Localization of eigenfunctions

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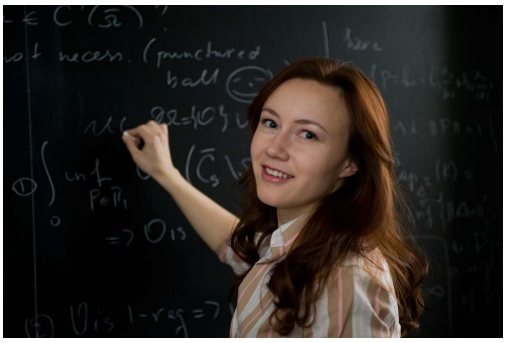

At the forefront of modern technology, matter structured at the nanometer and atomic levels increasingly reveals its wave-like nature. This provides new remarkable opportunities and challenges in analysis and partial differential equations. In recent years, pushing the limits of fundamental physics, cutting-edge experiments with ultracold atoms have created new states of matter and offer the possibility of controlling quantum entanglement, with major potential applications to cryptography and quantum computing. In engineering, electronic waves confined in quantum wells have yielded high efficiency LEDs, which are about to revolutionize the energetics of lighting, as recognized by the 2014 Nobel Prize. At these scales, even the slightest disorder or irregularity can trigger one the most puzzling and ill-understood phenomena – wave localization.

What is wave localization? It is an astonishing ability of physical systems to maintain vibrations in small portions of their original domains of activity while preventing extended propagation. One should not, in this context, think solely in terms of mechanical vibrations. Light is a particular example of an electromagnetic wave, wifi is delivered by waves, sound is a pressure wave, and, from the vantage point of quantum physics, even matter can be perceived as a type of wave. In mathematical terms, localized eigenfunctions φ of a self-adjoint elliptic operator $L = -\text{div}A\nabla + V$ in a domain Ω satisfy $L\varphi = \lambda\varphi$, where φ is extremely close to zero outside some small subset of Ω. This phenomenon can be triggered by irregularities of the coefficients of A, disorder in the potential V, special features of the shape of Ω, or an intricate mixture of all of the above. Over the past century this has been a source of persistent interest in condensed matter physics, engineering, and mathematics. However, it has remained a mystery whether it is possible to directly translate knowledge of A, V and Ω into the specific information on the location and frequencies of localized eigenfunctions, or better yet, to *design* systems with desired localization patterns.

In 2012, together with M. Filoche, we introduced a new concept of the "localization landscape" [FM]. This has turned out to have some remarkable and powerful features. Indeed, in a joint work with D. Arnold, G. David, M. Filoche, and D. Jerison, we show that the landscape function, which is defined as a solution to $Lu = 1$ on Ω, reveals a clear disjoint partition of Ω into independent regions, and this partition predicts localization domains with exponential accuracy, with the rate of decay governed by the so-called Agmon metric associated to $1/u$ [A+2016]. That is, given any system, one only needs to solve a simple equation $Lu = 1$ and determine the corresponding "valley lines" to obtain a map of subregions which host the localized eigenfunctions (see Figure 1). Furthermore, modulo this exponentially small error, the part of the spectrum of L on Ω not exceeding the maximum of $1/u$ can be fully diagonalized, that is, bijectively mapped on the combined spectrum of the subregions determined by the landscape. In particular, an eigenfunction can be massive simultaneously in two landscape subregions if and only if the eigenvalues of these subregions are exponentially close (a fairly unlikely event). In this sense, one could view $1/u$ as

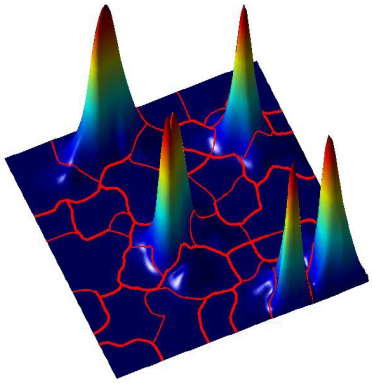

Figure 1. A network of valleys of a localization landscape (in red) and the first five localized eigenfunctions for the Schrödinger operator with disordered potential.

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an *effective* quantum potential which exhibits clear structure even when V is highly disordered or when V is absent and the localization is caused by the geometry and/or by coefficients of A. In other words, $1/u$ suitably quantifies the *uncertainty principle*.

Going further, the localization landscape furnishes a new version of the Weyl law. It appears to give the first universal estimate on the counting function and on the density of states for small eigenvalues, in the range where the classical Weyl law notoriously fails. In fact, numerical experiments show that already minima of $1/u$ give a very good approximation for the bottom of the spectrum, but these observations for now remain mathematically inaccessible.

Finally, in the intervening years, these results have found immediate applications in theoretical and experimental physics and in energy engineering, to predict the vibration of plates, the spectrum of the bilaplacian with Dirichlet data, the efficiency and quantum droop of GaN LEDs governed by the Poisson-Schrödinger self-consistent system, and the spectral properties of the Schrödinger operator with Anderson or Anderson-Bernoulli potentials in bounded domains.

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Figure 1: M. Filoche, S. Mayboroda, Universal mechanism for Anderson and weak localization, Proceedings of the National Academy of Sciences, (2012), 109 (37) 14761-14766, doi:10.1073/pnas.1120432109

References

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[FM] Filoche, M.; Mayboroda, S., *Universal mechanism for Anderson and weak localization*. Proc. Natl. Acad. Sci. USA 109 (2012), 14761–14766.