Strong Localization Induced by One Clamped Point in Thin Plate Vibrations

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We discover a strong localization of flexural (bi-Laplacian) waves in rigid thin plates. We show that clamping just one point inside such a plate not only perturbs its spectral properties, but essentially divides the plate into two independently vibrating regions. This effect progressively appears when increasing the plate eccentricity. Such a localization is qualitatively and quantitatively different from the results known for the Laplacian waves in domains of irregular boundary. It would allow us to control the confinement of mechanical vibrations in rigid plates and of eddies in the slow Stokes flow.

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It is well known that geometrical irregularities affect the vibrational properties of structures such as membranes or rigid plates. However, the phenomenon we report in the present Letter seems dramatically different from the weak localization of Laplacian waves observed previously in domains of irregular geometry [1]. It turns out that a very minor variation of the geometry (one interior clamped point in a rectangular plate) can almost completely annihilate *each* of the vibrational eigenmodes in a substantial portion of the plate. By way of comparison, a fixed point inside a rectangular membrane would not induce such a localization of Laplacian eigenmodes. Moreover, the localization of the stationary vibrations of a membrane observed so far has been associated with fairly substantial irregularities of the boundary (sawtooth or fractal) and affected only *some* of the eigenmodes [2].

Plate vibrations are governed by a biharmonic equation. Although a classical problem, the behavior of vibrations in rigid plates even of simple geometry is still generally poorly understood [3–7]. The mathematical model which describes the propagation of flexural deformations in thin plates with no curvature or strain is given by the following wave equation:

$$-\rho h \frac{\partial^2 u}{\partial t^2} = \kappa \Delta^2 u. \tag{1}$$

Here, h is the plate thickness (assumed to be much smaller than any other dimension of the plate), ρ is the material density, κ is the bending modulus of the elastic material [8], and the operator on the right-hand side is the bi-Laplacian ($\Delta^2 u = \Delta\Delta u$). The properties of the system can thus be entirely deduced from the spectral properties of the bi-Laplacian. We shall denote its eigenvalues and eigenfunctions by λ_i^2 and u_i, so that

$$\Delta^2 u_i = \lambda_i^2 u_i. \tag{2}$$

The boundary conditions are induced by clamping the edges of the plate, which amounts to requiring $u = 0$ and $\nabla u = 0$ along the external perimeter.

One should note that the biharmonic eigenvalue problem not only describes the vibrational properties of a rigid thin plate, but also the 2D Stokes flow of viscous fluids [3,9] and flow in porous media [10]. Over the last century it received considerable attention in engineering and applied mathematics. However, the known results are still scarce compared to the wealth of information available for the Laplacian. In particular, it turns out that the eigenfunctions of the biharmonic equation may behave dramatically differently from their counterparts corresponding to the case of the Laplacian. Here we discuss one such phenomenon.

In the present Letter we study the vibrational properties of a 2D clamped flat rectangular plate. In the following, the respective width and length of the plate are $1/\sqrt{e}$ and $\sqrt{e}$, where e is the plate eccentricity (its aspect ratio). All the plates have then unit area. Two types of plates are compared: those for which only the perimeter is clamped, and those with an additional point clamped inside. The point is located in the middle of a plate along the y direction and at $1/5$ of the length in the x direction (see Fig. 1). It has to be underlined that the precise location of the point has no qualitative impact on the occurrence of the phenomenon that is presented in this Letter, it may only affect the relevant values of the eccentricity. To compute the eigenmodes, the domain has been discretized, using Hermite rectangular finite elements with 16 degrees of freedom per element. This method ensures the continuity of both the solution and its derivative, and a good accuracy of the numerics. The eigenvalues and eigenmodes of the bi-Laplacian have been computed using MATLAB® software.

First of all, if one plots the amplitude distributions of several eigenmodes for an elongated plate without a clamped interior point (Fig. 2, left) and for the same plate with a clamped point inside (Fig. 2, right), one can effectively observe a drastic difference. In the former case, the

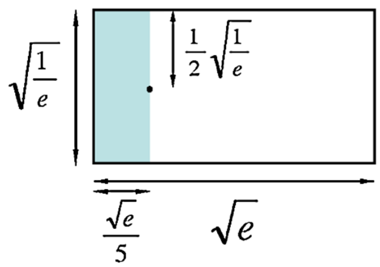

FIG. 1 (color online). Geometry of a clamped plate. The length and the width of a plate are, respectively, $\sqrt{e}$ and $1/\sqrt{e}$. Two types of plates are studied: those for which only the perimeter is clamped, and those with an additional interior clamped point. The coordinates of this point are $[\sqrt{e}/5,\ 1/(2\sqrt{e})]$. For further reference, two disjoint rectangular regions are introduced: region 1 (in gray) to the left of a clamped point, and region 2 (white) to the right of it.

amplitudes or intensities of all modes are distributed uniformly over the plate. In the latter, the modes appear to be almost completely confined either strictly within the region to the right of the clamped point or strictly within the region to the left of it. Because of space limitations we display only a few sample eigenmodes. However, this effect takes place for virtually all eigenmodes, with a few exceptions that also display localization, of a slightly different kind (see Fig. 3).

