity. $\square$

Given the result of Proposition 7.1, we can derive the estimates for biharmonic functions at infinity as well as those for Green's function in terms of the capacity of the complement of Ω, in the spirit of (7.2).

Proposition 7.2 *Let Ω be a bounded domain in $\mathbb{R}^3$, $O \in \mathbb{R}^3 \setminus \Omega$, $r > 0$ and assume that*

$$\Delta^2 u = f \text{ in } \Omega, \quad f \in C_0^\infty(B_{r/4} \cap \Omega), \quad u \in \mathring{W}_2^2(\Omega). \tag{7.19}$$

Fix some $a \geq 4$. Then for any $x \in \Omega \setminus B_{a^4 r}$

$$\begin{aligned} |\nabla u(x)|^2 + \frac{|u(x)|^2}{|x|^2} \quad &\leq \quad \frac{C}{|x|^2 \, r^3} \int_{C_{\frac{r}{4},r} \cap \Omega} |u(y)|^2\, dy \\ &\qquad \times \exp\left(-c \int_{a^2 r}^{|x|/a^2} \operatorname{Cap}\,(\overline{C_{s,as}} \setminus \Omega, \mathbb{R}^3 \setminus \{O\})\, ds\right). \end{aligned} \tag{7.20}$$

Proof. Recall the proof of Proposition 3.4. With the notation (3.26) the results (3.27)–(3.29), (3.32) allow to apply Proposition 7.1 to U_n, $R = 1/r$, in order to write

$$\begin{aligned}
|\nabla u_n(x)|^2 + \frac{|u_n(x)|^2}{|x|^2} &\leq C\, \frac{\big|(\nabla U_n)(x/|x|^2)\big|^2}{|x|^2} + \big|U_n(x/|x|^2)\big|^2 \\
&\leq C\, \frac{r^5}{|x|^2} \int_{C_{\frac{1}{r},\frac{4}{r}}} |U_n(z)|^2\, dz \;\times \exp\left(-c \int_{a^2/|x|}^{1/(a^2 r)} \mathrm{Cap}\,(\overline{C_{s,as}} \setminus \mathcal{I}(\Omega_n), \mathbb{R}^3 \setminus \{O\})\, ds \right) \\
&\leq \frac{C}{|x|^2\, r^3} \int_{C_{\frac{r}{4},r}} |u_n(z)|^2\, dz \;\times \exp\left(-c \int_{a^2 r}^{|x|/a^2} \mathrm{Cap}\,(\overline{C_{\frac{1}{s},\frac{a}{s}}} \setminus \mathcal{I}(\Omega_n), \mathbb{R}^3 \setminus \{O\})\, \frac{ds}{s^2} \right).
\end{aligned}$$

We claim that

$$\mathrm{Cap}\,(\overline{C_{\frac{1}{s},\frac{a}{s}}} \setminus \mathcal{I}(\Omega_n), \mathbb{R}^3 \setminus \{O\}) \approx s^2\, \mathrm{Cap}\,(\overline{C_{s,as}} \setminus \Omega_n, \mathbb{R}^3 \setminus \{O\}), \tag{7.21}$$

where the implicit constants are independent of s.

Indeed,

$$\mathrm{Cap}\,(\overline{C_{\frac{1}{s},\frac{a}{s}}} \setminus \mathcal{I}(\Omega_n), \mathbb{R}^3 \setminus \{O\}) \approx \mathrm{Cap}\,(\overline{C_{\frac{1}{s},\frac{a}{s}}} \setminus \mathcal{I}(\Omega_n), C_{\frac{1}{2s},\frac{2a}{s}}), \tag{7.22}$$

and for every $u \in \mathring{W}_2^2(C_{\frac{1}{2s},\frac{2a}{s}})$ the function $y \mapsto |y|\, u(y/|y|^2)$ belongs to $\mathring{W}_2^2(C_{\frac{s}{2a},2s})$ by (3.29) and therefore, if $U(y) := u(y/|y|^2)$ then $U \in \mathring{W}_2^2(C_{\frac{s}{2a},2s})$. In addition, if $u = P$ in a neighborhood of $\overline{C_{\frac{1}{s},\frac{a}{s}}} \setminus \mathcal{I}(\Omega_n)$ then $U(y) = P(y/|y|^2) = P(y)$ for all y in the corresponding neighborhood of $\overline{C_{\frac{s}{a},s}} \setminus \Omega_n$. Finally, by (3.28)

$$\int_{C_{\frac{1}{2s},\frac{2a}{s}}} |\Delta u(x)|^2\, dx = \int_{C_{\frac{s}{2a},2s}} |\Delta(|y|\, u(y/|y|^2))|^2\, dy \approx s^2 \int_{C_{\frac{s}{2a},2s}} |\Delta U(y)|^2\, dy, \tag{7.23}$$

since $u \in \mathring{W}_2^2(C_{\frac{1}{2s},\frac{2a}{s}})$. This proves the "$\geq$" inequality in (7.21). The opposite inequality reduces to the previous one taking $1/s$ in place of s and $\mathcal{I}(\Omega_n)$ in place of Ω_n, since $\mathcal{I}(\mathcal{I}(\Omega_n)) = \Omega_n$.

As a result, we have

$$\begin{aligned}
&|\nabla u_n(x)|^2 + \frac{|u_n(x)|^2}{|x|^2} \\
&\quad\leq \frac{C}{|x|^2\, r^3} \int_{C_{\frac{r}{4},r}} |u_n(z)|^2\, dz \;\times \exp\left(-c \int_{a^2 r}^{|x|/a^2} \mathrm{Cap}\,(\overline{C_{s,as}} \setminus \Omega_n, \mathbb{R}^3 \setminus \{O\})\, ds \right) \\
&\quad\leq \frac{C}{|x|^2\, r^3} \int_{C_{\frac{r}{4},r}} |u_n(z)|^2\, dz \;\times \exp\left(-c \int_{a^2 r}^{|x|/a^2} \mathrm{Cap}\,(\overline{C_{s,as}} \setminus \Omega, \mathbb{R}^3 \setminus \{O\})\, ds \right),
\end{aligned} \tag{7.24}$$

using the monotonicity property (5.4). Now the argument can be finished using the limiting procedure similar to the one in Proposition 3.4. □

The following Proposition is a more precise version of the estimate on Green's function we announced in the introduction after Theorem 1.2.

Proposition 7.3 *Let Ω be a bounded domain in $\mathbb{R}^3$, $O \in \partial\Omega$. Fix some $a \geq 4$ and let $c_a := 1/(32a^4)$. Then for $x, y \in \Omega$*

$$|\nabla_x \nabla_y G(x,y)|$$

$$\leq \begin{cases} \frac{C}{|x-y|} \times \exp\Big(-c\int_{32a^2|y|}^{|x|/a^2} \operatorname{Cap}(\overline{C_{s,as}} \setminus \Omega, \mathbb{R}^3 \setminus \{O\})\, ds\Big), & \text{if } \ |y| \leq c_a|x|, \\ \frac{C}{|x-y|} \times \exp\Big(-c\int_{32a^2|x|}^{|y|/a^2} \operatorname{Cap}(\overline{C_{s,as}} \setminus \Omega, \mathbb{R}^3 \setminus \{O\})\, ds\Big), & \text{if } \ |x| \leq c_a|y|, \\ \frac{C}{|x-y|}, & \text{if } \ c_a|y| \leq |x| \leq c_a^{-1}|y|, \end{cases}$$

and

$$\max\Big\{|\nabla_x G(x,y)|,\, |\nabla_y G(x,y)|\Big\}$$

$$\leq \begin{cases} C\, \exp\Big(-c\int_{32a^2|y|}^{|x|/a^2} \operatorname{Cap}(\overline{C_{s,as}} \setminus \Omega, \mathbb{R}^3 \setminus \{O\})\, ds\Big), & \text{if } \ |y| \leq c_a|x|, \\ C\, \exp\Big(-c\int_{32a^2|x|}^{|y|/a^2} \operatorname{Cap}(\overline{C_{s,as}} \setminus \Omega, \mathbb{R}^3 \setminus \{O\})\, ds\Big), & \text{if } \ |x| \leq c_a|y|, \\ C, & \text{if } \ c_a|y| \leq |x| \leq c_a^{-1}|y|. \end{cases}$$

Proof. Let us focus first on the estimates for the second mixed derivatives of G. The estimate for the case $c_a|y| \leq |x| \leq c_a^{-1}|y|$ was proved in Theorem 1.1, and the bound for $|x| \leq c_a|y|$ follows from the one for $|y| \leq c_a|x|$ by the symmetry of Green's function. Hence, it is enough to consider the case $|y| \leq c_a|x|$ only.