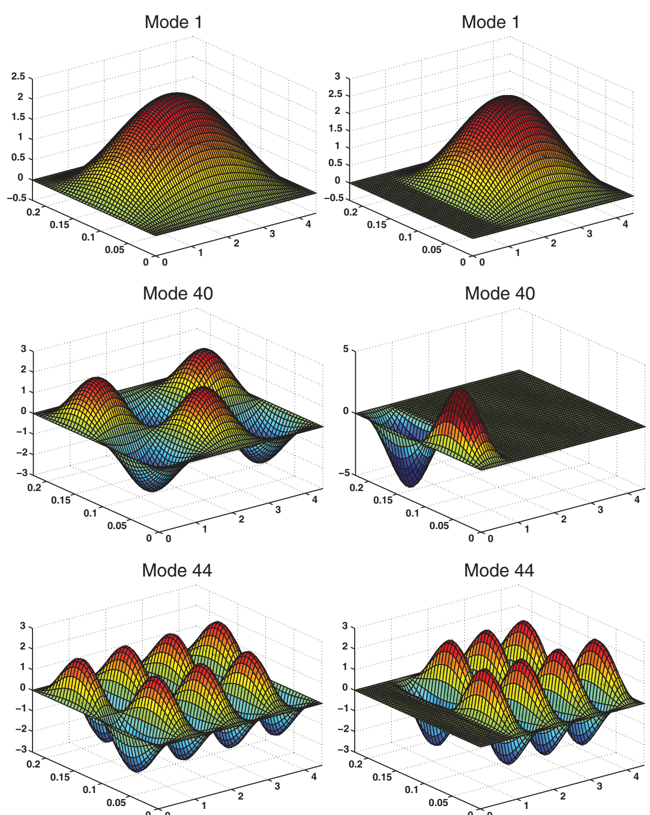

FIG. 2 (color). Several modes of the bi-Laplacian (from top to bottom, modes 1, 40, and 44) of an elongated plate of eccentricity 20. The charts on the left and on the right correspond, respectively, to a plate with no clamped interior point and a plate with one interior clamped point. One can observe that in the latter case the modes are almost entirely concentrated either to the left or to the right of the clamped point.

Let us now turn to the specifics. In order to quantify the localization of the eigenmodes, two scalar criteria are used. The first one involves the *existence area* defined as

$$\alpha = \left(\int_{\text{plate}} u^2 ds\right)^2 \left(\int_{\text{plate}} u^4 ds\right)^{-1}, \tag{3}$$

for each mode u [11,12]. This quantity has the dimensions of an area, and gives information regarding the size of the region on which the mode "lives." For instance, if function u takes value 1 on an area of size A and 0 elsewhere, then α is equal to A.

Figure 4 displays the distribution of the existence areas for the first 300 eigenmodes of the bi-Laplacian for two values of the eccentricity and two clamping conditions. On the top, the eccentricity e is equal to 1 (which corresponds to a square plate) and at the bottom $e = 20$. Furthermore, the plots on the left and on the right-hand side correspond to the cases of plates without or with an inside clamped point, respectively. In the square plate (top), the existence areas are distributed around 0.35, whether an interior point is clamped or not. On the other hand, in the elongated plate (bottom), one can observe a clear separation of the existence areas around two different values (0.07 and 0.3) when the inside point is clamped (bottom right). Roughly speaking, the two bumps of the distribution of the existence area in the bottom right picture correspond to localization of the eigenmodes in the regions to the left and to the right of a clamped point (cf. Fig. 2). Overall, the distribution of the existence areas clearly indicates that for an elongated plate the localization of the eigenmodes has been dramatically affected by just one new boundary point inside the domain.