The function $x \mapsto \nabla_y G(x,y)$ is biharmonic in $\Omega \setminus \{y\}$. We use Proposition 7.2 with $r = 32|y|$ to write

$$\begin{aligned} |\nabla_x \nabla_y G(x,y)|^2 \ &\leq \ \frac{C}{|x|^2\,|y|^3} \int_{C_{8|y|,32|y|}} |\nabla_y G(z,y)|^2\, dz \\ &\qquad \times \exp\left(-c\int_{32a^2|y|}^{|x|/a^2} \operatorname{Cap}(\overline{C_{s,as}} \setminus \Omega, \mathbb{R}^3 \setminus \{O\})\, ds\right), \end{aligned} \tag{7.25}$$

for $x \in \Omega \setminus B_{c_a^{-1}|y|}$. Recall now the function $\mathcal{R}$ introduced in the proof of Proposition 4.1. If y_0 is a point on $\partial\Omega$ such that $|y - y_0| = d(y)$, then

$$C_{8|y|,32|y|} \subset C_{6|y|,34|y|}(y_0), \tag{7.26}$$

and $\nabla_y G(z,y) = \mathcal{R}(z,y)$ for every $z \in C_{6|y|,34|y|}(y_0)$. Therefore,

$$\begin{aligned} \frac{1}{|x|^2\,|y|^3} \int_{C_{8|y|,32|y|}} |\nabla_y G(z,y)|^2\, dz &\leq \frac{1}{|x|^2\,|y|^3} \int_{C_{6|y|,34|y|}(y_0)} |\mathcal{R}(z,y)|^2\, dz \\ &\leq \frac{C}{|x|^2\, d(y)^3} \int_{C_{3d(y)/2,6d(y)}(y_0)} |\mathcal{R}(z,y)|^2\, dz \leq \frac{C}{|x|^2} \leq \frac{C}{|x-y|^2}. \end{aligned} \tag{7.27}$$

The second inequality above follows from Proposition 3.4, the third one has been proved in (4.13)–(4.14) and the last one owes to the observation that

$$|x - y| \leq |x| + |y| \leq (1 + c_a)|x| \quad \text{whenever} \quad |y| \leq c_a|x|. \tag{7.28}$$

Combining (7.25)–(7.27), we finish the proof of the bound for the second mixed derivatives of Green's function.

The proof of the estimate for $\nabla_y G$ follows a similar path, and then the estimate for $\nabla_x G$ is a consequence of the symmetry of Green's function. $\square$

Analogously to (4.26)–(4.31), Proposition 7.3 yields the following Corollary.

Corollary 7.4 *Suppose u satisfies*

$$\Delta^2 u = \operatorname{div} f + h, \quad u \in \mathring{W}_2^2(\Omega), \tag{7.29}$$

for some functions $f = (f_1, f_2, f_3)$ and h. Fix some $a \geq 4$ and let $c_a := 1/(32a^4)$. Then for any $x \in \Omega$

$$\begin{aligned}
&|\nabla u(x)| \\
&\leq C \int_{y\in\Omega:\, |y|\leq c_a|x|} \exp\left(-c\int_{32a^2|y|}^{|x|/a^2} \operatorname{Cap}\left(\overline{C_{s,as}} \setminus \Omega, \mathbb{R}^3 \setminus \{O\}\right) ds\right) \left(\frac{|f(y)|}{|x|} + |h(y)|\right) dy \\
&+ C \int_{y\in\Omega:\, |x|\leq c_a|y|} \exp\left(-c\int_{32a^2|x|}^{|y|/a^2} \operatorname{Cap}\left(\overline{C_{s,as}} \setminus \Omega, \mathbb{R}^3 \setminus \{O\}\right) ds\right) \left(\frac{|f(y)|}{|y|} + |h(y)|\right) dy \\
&+ C \int_{y\in\Omega:\, c_a|y|\leq|x|\leq c_a^{-1}|y|} \left(\frac{|f(y)|}{|x-y|} + |h(y)|\right) dy.
\end{aligned}$$

8 Necessary condition for 1-regularity

This section will be entirely devoted to the proof of the second part of Theorem 1.2, i.e. the necessary condition for 1-regularity. We recall that $\operatorname{Cap}_P(K) = \operatorname{Cap}_P(K, \mathbb{R}^3 \setminus \{0\})$ for any compactum K by definition, and begin with

Step I: setup. Suppose that for some $P \in \Pi_1$, $c > 0$, $a \geq 8$ the integral in (1.11) is convergent. Then for every $\varepsilon > 0$ there exists a number $N \in \mathbb{N}$ such that

$$\int_0^{2^{-N}} \operatorname{Cap}_P\left(\overline{C_{s,as}} \setminus \Omega, \mathbb{R}^3 \setminus \{O\}\right) ds < \varepsilon. \tag{8.1}$$

However, the integral above is bounded from below by

$$\begin{aligned}
\sum_{j=N}^{\infty} \int_{2^{-j-1}}^{2^{-j}} \operatorname{Cap}_P\left(\overline{C_{s,as}} \setminus \Omega, \mathbb{R}^3 \setminus \{O\}\right) ds &\geq C \sum_{j=N}^{\infty} 2^{-j} \operatorname{Cap}_P\left(\overline{C_{2^{-j},2^{-j-1}a}} \setminus \Omega, \mathbb{R}^3 \setminus \{O\}\right) \\
&\geq C \sum_{j=N}^{\infty} 2^{-j} \operatorname{Cap}_P\left(\overline{C_{2^{-j},2^{-j+2}}} \setminus \Omega, \mathbb{R}^3 \setminus \{O\}\right),
\end{aligned} \tag{8.2}$$

using the monotonicity property (5.4). Therefore, for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$\sum_{j=N}^{\infty} 2^{-j}\,\mathrm{Cap}_P\,(\overline{C_{2^{-j},2^{-j+2}}} \setminus \Omega, \mathbb{R}^3 \setminus \{O\}) < \varepsilon. \tag{8.3}$$

Now let $K := \overline{B_{2^{-N}}} \setminus \Omega$ and $D := \mathbb{R}^3 \setminus K$. We shall prove that the point O is not 1-regular with respect to D, and therefore with respect to Ω, since D coincides with Ω in a fixed neighborhood of O (see Proposition 6.1).

To this end, fix $P \in \Pi_1$ and let $\mathbb{P}(x) := |x|P(x)$, $x \in \mathbb{R}^3$. Then take some cut-off $\eta \in C_0^\infty(B_2)$ equal to 1 on $B_{3/2}$ and denote $f := -[\Delta^2, \eta]\mathbb{P} \in C_0^\infty(B_2 \setminus B_{3/2})$. Finally, let V be a solution of the boundary value problem

$$\Delta^2 V = f \text{ in } D, \quad V \in \mathring{W}_2^2(D). \tag{8.4}$$

Our goal is to show that $|\nabla V|$ does not vanish as $x \to O$, $x \in D$.

Let us also consider the function $U := V + \eta\,\mathbb{P}$. One can check that

$$\Delta^2 U = 0 \text{ in } D, \quad U = \mathbb{P} \text{ on } K, \quad U \in \mathring{W}_2^2(\mathbb{R}^3). \tag{8.5}$$

Therefore, U can be seen as a version of a biharmonic potential. In fact, it is (8.5) that gave an original idea for the above definition of V.

Step II: main identity. Let $\mathcal{B}$ denote the bilinear form associated to the quadratic form in (2.3), i.e.

$$\begin{aligned}\mathcal{B}(v,w) \;=\; & \int_{\mathbb{R}}\int_{S^2}\Big[(\delta_\omega v)(\delta_\omega w)\,\mathcal{G} + 2(\partial_t\nabla_\omega v)\cdot(\partial_t\nabla_\omega w)\,\mathcal{G} + (\partial_t^2 v)(\partial_t^2 w)\,\mathcal{G} \\ & -(\nabla_\omega v)\cdot(\nabla_\omega w)\Big(\partial_t^2\mathcal{G} + \partial_t\mathcal{G} + 2\mathcal{G}\Big) - (\partial_t v)(\partial_t w)\Big(2\partial_t^2\mathcal{G} + 3\partial_t\mathcal{G} - \mathcal{G}\Big) \\ & +\frac{1}{2}\,v\,w\Big(\partial_t^4\mathcal{G} + 2\partial_t^3\mathcal{G} - \partial_t^2\mathcal{G} - 2\partial_t\mathcal{G}\Big)\Big]\,d\omega dt. \end{aligned} \tag{8.6}$$

As before, we fix some point $\xi \in \mathbb{R}^3$, $\tau := \log|\xi|^{-1}$ and let $\mathcal{G}(t) = g(t-\tau)$, $t \in \mathbb{R}$. By $\mathcal{B}_\tau(v,w)$ we shall denote $\mathcal{B}(v,w)$ for this particular choice of $\mathcal{G}$. Then

$$\begin{aligned} & \int_{\mathbb{R}^3} \Delta U(x)\,\Delta\Big(\mathbb{P}(x)|x|^{-1}\,g(\log(|\xi|/|x|))\Big)\,dx \\ & \qquad + \int_{\mathbb{R}^3} \Delta\mathbb{P}(x)\,\Delta\Big(U(x)|x|^{-1}\,g(\log(|\xi|/|x|))\Big)\,dx = 2\mathcal{B}_\tau(u,q), \end{aligned} \tag{8.7}$$

where $u = e^t(U \circ \varkappa^{-1})$ and $q = e^t(\mathbb{P} \circ \varkappa^{-1}) = P \circ \varkappa^{-1}$.

The identity above can be proved along the lines of the argument for Lemma 2.1, as long as the integration by parts and absence of the boundary terms is justified. To this end, we note that for any fixed $\xi \in \mathbb{R}^3$ the function $x \mapsto g(\log(|\xi|/|x|))$ is bounded by a constant as $|x| \to \infty$, while $x \mapsto |x|^{-1}\,g(\log(|\xi|/|x|))$ is bounded by a constant as $x \to O$. If $v_s \in C_0^\infty(D)$, $s \in \mathbb{N}$, is a collection of functions approximating V in the $\mathring{W}_2^2(D)$-norm, we let $u_s := v_s + \eta\mathbb{P}$.

Then u_s converges to U in $\mathring{W}_2^2(\mathbb{R}^3)$. This, combined with the above observation about the behavior of g, shows that it suffices to prove (8.7) for u_s in place of U. Finally, since u_s is compactly supported in $\mathbb{R}^3$ and is equal to $\mathbb{P}$ in a neighborhood of 0, it is a matter of direct calculation to establish (8.7).

Since $(8\pi)^{-1}|x|$ is the fundamental solution of the bilaplacian,

$$\Delta^2\mathbb{P}(x) = \Delta^2(b_0|x| + b_1x_1 + b_2x_2 + b_3x_3) = (8\pi)^{-1}b_0\,\delta(x), \tag{8.8}$$

where δ is the Dirac delta function. Therefore, the second term on the left-hand side of (8.7) is equal (modulo a multiplicative constant) to $U(0) = 0$.

Going further, we show that

$$\int_{\mathbb{R}^3} \Delta U(x)\,\Delta\Big((U(x) - \mathbb{P}(x))|x|^{-1}\,g(\log(|\xi|/|x|))\Big)\,dx = 0. \tag{8.9}$$

Indeed, the expression in (8.9) is equal to

$$\int_{\mathbb{R}^3} \Delta U(x)\,\Delta\Big(V(x)|x|^{-1}\,g(\log(|\xi|/|x|))\Big)\,dx$$
$$+\int_{\mathbb{R}^3} \Delta U(x)\,\Delta\Big((\eta(x) - 1)\mathbb{P}(x)|x|^{-1}\,g(\log(|\xi|/|x|))\Big)\,dx. \tag{8.10}$$

Then, using the aforementioned approximation by v_s, $s \in \mathbb{N}$, in the first integral, an observation that $\operatorname{supp}(\eta - 1)\mathbb{P} \subset D$ in the second one, and the biharmonicity of U in D we arrive at (8.9).

Now the combination of (8.7)–(8.10) leads to the identity

$$\int_{\mathbb{R}^3} \Delta U(x)\,\Delta\Big(U(x)|x|^{-1}\,g(\log(|\xi|/|x|))\Big)\,dx = 2\mathcal{B}_\tau(u, q). \tag{8.11}$$

Finally, since the identity (2.3) holds for the function U, (8.11) implies that

$$\mathcal{B}_\tau(u, u) = 2\mathcal{B}_\tau(u, q). \tag{8.12}$$

Recall now that g is a fundamental solution of the equation (2.7), and therefore with the notation

$$\begin{aligned}\widetilde{\mathcal{B}}_\tau(v, w) = \int_{\mathbb{R}}\int_{S^2} \Big[&(\delta_\omega v)(\delta_\omega w)\,g(t - \tau) + 2(\partial_t\nabla_\omega v)\cdot(\partial_t\nabla_\omega w)\,g(t - \tau)\\ &+(\partial_t^2 v)(\partial_t^2 w)\,g(t - \tau) - (\nabla_\omega v)\cdot(\nabla_\omega w)\Big(\partial_t^2 g(t - \tau) + \partial_t g(t - \tau) + 2g(t - \tau)\Big)\\ &-(\partial_t v)(\partial_t w)\Big(2\partial_t^2 g(t - \tau) + 3\partial_t g(t - \tau) - g(t - \tau)\Big)\Big]\,d\omega dt,\end{aligned} \tag{8.13}$$

we have

$$\mathcal{B}_\tau(v, w) = \widetilde{\mathcal{B}}_\tau(v, w) + \frac{1}{2}\int_{S^2} v(\tau, \omega)w(\tau, \omega)\,d\omega. \tag{8.14}$$

Then the equality (8.12) can be written as

$$\int_{S^2}(u(\tau,\omega)-q(\tau,\omega))^2\,d\omega=\int_{S^2}q^2(\tau,\omega)\,d\omega+4\widetilde{\mathcal{B}}_\tau(u,q)-2\widetilde{\mathcal{B}}_\tau(u,u),\tag{8.15}$$

so that if $|\xi|<3/2$, $\tau=\log|\xi|^{-1}$,

$$\int_{S^2}v^2(\tau,\omega)\,d\omega=\int_{S^2}q^2(\tau,\omega)\,d\omega+4\widetilde{\mathcal{B}}_\tau(u,q)-2\widetilde{\mathcal{B}}_\tau(u,u),\tag{8.16}$$

where $v=e^t(V\circ\varkappa^{-1})$.

The identity (8.16) is our major starting point. We shall show that $\widetilde{\mathcal{B}}_\tau(u,q)$ and $\widetilde{\mathcal{B}}_\tau(u,u)$ can be estimated in terms of the series in (8.3) and hence, can be made arbitrarily small by shrinking ε in (8.3). On the other hand,

$$\int_{S^2}q^2(\tau,\omega)\,d\omega=\int_{S^2}\Big(b_0^2+\sum_{i=1}^3 b_i^2\omega_i^2\Big)\,d\omega=4\pi b_0^2+\frac{4\pi}{3}\sum_{i=1}^3 b_i^2,\tag{8.17}$$

so that

$$\frac{4\pi}{3}\le\int_{S^2}q^2(\tau,\omega)\,d\omega\le 4\pi.\tag{8.18}$$

Therefore, by (8.16),

$$\int_{S^2}v^2(\tau,\omega)\,d\omega=\frac{C}{|\xi|^4}\int_{S_{|\xi|}}V^2(\xi)\,d\sigma_\xi\tag{8.19}$$

does not vanish as $\xi\to O$, which means that ∇V does not vanish at O either, as desired. It remains to estimate $\widetilde{\mathcal{B}}_\tau(u,q)$ and $\widetilde{\mathcal{B}}_\tau(u,u)$.

Step III: estimate for $\widetilde{\mathcal{B}}_\tau(u,q)$. Since $q=P\circ\varkappa^{-1}$ is independent of t,

$$\begin{aligned}\widetilde{\mathcal{B}}_\tau(u,q)\;=\;&\int_{\mathbb{R}}\int_{S^2}\Big[(\delta_\omega u)(\delta_\omega q)\,g(t-\tau)\\&-(\nabla_\omega u)\cdot(\nabla_\omega q)\Big(\partial_t^2 g(t-\tau)+\partial_t g(t-\tau)+2g(t-\tau)\Big)\Big]\,d\omega dt.\end{aligned}\tag{8.20}$$

Next, $\delta_\omega\omega_i=-2\omega_i$ for $i=1,2,3$, and therefore $\delta_\omega q=-2\sum_{i=1}^3 b_i\omega_i$, so that

$$\begin{aligned}\widetilde{\mathcal{B}}_\tau(u,q)\;&=\;\int_{\mathbb{R}}\int_{S^2}\Big[2b_0\delta_\omega u\,g(t-\tau)-(\nabla_\omega u)\cdot(\nabla_\omega q)\Big(\partial_t^2 g(t-\tau)+\partial_t g(t-\tau)\Big)\Big]\,d\omega dt\\&=\;-\int_{\mathbb{R}}\int_{S^2}\Big[(\nabla_\omega u)\cdot(\nabla_\omega q)\Big(\partial_t^2 g(t-\tau)+\partial_t g(t-\tau)\Big)\Big]\,d\omega dt\\&\le\;\left(\int_{\mathbb{R}}\int_{S^2}\Big[|\nabla_\omega u|^2\Big(-\partial_t^2 g(t-\tau)-\partial_t g(t-\tau)\Big)\Big]\,d\omega dt\right)^{1/2}\\&\quad\times\left(\int_{S^2}|\nabla_\omega q|^2\,d\omega\int_{\mathbb{R}}\Big(-\partial_t^2 g(t-\tau)-\partial_t g(t-\tau)\Big)\,dt\right)^{1/2}=:I_1\times I_2,\end{aligned}\tag{8.21}$$

using the Cauchy-Schwarz inequality and the positivity of the weight function (see (2.17)).