Going further, we introduce the second localization criterion. It is tailored to the particular domain we are considering and is designated to quantify the strength of

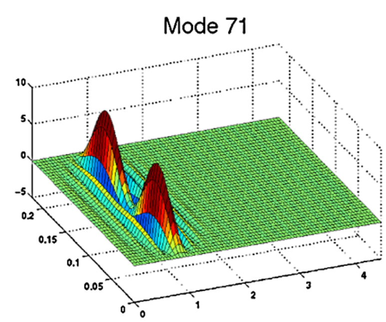

FIG. 3 (color online). Mode number 71 of a clamped plate of eccentricity 20 with a clamped point inside. An analogous effect is observed in all (very rare) cases when the energy is evenly distributed between the regions to the left and to the right of a clamped point.

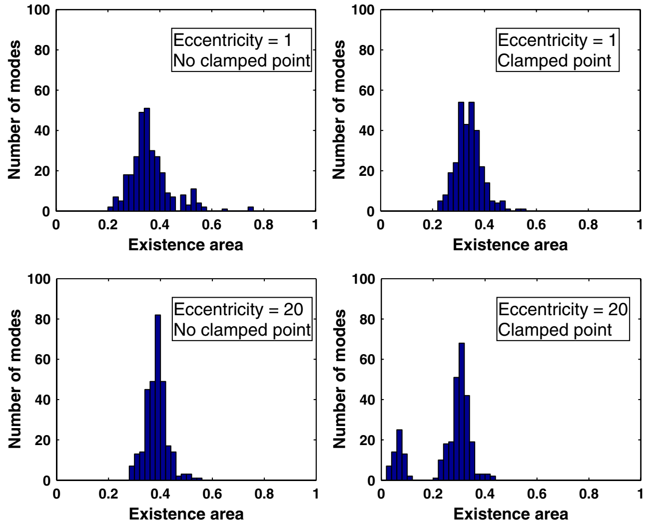

FIG. 4 (color online). Distributions of the existence areas for 300 first eigenmodes for two different plate eccentricities (top, $e = 1$, bottom, $e = 20$) and two different boundary cases (left, no interior point clamped, right, one interior point clamped). One can observe that clamping a point does not split the eigenmode population in the case of a square plate (top). On the other hand, in the case of an elongated plate (bottom), clamping a point clearly separates the eigenmodes into two distinct groups.

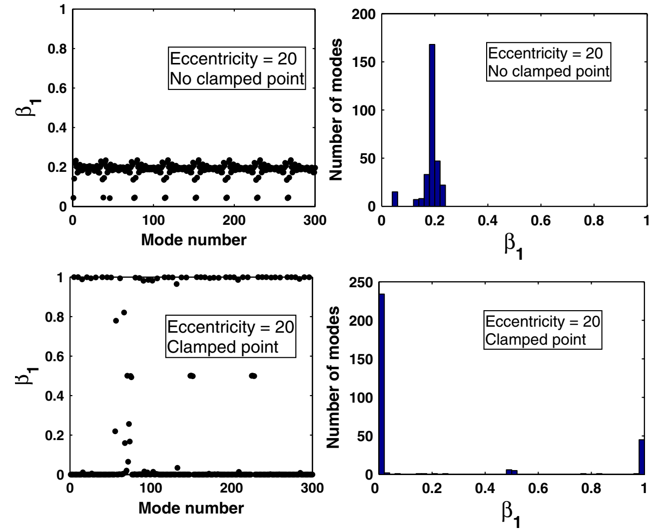

FIG. 5 (color online). Figures on the left-hand side depict the coefficients β_1 for the first 300 modes of a clamped plate of eccentricity 20. Figures on the right-hand side display the distribution of these coefficients. The charts on the top and on the bottom correspond, respectively, to a plate with no clamped interior point and a plate with one interior clamped point. One can observe that on the top β_1 is very close to 0.2 for almost all modes, which signifies a distribution of energy proportional to the surface area within the plate (hence, no localization). At the bottom, the values of β_1 are concentrated around 0 and 1; therefore, the mode energies are either entirely confined to region 1 or to region 2.

the confinement of eigenmodes to one of the two regions: region 1 (to the left of a clamped point) or region 2 (to the right of it). To this end, let us define β_1 as the ratio between the energy integral over region 1 and the total energy of the mode:

$$\beta_1 = \left(\int_{\text{region 1}} (\Delta u)^2 ds\right)\left(\int_{\text{plate}} (\Delta u)^2 ds\right)^{-1}. \tag{4}$$

This quantity assumes values between 0 and 1 and measures the relative amount of energy of a given mode u concentrated in region 1. The analogous quantity for region 2 is thus simply $\beta_2 = 1 - \beta_1$.