Inspecting the argument of Lemma 2.3 one can see that

$$I_1 \leq (\widetilde{\mathcal{B}}_\tau(u,u))^{1/2}. \tag{8.22}$$

On the other hand,

$$\begin{aligned} I_2^2 &= \frac{8\pi}{3}\sum_{i=1}^{3} b_i^2 \int_{\mathbb{R}} \Big(-\partial_t^2 g(t-\tau) - \partial_t g(t-\tau)\Big)\,dt \\ &= \frac{8\pi}{9}\sum_{i=1}^{3} b_i^2 \left(\int_{-\infty}^{\tau} e^{t-\tau}\,dt + \int_{\tau}^{\infty} e^{-2(t-\tau)}\,dt\right) = \frac{4\pi}{9}\sum_{i=1}^{3} b_i^2 \leq \frac{4\pi}{9}. \end{aligned} \tag{8.23}$$

Therefore,

$$\widetilde{\mathcal{B}}_\tau(u,q) \leq \frac{2\sqrt{\pi}}{3}(\widetilde{\mathcal{B}}_\tau(u,u))^{1/2}. \tag{8.24}$$

Step IV: estimate for $\widetilde{\mathcal{B}}_\tau(u,u)$, the setup. Let us now focus on the estimate for $\widetilde{\mathcal{B}}_\tau(u,u)$. To this end, consider the covering of $K = \overline{B_{2^{-N}}} \setminus \Omega$ by the sets $K \cap C_{2^{-j},2^{-j+2}}$, $j \geq N$, and observe that

$$K \cap \overline{C_{2^{-j},2^{-j+2}}} = \overline{C_{2^{-j},2^{-j+2}}} \setminus \Omega, \quad j \geq N+2, \tag{8.25}$$

$$K \cap \overline{C_{2^{-j},2^{-j+2}}} \subseteq \overline{C_{2^{-j},2^{-j+2}}} \setminus \Omega, \quad j = N, N+1. \tag{8.26}$$

Let $\{\eta^j\}_{j=N}^{\infty}$ be the corresponding partition of unity such that

$$\eta^j \in C_0^\infty(C_{2^{-j},2^{-j+2}}), \quad |\nabla^k \eta^j| \leq C 2^{kj}, \quad k = 0,1,2, \quad \text{and} \quad \sum_{j=N}^{\infty} \eta^j = 1. \tag{8.27}$$

By U^j we denote the capacitary potential of $K \cap \overline{C_{2^{-j},2^{-j+2}}}$ with the boundary data P, i.e. the minimizer for the optimization problem

$$\begin{aligned} \inf \Bigg\{ \int_{C_{2^{-j-2},2^{-j+4}}} (\Delta u(x))^2\,dx : \ u \in \mathring{W}_2^2(C_{2^{-j-2},2^{-j+4}}), \\ u = P \text{ in a neighborhood of } K \cap \overline{C_{2^{-j},2^{-j+2}}} \Bigg\}. \end{aligned} \tag{8.28}$$

Such U^j always exists and belongs to $\mathring{W}_2^2(C_{2^{-j-2},2^{-j+4}})$ since P is an infinitely differentiable function in a neighborhood of $K \cap \overline{C_{2^{-j},2^{-j+2}}}$. The infimum above is equal to

$$\mathrm{Cap}_P\{K \cap \overline{C_{2^{-j},2^{-j+2}}}, C_{2^{-j-2},2^{-j+4}}\} \approx \mathrm{Cap}_P\{K \cap \overline{C_{2^{-j},2^{-j+2}}}, \mathbb{R}^3 \setminus \{O\}\}. \tag{8.29}$$

Let us now define the function

$$T(x) := \sum_{j=N}^{\infty} |x| \eta^j(x) U^j(x), \quad x \in \mathbb{R}^3, \tag{8.30}$$

and let $\vartheta := e^t(T \circ \varkappa^{-1})$. Then by the Cauchy-Schwarz inequality

$$\widetilde{\mathcal{B}}_\tau(\vartheta, \vartheta) \leq C \sum_{k=0}^{2} \sum_{j=N}^{\infty} \int_{C_{2^{-j},2^{-j+2}}} \frac{|\nabla^k(U^j(x))|^2}{|x|^{3-2k}}\, dx. \tag{8.31}$$

Next, since $U^j \in \mathring{W}_2^2(C_{2^{-j-2},2^{-j+4}})$, the Hardy's inequality allows us to write

$$\begin{aligned}
\widetilde{\mathcal{B}}_\tau(\vartheta, \vartheta) &\leq C \sum_{j=N}^{\infty} 2^{-j} \int_{C_{2^{-j-2},2^{-j+4}}} |\nabla^2 U^j(x)|^2\, dx \leq C \sum_{j=N}^{\infty} 2^{-j} \int_{C_{2^{-j-2},2^{-j+4}}} |\Delta U^j(x)|^2\, dx \\
&\leq C \sum_{j=N}^{\infty} 2^{-j}\, \mathrm{Cap}_P\{K \cap \overline{C_{2^{-j},2^{-j+2}}}, \mathbb{R}^3 \setminus \{O\}\} \\
&\leq C \sum_{j=N}^{\infty} 2^{-j}\, \mathrm{Cap}_P\{\overline{C_{2^{-j},2^{-j+2}}} \setminus \Omega, \mathbb{R}^3 \setminus \{O\}\} < C\varepsilon, \qquad (8.32)
\end{aligned}$$

by (8.29), (8.25)–(8.26), the monotonicity property (5.4), and (8.3).

Having (8.32) at hand, we need to consider the difference $U - T$ in order to obtain the estimate for $\widetilde{\mathcal{B}}_\tau(u, u)$. Let us denote $W := U - T$, $w := e^t(W \circ \varkappa^{-1})$.

Step V: estimate for $\mathcal{B}_\tau(w, w)$. First of all, one can show that $W \in \mathring{W}_2^2(D)$. Indeed, both U and T belong to $\mathring{W}_2^2(\mathbb{R}^3)$. For U this was pointed out in (8.5), the statement about T can be proved along the lines of (8.31)–(8.32):

$$\begin{aligned}
\|T\|_{\mathring{W}_2^2(\mathbb{R}^3)} &\leq C \sum_{k=0}^{2} \sum_{j=N}^{\infty} 2^{j(4-2k)} \int_{C_{2^{-j},2^{-j+2}}} |\nabla^k(|x| U^j(x))|^2\, dx \\
&\leq C \sum_{j=N}^{\infty} 2^{-2j} \int_{C_{2^{-j-2},2^{-j+4}}} |\Delta U^j(x)|^2\, dx \\
&\leq C \sum_{j=N}^{\infty} 2^{-2j}\, \mathrm{Cap}_P\{\overline{C_{2^{-j},2^{-j+2}}} \setminus \Omega, \mathbb{R}^3 \setminus \{O\}\} < C\varepsilon. \qquad (8.33)
\end{aligned}$$

In addition to (8.33), we know that $U = \mathbb{P}$ on the boundary of K by definition, and $\eta^j U^j = U^j = P = \mathbb{P}/|x|$ on the boundary of $K \cap \overline{C_{2^{-j},2^{-j+2}}}$. Since by (8.30) the function W is given by $\sum_{j=N}^{\infty} \eta^j (U - |x| U^j)$ in a neighborhood of K, it vanishes on ∂K (in the sense of $\mathring{W}_2^2(D)$).

Furthermore, $\Delta^2 W = -\Delta^2 T$ in D by (8.5). Then, with the notation $w := e^t(W \circ \varkappa^{-1})$ we have the formula

$$\begin{aligned}
\mathcal{B}_\tau(w, w) &= \int_{\mathbb{R}^3} \Delta W(x)\, \Delta\Big(W(x)|x|^{-1}\, g(\log(|\xi|/|x|))\Big)\, dx \\
&= -\int_{\mathbb{R}^3} \Delta T(x)\, \Delta\Big(W(x)|x|^{-1}\, g(\log(|\xi|/|x|))\Big)\, dx. \qquad (8.34)
\end{aligned}$$

In what follows we will show that

$$-\int_{\mathbb{R}^3} \Delta T(x)\, \Delta\Big(W(x)|x|^{-1}\, g(\log(|\xi|/|x|))\Big)\, dx \le C\varepsilon^{1/2}(\mathcal{B}_\tau(w,w))^{1/2}. \tag{8.35}$$

Observe that according to (8.34) and (2.11) the expression on the left-hand side of (8.35) is positive. Next, analogously to (2.5),

$$\begin{aligned}
&-\int_{\mathbb{R}^3} \Delta T(x)\, \Delta\Big(W(x)|x|^{-1}\, g(\log(|\xi|/|x|))\Big)\, dx\\
&\quad = -\int_{\mathbb{R}}\int_{S^2} \Big(\partial_t^2\vartheta - 3\partial_t\vartheta + 2\vartheta + \delta_\omega\vartheta\Big)\Big(g(t-\tau)\,\delta_\omega w + g(t-\tau)\,\partial_t^2 w\\
&\qquad\qquad + (2\partial_t g(t-\tau) - g(t-\tau))\,\partial_t w + (\partial_t^2 g(t-\tau) - \partial_t g(t-\tau))\, w\Big)\, d\omega dt.
\end{aligned} \tag{8.36}$$

Now recall the formula for $-(2\partial_t^2 g + 3\partial_t g - g)$ from (2.18). It is evident that for any $D_t = \sum_{i=0}^4 \alpha_i \partial_t^i$, $\alpha_i \in \mathbb{R}$, we have

$$|D_t g| \le C(-2\partial_t^2 g - 3\partial_t g + g), \tag{8.37}$$

where C generally depends on D_t, i.e. on $\{\alpha_i\}_{i=0}^4$. Hence, for every such D_t

$$\begin{aligned}
&\int_{\mathbb{R}}\int_{S^2} (\partial_t w)^2\, |D_t g(t-\tau)|\, d\omega dt\\
&\quad \le -C\int_{\mathbb{R}}\int_{S^2} (\partial_t w)^2 (2\partial_t^2 g(t-\tau) + 3\partial_t g(t-\tau) - g(t-\tau))\, d\omega dt \le C\widetilde{\mathcal{B}}_\tau(w,w),
\end{aligned} \tag{8.38}$$

where the last inequality follows from the calculations in Lemma 2.3. Then, using (8.31)–(8.32), we have

$$\begin{aligned}
&\int_{\mathbb{R}}\int_{S^2} |\partial_t^k \nabla_\omega^i \vartheta|\, |\partial_t w|\, |D_t g(t-\tau)|\, d\omega dt\\
&\quad \le C\left(\int_{\mathbb{R}}\int_{S^2} |\partial_t^k \nabla_\omega^i \vartheta|^2\, |D_t g(t-\tau)|\, d\omega dt\right)^{1/2} (\widetilde{\mathcal{B}}_\tau(w,w))^{1/2}\\
&\quad \le C\sum_{j=0}^2 \left(\int_{\mathbb{R}^3} \frac{|\nabla^j T(x)|^2}{|x|^{5-2j}}\, dx\right)^{1/2} (\widetilde{\mathcal{B}}_\tau(w,w))^{1/2} \le C\varepsilon^{1/2}(\widetilde{\mathcal{B}}_\tau(w,w))^{1/2},
\end{aligned} \tag{8.39}$$

for $0 \le i+k \le 2$.

For similar reasons,

$$\int_{\mathbb{R}}\int_{S^2} |\partial_t^k \nabla_\omega^i \vartheta|\, |\partial_t^2 w|\, g(t-\tau)\, d\omega dt \le C\varepsilon^{1/2}(\widetilde{\mathcal{B}}_\tau(w,w))^{1/2}, \tag{8.40}$$

and

$$\int_{\mathbb{R}}\int_{S^2} |\partial_t^k \nabla_\omega^i \vartheta|\, |\partial_t \nabla_\omega w|\, g(t-\tau)\, d\omega dt \le C\varepsilon^{1/2}(\widetilde{\mathcal{B}}_\tau(w,w))^{1/2}, \tag{8.41}$$

for $0 \le i+k \le 2$.

Invoking (8.39)–(8.41) and integrating by parts, we see that the expression in (8.36) is bounded by

$$\begin{aligned}&\Big| \int_{\mathbb{R}} \int_{S^2} \Big(\delta_\omega \vartheta \delta_\omega w\, g(t-\tau) - \nabla_\omega \vartheta \cdot \nabla_\omega w\, (2\partial_t^2 g(t-\tau) + 2\partial_t g(t-\tau) + 2g(t-\tau)) \\ &\qquad + \vartheta w (\partial_t^4 g(t-\tau) + 2\partial_t^3 g(t-\tau) - \partial_t^2 g(t-\tau) - 2\partial_t g(t-\tau)) \Big)\, dt d\omega \Big| \\ &\qquad + C\varepsilon^{1/2} (\widetilde{\mathcal{B}}_\tau(w,w))^{1/2}. \end{aligned} \tag{8.42}$$

Also,

$$\begin{aligned}&\Big| \int_{\mathbb{R}} \int_{S^2} (\delta_\omega \vartheta \cdot \delta_\omega w - 2\nabla_\omega \vartheta \cdot \nabla_\omega w)\; g\, dt\, d\omega \Big| \\ &\le \left(\int_{\mathbb{R}} \int_{S^2} \big[(\delta_\omega \vartheta)^2 - 2(\nabla_\omega \vartheta)^2 \big]\; g\, dt\, d\omega \right)^{1/2} \left(\int_{\mathbb{R}} \int_{S^2} \big[(\delta_\omega w)^2 - 2(\nabla_\omega w)^2 \big]\; g\, dt\, d\omega \right)^{1/2} \\ &\le C\varepsilon^{1/2} (\widetilde{\mathcal{B}}_\tau(w,w))^{1/2}, \end{aligned} \tag{8.43}$$

using (2.12) and the Cauchy-Schwarz inequality for the bilinear form on the left-hand side of (8.43). In view of (8.43) and (2.7) the expression in (8.42) is controlled by

$$\begin{aligned}&\Big| \int_{\mathbb{R}} \int_{S^2} \nabla_\omega \vartheta \cdot \nabla_\omega w\, (-2\partial_t^2 g(t-\tau) - 2\partial_t g(t-\tau))\, dt d\omega \Big| \\ &\qquad + \frac{1}{2} \Big| \int_{S^2} \vartheta(\tau,\omega) w(\tau,\omega)\, d\omega \Big| + C\varepsilon^{1/2} (\widetilde{\mathcal{B}}_\tau(w,w))^{1/2} \\ &\le C\varepsilon^{1/2} (\widetilde{\mathcal{B}}_\tau(w,w))^{1/2} + \frac{1}{2} \left(\int_{S^2} \vartheta^2(\tau,\omega)\, d\omega \right)^{1/2} (\mathcal{B}_\tau(w,w))^{1/2}. \end{aligned} \tag{8.44}$$

Here we used the positivity of $-2\partial_t^2 g - 2\partial_t g$ (see (2.17)) and the argument similar to (8.38)–(8.39) to estimate the first term. The bound for the second one follows from the Cauchy-Schwarz inequality and (2.11).

Finally, we claim that

$$\int_{S^2} \vartheta^2(\tau,\omega)\, d\omega < C\varepsilon. \tag{8.45}$$

Indeed, by definition (8.45) is equal to

$$\begin{aligned}\frac{1}{|\xi|^4} \int_{S_{|\xi|}} T^2(\xi)\, d\sigma_\xi &\le C \sum_{j:\, 2^{-j} \le |\xi| \le 2^{-j+2}} \frac{1}{|\xi|^2} \int_{S_{|\xi|}} (U^j(\xi))^2\, d\sigma_\xi \\ &\le C \sum_{j:\, 2^{-j} \le |\xi| \le 2^{-j+2}} \int_{\mathbb{R}^3} \Delta\Big(|x| U^j(x)\Big) \Delta\Big(U^j(x)\, g(\log(|\xi|/|x|))\Big)\, dx, \end{aligned} \tag{8.46}$$

using (2.11) for the function $x \mapsto |x|\, U^j(x)$ in $\mathring{W}_2^2(C_{2^{-j-2},2^{-j+4}})$. Finally, the right-hand side of (8.46) is bounded by

$$C \sum_{j:\, 2^{-j} \le |\xi| \le 2^{-j+2}} \sum_{k=0}^{2} \int_{C_{2^{-j},2^{-j+2}}} \frac{|\nabla^k(|x| U^j(x))|^2}{|x|^{5-2k}}\, dx < C\varepsilon, \tag{8.47}$$

by the estimate following (8.31). This completes the proof of (8.35), which together with (8.34) yields $\mathcal{B}_\tau(w,w) < \varepsilon$. and therefore,

$$\widetilde{\mathcal{B}}_\tau(w,w) < \mathcal{B}_\tau(w,w) < C\varepsilon. \tag{8.48}$$

The last estimate, in turn, implies that $\mathcal{B}_\tau(u,u) < C\varepsilon$ owing to the results of Step IV. At last, the combination with (8.24) finishes the argument. □

9 Examples and further properties of Cap_P and Cap.

Lemma 9.1 *Consider a domain Ω shaped as an exterior of a cusp in some neighborhood of $O \in \partial\Omega$, i.e.*

$$\Omega \cap B_c = \{(r,\theta,\phi) : 0 < r < c,\, \theta > h(r)\}, \quad \textit{for some} \quad c > 0, \tag{9.1}$$

where (r,θ,ϕ), $r \in (0,c)$, $\theta \in [0,\pi]$, $\phi \in [0,2\pi)$, are spherical coordinates in $\mathbb{R}^3$ and $h(r) : (0,c) \to \mathbb{R}$ is a nondecreasing function satisfying the condition $h(br) \le Ch(r)$ for some $b > 1$ and all $r \in (0,c)$.