The ratio β_1 has been computed for the first 300 modes for plates of eccentricity 20 with and without an interior clamped point. Figure 5 represents the values of β_1 as a function of mode number (left) and the distribution of these values between 0 and 1 (right). One can immediately notice how strong is the separation phenomenon. In the first case, almost all the modes have the energy partition close to $\beta_1 = 0.2$, $\beta_2 = 1 - \beta_1 = 0.8$, proportional to the respective surface areas of regions 1 and 2. This signifies a rather spatially uniform distribution of energy for all modes. In contrast, with a point clamped inside, β_1 either takes the values close to 0 or the values close to 1. Hence, almost all the modes are concentrated entirely in one of the two regions, region 1 or region 2 (cf. Fig. 2).

Finally, let us discuss the dependence of localization on the plate eccentricity in more precise terms. To this end, we define the *connection coefficient*

$$C = \langle \min\{\beta_1, \beta_2\}\rangle_{\text{modes}}, \tag{5}$$

the average of $\min\{\beta_1, \beta_2\} = \min\{\beta_1, 1 - \beta_1\}$ over a large number of eigenmodes (the first 300 in our simulations). This quantity estimates the average proportion of energy in the region complimentary to the one where the modes "live." One can understand it in simple terms. Let us assume that there exists a slight damping in the system. Assume that vibrations are triggered in the plate by a stationary wide spectrum excitation in one of the regions, for instance region 1. Eventually, a steady state vibration will settle in the whole plate. The coefficient C then gives the proportion of energy confined in the complementary region (region 2). In a sense, C measures the "connection" between the two regions.

When plotted against the eccentricity (Fig. 6), the coefficient C exhibits a steady decrease from $e = 1$ (the square plate) to $e = 20$. The value of C close to 0.2 for the square plate corresponds to the case in which almost all eigenmodes have a rather uniform spatial distribution of energy. There is no energy or vibration localization induced by clamping a point. On the other hand, for eccentricities larger than 10, the value of C drops under 0.1 and gets close to 0 when the eccentricity approaches 20. This corresponds to a situation when almost all modes are entirely confined either to region 1 or to region 2: a stationary vibration induced in one of the two regions would essen-

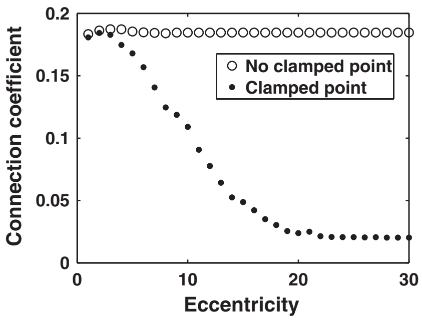

FIG. 6. The ''connection'' coefficient C [as defined in Eq. (5)] as a function of the plate eccentricity. If all the eigenmodes have a rather uniform spatial distribution of energy, then the coefficient C is close to 0.2. On the other hand, if the eigenmode energy is concentrated either in region 1 or in region 2, the coefficient C is close to 0. For the classical rectangular plate, C almost does not depend on the eccentricity, which means that all the modes are more or less uniformly distributed. With one clamped point inside the structure, the steady decrease of C for the values of eccentricity between 1 and 20 expresses the increasing disconnection between regions 1 and 2.

tially remain confined in the same region. Thus, the regions appear to be almost disconnected except for a very limited number of frequencies, corresponding to the case depicted in Fig. 3.

At this point, let us say that in mathematics very little is known about the relation between the shape of a domain and the properties of the bi-Laplacian eigenmodes. However, it is interesting to note that exact results exist regarding the role of the boundary geometry in the regularity of the solutions to the Dirichlet problem. The eigenmodes correspond to the response of the system under a periodic excitation at a given frequency, while the solution of the Dirichlet problem (or more precisely, the Green function) corresponds to static flexions induced by a point load. At the moment we understand quite well the behavior of the Green function near the boundary and how it is affected by a cusp, a crack, or by a clamped point [13–15]. These results were the initial grounds for our intuition in the present study, although the direct connections between the properties of the eigenmodes and the shape of the domain are yet to be explored.

In summary, our simulations have shown that clamping only one point inside a rectangular clamped plate may be sufficient to trigger a dramatic change in the spatial localization of its vibrational eigenmodes. In a plate of eccentricity 20, this localization is so strong that almost all the modes are confined either strictly to the left or strictly to the right of the clamped point. Such an effect may be of great importance in the control of vibrations, as it reveals a very simple way to achieve an almost complete mechanical isolation between two regions of the same plate. Moreover, the same property should be observed with respect to the localization of eddies in 2D flow of viscous fluids, as these systems share the same mathematical frame.

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