Then

$$O \textit{ is 1-regular} \quad \textit{if and only if} \quad \int_0^c s^{-1} h(s)^2\, ds = +\infty. \tag{9.2}$$

Proof. We claim that for every $P \in \Pi_1$ and every $a \ge 4$

$$\mathrm{Cap}_P\,(\overline{C_{s,as}} \setminus \Omega, \mathbb{R}^3 \setminus \{O\}) \ge Cs^{-1}h(s)^2, \qquad 0 < s < c/a. \tag{9.3}$$

Indeed, recall from Lemma 5.2 that

$$\mathrm{Cap}_P\,(\overline{C_{s,as}} \setminus \Omega, \mathbb{R}^3 \setminus \{O\}) \approx \mathrm{Cap}_P\,(\overline{C_{s,as}} \setminus \Omega, C_{s/2,2as}). \tag{9.4}$$

By definition of the capacity Cap_P, for every $\varepsilon > 0$ there exists some $u \in \mathring{W}_2^2(C_{s/2,2as})$ such that

$$\mathrm{Cap}_P\,(\overline{C_{s,as}} \setminus \Omega, C_{s/2,2as}) + \varepsilon \ge C \int_{C_{s/2,2as}} (\Delta u(x))^2\, dx, \tag{9.5}$$

and $u = P$ in a neighborhood of $\overline{C_{s,as}} \setminus \Omega$. Since $u \in \mathring{W}_2^2(C_{s/2,2as})$, by Hardy's inequality

$$\begin{aligned}
\int_{C_{s/2,2as}} (\Delta u(x))^2\, dx &= \int_{C_{s/2,2as}} |\nabla^2 u(x)|^2\, dx \\
&\ge C \int_{C_{s/2,2as}} \left(\frac{|u(x)|^2}{|x|^4} + \frac{|\nabla u(x)|^2}{|x|^2} \right) dx \ge C \int_{C_{s,as}\setminus\Omega} \left(\frac{|P(x)|^2}{|x|^4} + \frac{|\nabla P(x)|^2}{|x|^2} \right) dx \\
&\ge C \int_{C_{s,as}\setminus\Omega} \left(\frac{|P(x)|^2}{|x|^4} + \frac{|\nabla\left(|x|\, P(x)\right)|^2}{|x|^4} \right) dx.
\end{aligned} \tag{9.6}$$

The contribution from $\frac{|P(x)|^2}{|x|^4}$ amounts to

$$\begin{aligned}
&\frac{C}{s^4}\int_s^{as}\int_0^{h(r)}\int_0^{2\pi}\left(b_0+b_1\sin\theta\cos\phi+b_2\sin\theta\sin\phi+b_3\cos\theta\right)^2\ sin\theta\, r^2\,d\phi d\theta dr\\
&\quad\geq \frac{C}{s}\int_0^{h(s)}\left(b_0^2+b_3^2\cos^2\theta+2b_0b_3\cos\theta\right)\ sin\theta\,d\theta\\
&\quad\geq \frac{C}{s}\left(\cos\theta\Big(b_0^2+\frac{b_3^2}{3}\cos^2\theta+b_0b_3\cos\theta\Big)\right)\Bigg|_{h(s)}^0\\
&\quad\geq \frac{C}{s}\left(\cos\theta\Big(\frac{1}{4}\,b_0^2+\Big(\frac{\sqrt{3}}{2}\,b_0-\frac{\cos\theta}{\sqrt{3}}\,b_3\Big)^2\Big)\right)\Bigg|_{h(s)}^0\geq\frac{C}{s}\,b_0^2\cos\theta\Bigg|_{h(s)}^0\geq\frac{C}{s}\,b_0^2h(s)^2. \quad (9.7)
\end{aligned}$$

On the other hand,

$$|\nabla(|x|\,P(x))|^2=\sum_{i=1}^3\left(b_0\frac{x_i}{|x|}+b_i\right)^2 \tag{9.8}$$

and for every $i=1,2,3$

$$\left(b_0\frac{x_i}{|x|}+b_i\right)^2+b_0^2\approx b_i^2+b_0^2. \tag{9.9}$$

Hence,

$$\int_{C_{s,as}\setminus\Omega}\left(\frac{|P(x)|^2}{|x|^4}+\frac{|\nabla\left(|x|\,P(x)\right)|^2}{|x|^4}\right)dx\geq\frac{C}{s}\,h(s)^2\sum_{i=0}^3 b_i^2\geq\frac{C}{s}\,h(s)^2. \tag{9.10}$$

Now one can combine (9.5), (9.6), (9.10) and let $\varepsilon\to 0$ to obtain (9.3).

Therefore, the divergence of the integral in (9.2) implies that

$$\int_0^{c/a}\inf_{P\in\Pi_1}\mathrm{Cap}_P\left(\overline{C_{s,as}}\setminus\Omega,\mathbb{R}^3\setminus\{O\}\right)ds=+\infty, \tag{9.11}$$

which, in turn, shows that the point O is 1-regular by Theorem 1.2.

Conversely, we claim that there exists $P\in\Pi_1$ such that for every $s\in(0,c/a)$

$$\mathrm{Cap}_P\left(\overline{C_{s,as}}\setminus\Omega,\mathbb{R}^3\setminus\{O\}\right)\leq Cs^{-1}h(s)^2. \tag{9.12}$$

Indeed, let us take

$$P(x):=\frac{1}{2}\Big(1-\frac{x_3}{|x|}\Big),\qquad x\in\mathbb{R}^3. \tag{9.13}$$

Clearly, $P\in\Pi_1$. Next, we choose a function $U\in\mathring{W}_2^2(C_{s/2,2as})$ that is given by P in a neighborhood of $C_{s,as}\setminus\Omega$. To do this, let us introduce two cut-off functions, ζ^θ and ζ^r, such that

$$\zeta^\theta\in C_0^\infty\left(-1/2,2\right),\ \zeta^\theta=1\text{ on }[0,3/2];\quad \zeta^r\in C_0^\infty\left(1/2,2a\right),\ \zeta^r=1\text{ on }[3/4,3a/2]. \tag{9.14}$$

Then let

$$u(r,\phi,\theta) := \frac{1}{2}(1-\cos\theta)\,\zeta^\theta\Big(\frac{\theta}{h(as)}\Big)\,\zeta^r\Big(\frac{r}{s}\Big), \tag{9.15}$$

so that

$$u(r,\phi,\theta) = 1 \quad \text{whenever} \quad 0\le\theta\le\frac{3h(as)}{2} \quad \text{and} \quad \frac{3s}{4}\le r\le\frac{3as}{2}, \tag{9.16}$$

and

$$u(r,\phi,\theta) = 0 \quad \text{whenever} \quad 2h(as)\le\theta\le\pi \quad \text{or} \quad r\notin\Big(\frac{s}{2}, 2as\Big). \tag{9.17}$$

Finally, let $U := u\circ\kappa$, where κ is the change of coordinates in (2.2). Then

$$\int_{C_{s/2,2as}} |\Delta U(x)|^2\,dx = C\int_{s/2}^{2as}\int_0^{2h(as)} \left|\frac{1}{r^2}\,\partial_r(r^2\partial_r u) + \frac{1}{r^2\sin\theta}\,\partial_\theta(\sin\theta\,\partial_\theta u)\right|^2 \sin\theta\,d\theta\,r^2\,dr,$$

since u is independent of ϕ. A straightforward calculation shows that for r and θ as above

$$\left|\frac{1}{r^2}\,\partial_r(r^2\partial_r u) + \frac{1}{r^2\sin\theta}\,\partial_\theta(\sin\theta\,\partial_\theta u)\right| \le \frac{C}{s^2}, \tag{9.18}$$

and therefore,

$$\int_{C_{s/2,2as}} |\Delta U(x)|^2\,dx \le Cs^{-1}h(as)^2 \le Cs^{-1}h(s)^2. \tag{9.19}$$

If $a\le b$, the last inequality follows from the fact that h is nondecreasing. If $a>b$, we have $h(ar)\le C^m h\Big(ab^{-m}r\Big)\le C^{m+1}h(r)$ for $m\ge\log_b a-1$.

Finally, if the point O is 1-regular, then by Theorem 1.2 the integral in (1.11) diverges for every $P\in\Pi_1$ and therefore, by (9.12), the integral in (9.2) diverges. $\square$

In order to state the next result, let us recall one of the definitions of the harmonic capacity of a compact set. For an open set $\Omega\subset\mathbb{R}^3\setminus\{O\}$ and a compactum $e\subset\Omega$

$$\operatorname{cap}(e,\Omega) := \inf\Big\{\int_\Omega (\nabla u(x))^2\,dx:\ u\in \mathring{W}_2^1(\Omega),\ u=1 \text{ in a neighborhood of } e\Big\}, \tag{9.20}$$

is a harmonic capacity of the set e relative to Ω. If $\Omega=\mathbb{R}^3\setminus\{0\}$ then (9.20) coincides with (1.6).

Lemma 9.2 *Let K be a compactum situated on the set*

$$\{x\in\mathbb{R}^3:\ b_0|x| + b_1x_1 + b_2x_2 + b_3x_3 = 0\}, \quad b_i\in\mathbb{R},\ i=0,1,2,3, \tag{9.21}$$

such that $O\notin K$. If the harmonic capacity of K equals zero, then

$$\operatorname{Cap}_P(K,\mathbb{R}^3\setminus\{0\}) = 0 \tag{9.22}$$

for

$$P(x) = \frac{1}{\sqrt{b_0^2+b_1^2+b_2^2+b_3^2}}\Big(b_0 + b_1\frac{x_1}{|x|} + b_2\frac{x_2}{|x|} + b_3\frac{x_3}{|x|}\Big), \qquad x\in\mathbb{R}^3\setminus\{O\}. \tag{9.23}$$

In particular, $\operatorname{Cap}(K,\mathbb{R}^3\setminus\{0\}) = 0$.

Proof. By current assumptions, $O \notin K$. Therefore, there exist $s > 0$ and $a > 1$ such that $K \subset \overline{C_{s,as}}$. In the course of proof some constants will depend on s and a. That, however, does not influence the result.

Since

$$\operatorname{cap}(K, C_{s/2,2as}) \approx \operatorname{cap}(K, \mathbb{R}^3 \setminus \{0\}) = 0, \tag{9.24}$$

for every $\varepsilon > 0$ there exists a compactum K_ε with a smooth boundary contained in the set (9.21) and such that

$$K \subset K_\varepsilon \subset C_{s/2,2as} \quad \text{and} \quad \operatorname{cap}(K_\varepsilon, C_{s/2,2as}) < \varepsilon. \tag{9.25}$$

Let u denote the harmonic potential of K_ε, so that

$$u \in \mathring{W}_2^1(C_{s/2,2as}), \quad u = 1 \text{ in } K_\varepsilon, \quad \Delta u = 0 \text{ in } \mathbb{R}^3 \setminus K_\varepsilon, \quad \int_{C_{s/2,2as}} |\nabla u(x)|^2\, dx < \varepsilon. \tag{9.26}$$

Next, given $\alpha < 1$ let

$$v_\alpha(x) = \begin{cases} \alpha^{-4} P(x) u^2(x)(2\alpha - u(x))^2, & \text{if} \quad u(x) \le \alpha, \\ P(x), & \text{if} \quad u(x) > \alpha, \end{cases} \tag{9.27}$$

where $x \in C_{s/2,2as}$ and P is defined by (9.23). Then $v_\alpha \in \mathring{W}_2^2(C_{s/2,2as})$ by (9.26) and $v_\alpha = P$ in a neighborhood of K. Therefore,

$$\begin{aligned} \operatorname{Cap}_P(K, \mathbb{R}^3 \setminus \{0\}) &\approx \operatorname{Cap}_P(K, C_{s/2,2as}) \le \int_{C_{s/2,2as}} |\Delta v_\alpha(x)|^2\, dx \\ &= \alpha^{-8} \int_{x:\, u(x) \le \alpha} \left| \Delta \Big(P(x) u^2(x)(2\alpha - u(x))^2 \Big) \right|^2 dx + \int_{x:\, u(x) > \alpha} |\Delta P(x)|^2\, dx. \end{aligned} \tag{9.28}$$

We take $\alpha = \alpha(\varepsilon) < 1$ (close to 1) such that the last term above is less than ε. In addition, on the set $\{x :\ u(x) \le \alpha\}$

$$\begin{gathered} \left| \Delta \Big(u^2(x)(2\alpha - u(x))^2 \Big) \right| \le C |\nabla u|^2, \qquad \left| \nabla P \cdot \nabla \Big(u^2(x)(2\alpha - u(x))^2 \Big) \right| \le C |\nabla u|, \\ \left| \Delta P \Big(u^2(x)(2\alpha - u(x))^2 \Big) \right| \le C |u|, \end{gathered} \tag{9.29}$$

so that

$$\int_{x:\, u(x) \le \alpha} \left| \Delta \Big(P(x) u^2(x)(2\alpha - u(x))^2 \Big) \right|^2 dx \le C\varepsilon + C \int_{x:\, u(x) \le \alpha} |P(x)|^2 |\nabla u|^4\, dx, \tag{9.30}$$

by (9.26).

It remains to estimate the last integral above. Let us denote by $\{B_i\}_{i=1}^\infty$ a Whitney decomposition of the set $C_{s/2,2as} \setminus K_\varepsilon$, i.e. a collection of balls such that

$$\bigcup_{i=1}^\infty B_i = C_{s/2,2as} \setminus K_\varepsilon, \quad \sum_{i=1}^\infty \chi_{B_i} \le C, \quad r(B_i) \approx \operatorname{dist}\Big(B_i, \partial (C_{s/2,2as} \setminus K_\varepsilon) \Big), \tag{9.31}$$

where $r(B_i)$ denotes the radius of B_i. Observe that

$$|u(x)| \le 1, \quad |P(x)| \le Cr_i, \quad \text{if } x \in B_i \text{ such that } \operatorname{dist}(B_i, \partial C_{s/2,2as}) \ge \operatorname{dist}(B_i, K_\varepsilon),$$
$$|u(x)| \le Cr_i, \quad |P(x)| \le C, \quad \text{if } x \in B_i \text{ such that } \operatorname{dist}(B_i, \partial C_{s/2,2as}) \le \operatorname{dist}(B_i, K_\varepsilon).$$

Since u is harmonic in $C_{s/2,2as} \setminus K_\varepsilon$,

$$|\nabla u|^2 \le \frac{C}{r_i^5} \int_{B_i} |u(x)|^2\, dx. \tag{9.32}$$

Therefore, $|P||\nabla u| \le C$ on $C_{s/2,2as} \setminus K_\varepsilon$ and

$$\int_{C_{s/2,2as}} |P(x)|^2 |\nabla u|^4\, dx \le \int_{C_{s/2,2as}} |\nabla u|^2\, dx < \varepsilon. \tag{9.33}$$

Letting $\varepsilon \to 0$, we finish the argument. □

Corollary 9.3 *Let Ω be a domain in $\mathbb{R}^3$ such that $O \in \partial\Omega$ and the complement of Ω is a compactum of zero harmonic capacity situated on the set (9.21). Then the point O is not 1-regular.*

Proof. By Lemma 9.2 for the choice of P in (9.23)

$$\operatorname{Cap}_P(\overline{C_{s,as}} \setminus \Omega, \mathbb{R}^3 \setminus \{O\})\, ds = 0, \tag{9.34}$$

for every $s > 0$, $a > 1$. One can see that such P does not depend on s and a, but only on the initial cone containing the complement of Ω. Therefore,

$$\inf_{P \in \Pi_1} \int_0^c \operatorname{Cap}_P(\overline{C_{s,as}} \setminus \Omega, \mathbb{R}^3 \setminus \{O\})\, ds = 0, \tag{9.35}$$

and hence O is not 1-regular by Theorem 1.2. □

Remark. The set defined by (9.21) is either a circular cone with the vertex at O or a plane containing O. Indeed, the set (9.21) is formed by the rays originating at O and passing through the intersection of the plane $b_0 + b_1x_1 + b_2x_2 + b_3x_3 = 0$ with the unit sphere. If this plane passes through the origin ($b_0 = 0$), it is actually the set (9.21). If it does not, then its intersection with S^2 is a circle giving rise to the corresponding circular cone.

Due to the particular form of elements in the space Π_1 such sets play a special role for our concept of the capacity and for 1-regularity. This observation is, in particular, supported by Lemma 9.2 and the upcoming example.

We consider a domain whose complement consists of a set of points such that in each dyadic spherical layer three of the points belong to a fixed circular cone, while the fourth one does not. The result below shows that in this case the origin is 1-regular provided the deviation of the fourth point is large enough in a certain sense. The details are as follows.

Lemma 9.4 *Fix some $a \geq 4$. Consider a domain Ω such that in some neighborhood of the origin its complement consists of the set of points*

$$\bigcup_k \left\{ A_1^k = (a^{-k}, 0, \alpha),\ A_2^k = (a^{-k}, \pi/2, \alpha),\ A_3^k = (a^{-k}, \pi, \alpha),\ A_4^k = (a^{-k+1/2}, 3\pi/2, \beta_k) \right\}, \tag{9.36}$$

where the points are represented in spherical coordinates (r, ϕ, θ), $r \in (0, c)$ for some $c > 0$, $\theta \in [0, \pi]$, $\phi \in [0, 2\pi)$, $k \in \mathbb{N} \cap (1/2 - \log_a c, \infty)$. Assume, in addition, that

$$0 < \alpha < \pi/2, \quad 0 \leq |\beta_k - \alpha| < \alpha/2, \quad \forall\ k \in \mathbb{N} \cap (1/2 - \log_a c, \infty). \tag{9.37}$$

Then

$$\int_0^{c/a} \inf_{P \in \Pi_1} \mathrm{Cap}_P\, (\overline{C_{s,as}} \setminus \Omega, \mathbb{R}^3 \setminus \{O\})\, ds \geq C \sum_k (\beta_k - \alpha)^2, \tag{9.38}$$

where $C = C(\alpha) > 0$. In particular,

$$\text{if} \quad \sum_k (\beta_k - \alpha)^2 = +\infty \quad \text{then } O \text{ is 1-regular.} \tag{9.39}$$

Proof. To begin, let us observe that

$$\int_0^{c/a} \mathrm{Cap}\, (\overline{C_{s,as}} \setminus \Omega, \mathbb{R}^3 \setminus \{O\})\, ds \geq \sum_k \int_{a^{-k-1/2}}^{a^{-k}} \mathrm{Cap}\, (\overline{C_{s,as}} \setminus \Omega, \mathbb{R}^3 \setminus \{O\})\, ds$$

$$\geq \sum_k a^{-k} \min_{s \in (a^{-k-1/2}, a^{-k})} \mathrm{Cap}\, (\overline{C_{s,as}} \setminus \Omega, \mathbb{R}^3 \setminus \{O\}). \tag{9.40}$$

For every $s \in (a^{-k-1/2}, a^{-k})$ in the spherical layer $\overline{C_{s,as}}$ there are exactly four points that belong to the complement of Ω, namely, A_i^k, $i = 1, 2, 3, 4$. We aim to show that for each $k \geq 1/2 - \log_a c$

$$\mathrm{Cap}\, (\overline{C_{s,as}} \setminus \Omega, \mathbb{R}^3 \setminus \{O\}) \geq C a^k (\beta_k - \alpha)^2, \tag{9.41}$$

provided $s \in (a^{-k-1/2}, a^{-k})$.

Take some $P \in \Pi_1$ and consider the distribution

$$T^k(x) := \sum_{i=1}^4 P(A_i^k) \delta(x - A_i^k). \tag{9.42}$$

Then for every $u \in \mathring{W}_2^2(C_{s/2,2as})$ such that $u = P$ in a neighborhood of $\{A_i^k, i = 1, 2, 3, 4\}$, we have

$$\langle T^k, P \rangle = \sum_{i=1}^4 P(A_i^k)^2. \tag{9.43}$$

On the other hand, since T^k is supported in the set $\{A_i^k, i = 1, 2, 3, 4\}$,

$$\langle T^k, P \rangle = -\langle \Delta E * T^k, u \rangle = -\langle E * T^k, \Delta u \rangle, \tag{9.44}$$

where $E(x) = 1/(4\pi|x|)$ is the fundamental solution for the Laplacian. By the Cauchy-Schwarz inequality

$$\begin{aligned} |\langle T^k, P\rangle|^2 &\leq \|E * T^k\|^2_{L^2(C_{s/2,2as})}\|\Delta u\|^2_{L^2(C_{s/2,2as})} \\ &\leq Cs\sum_{i=1}^{4} P(A_i^k)^2 \operatorname{Cap}_P(\overline{C_{s,as}} \setminus \Omega, C_{s/2,2as}). \end{aligned} \tag{9.45}$$

Therefore, combining (9.43)–(9.45) and taking the infimum in P, we obtain the estimate

$$\operatorname{Cap}(\overline{C_{s,as}} \setminus \Omega, \mathbb{R}^3 \setminus \{O\}) \geq Ca^k \inf_{P\in\Pi_1} \sum_{i=1}^{4} P(A_i^k)^2 = Ca^k \inf_{b\in\mathbb{R}^4:\,\|b\|=1} b\, MM^\perp\, b^\perp, \tag{9.46}$$

where $b = (b_0, b_1, b_2, b_3)$,

$$M = \begin{pmatrix} 1 & 1 & 1 & 1 \\ \sin\alpha & 0 & -\sin\alpha & 0 \\ 0 & \sin\alpha & 0 & -\sin\beta_k \\ \cos\alpha & \cos\alpha & \cos\alpha & \cos\beta_k \end{pmatrix} \tag{9.47}$$

and the superindex $\perp$ denotes matrix transposition. Then the infimum in (9.46) is bounded from below by the smallest eigenvalue of $MM^\perp$. The characteristic equation of $MM^\perp$ is

$$\begin{aligned} &-\lambda^4 + 8\lambda^3 - \frac{1}{4}\Big(55 - 22\cos(2\alpha) - 3\cos(4\alpha) - 8\cos(\alpha-\beta_k) - \cos(2\alpha - 2\beta_k) - 2\cos(2\beta_k) \\ &-16\cos(\alpha+\beta_k) - 3\cos(2\alpha+2\beta_k)\Big)\lambda^2 - \frac{1}{2}\sin^2\alpha\Big(-4\cos(2\alpha) + \cos(4\alpha) + 12\cos(\alpha-\beta_k) \\ &-33 + \cos(2\alpha-2\beta_k) + 20\cos(\alpha+\beta_k) + 3\cos(2\alpha+2\beta_k)\Big)\lambda = 4\sin^2\alpha(\cos\alpha - \cos\beta_k)^2. \end{aligned}$$

By the Mean Value Theorem for the function arccos and our assumptions on α, β_k there exists $C_0(\alpha)$ independent of β_k such that for all k

$$|\alpha - \beta_k| \leq C_0(\alpha)|\cos\alpha - \cos\beta_k|, \tag{9.48}$$

and therefore,

$$4\sin^2\alpha(\cos\alpha - \cos\beta_k)^2 \geq 4\sin^2\alpha(C_0(\alpha))^{-2}|\alpha-\beta_k|^2. \tag{9.49}$$

It follows that

$$\lambda \geq \frac{\sin^2\alpha(C_0(\alpha))^{-2}}{100}|\alpha - \beta_k|^2, \tag{9.50}$$

because otherwise the left-hand side of (9.48) is strictly less than its right-hand side. Combined with (9.46), this finishes the proof of (9.38). The statement (9.39) follows from (9.38) and Theorem 1.2. □

Remark. Retain the conditions of Lemma 9.4. By our construction, for every $s \in (0, c/a^{1/3})$ in the spherical layer $\overline{C_{s,a^{1/3}s}}$ there are either

(i) exactly three points A_i^k, $i = 1, 2, 3$ for some $k = k(s)$,
(ii) or exactly one point A_4^k, $k = k(s)$,
(iii) or no points from the complement of Ω.

By Lemma 9.2 it follows that in either case

$$\mathrm{Cap}\,(\overline{C_{s,a^{1/3}s}} \setminus \Omega, \mathbb{R}^3 \setminus \{O\}) = 0 \tag{9.51}$$

and hence,

$$\int_0^{c/a^{1/3}} \inf_{P \in \Pi_1} \mathrm{Cap}_P\,(\overline{C_{s,a^{1/3}s}} \setminus \Omega, \mathbb{R}^3 \setminus \{O\})\, ds = 0. \tag{9.52}$$

At the same time, if $\sum_k (\alpha - \beta_k)^2$ diverges, then so does the integral in (9.38).

It follows that for the *same* domain Ω the convergence of the integral in (1.10) might depend on the choice of a.

Alternatively, one can say that for the same a the convergence of the integral in (1.10) might depend on the dilation of the domain. In particular, (1.10) *can not* be a necessary condition for the 1-regularity since the concept of 1-regularity is dilation invariant.

Conversely, our proof of the first statement in Theorem 1.2 and Proposition 7.1 relies on Proposition 5.4 which, in turn, follows from the Poincaré-type inequality (5.21). In fact, for every s our choice of P, that allows to estimate the infimum under the integral sign in (1.10), is dictated by the approximating constants in the Poincaré's inequality on (s, as) (see the proof of Lemma 5.3). Therefore, in our argument one can not make a uniform choice of P for all s to substitute (1.10) with (1.11).

Corollary 9.5 *The 1-irregularity may be unstable under the affine transformation of coordinates.*

Proof. The proof is based on Corollary 9.3 and Lemma 9.4. Indeed, given the assumptions of Lemma 9.4, if $\beta_k = \alpha$ for all k, then the complement of Ω is entirely contained in the circular cone of aperture α with the vertex at the origin and hence, by virtue of Corollary 9.3, the point O is not 1-regular.

However, if $\beta_k = \alpha + \varepsilon$ for all k, then the series in (9.39) diverges for arbitrary small $\varepsilon > 0$, which entails 1-regularity of O. □

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