Pointwise estimates for the polyharmonic Green function in general domains

Svitlana Mayboroda and Vladimir Maz'ya

Mathematics Subject Classification (2000). 35J40, 35J30, 35B65.

Keywords. polyharmonic equation, Green function, Dirichlet problem, general domains.

1. Introduction

Let Ω be a arbitrary bounded domain in $\mathbb{R}^n$. The Green function for the polyharmonic equation is a function $G : \Omega \times \Omega \to \mathbb{R}$ which for every fixed $y \in \Omega$ solves the equation

$$(-\Delta_x)^m G(x,y) = \delta(x-y), \qquad x \in \Omega, \tag{1.1}$$

in the space $\mathring{W}^{m,2}(\Omega)$, a completion of $C_0^\infty(\Omega)$ in the norm given by $\|u\|_{\mathring{W}^{m,2}(\Omega)} = \|\nabla^m u\|_{L^2(\Omega)}$. The case $m = 2$ corresponds to the biharmonic equation, and respectively, (1.1) gives rise to the biharmonic Green function.

In dimension two the biharmonic Green function can be interpreted as a deflection of a thin clamped plate under a point load. Numerous applications in structural engineering, emerging from this fact, have stimulated considerable interest to the biharmonic equation and its Green function as early as in the beginning of 20th century. In 1908 Hadamard has published a volume devoted to properties of the solutions to the biharmonic equation [8], where, in particular, he conjectured that the corresponding Green function must be positive, at least, in convex domains. However, several counterexamples to Hadamard's conjecture have been found later on ([5], [6], [7], [12], [20], [3], [9]) and it was proved that the biharmonic Green function may change sign even in a smooth convex domain, in a sufficiently eccentric ellipse ([7], [3]). Moreover, in a rectangle the first eigenfunction of the biharmonic operator has infinitely many changes of sign near each of the vertices ([2], [9]).

The first author is partially supported by the NSF grant DMS 0758500.

During the past century, the biharmonic and more generally, the polyharmonic Green function has been thoroughly studied, and a variety of upper estimates has been obtained. In particular, we would like to point out the results in smooth domains [4], [11], [18], [19], in conical domains [16], [10], and in polyhedra [17].

The objective of the present paper is to establish sharp estimates on the polyharmonic Green function and its derivatives without any geometric assumptions, in an arbitrary bounded open set.

For example, we show that, whenever the dimension $n \in [3, 2m+1] \cap \mathbb{N}$ is odd, the regular part of the Green function admits the estimate

$$|\nabla_x^{m-\frac{n}{2}+\frac{1}{2}} \nabla_y^{m-\frac{n}{2}+\frac{1}{2}} (G(x,y) - \Gamma(x-y))| \leq \frac{C}{\max\{d(x), d(y), |x-y|\}}, \quad x, y \in \Omega, \tag{1.2}$$

where $\Gamma(x) = C_{m,n}|x|^{2m-n}$, $x \in \Omega$, is a fundamental solution for the polyharmonic operator, $d(x)$ is the distance from $x \in \Omega$ to $\partial\Omega$ and the constant C depends on n and m only. Hence, in particular,

$$|\nabla_x^{m-\frac{n}{2}+\frac{1}{2}} \nabla_y^{m-\frac{n}{2}+\frac{1}{2}} G(x,y)| \leq \frac{C}{|x-y|}, \qquad x, y \in \Omega, \tag{1.3}$$

and similar results are established for the lower order derivatives.

Furthermore, the estimates on the Green function allow us to derive optimal bounds for the solution u of the Dirichlet boundary value problem

$$(-\Delta)^m u = \sum_{|\alpha| \leq m-\frac{n}{2}+\frac{1}{2}} c_\alpha \partial^\alpha f_\alpha, \quad u \in \mathring{W}^{m,2}(\Omega). \tag{1.4}$$

Specifically,

$$|\nabla^{m-\frac{n}{2}+\frac{1}{2}} u(x)| \quad \leq \quad C \sum_{|\alpha| \leq m-\frac{n}{2}+\frac{1}{2}} \int_\Omega d(y)^{m-\frac{n}{2}+\frac{1}{2}-|\alpha|} \frac{|f_\alpha(y)|}{|x-y|}\, dy, \ x \in \Omega, \tag{1.5}$$

whenever the integrals on the right-hand side of (1.5) are finite. In particular, there exists a constant $C_\Omega > 0$ depending on m, n and the domain Ω such that

$$\|\nabla^{m-\frac{n}{2}+\frac{1}{2}} u\|_{L^\infty(\Omega)} \quad \leq \quad C_\Omega \sum_{|\alpha| \leq m-\frac{n}{2}+\frac{1}{2}} \|d(\cdot)^{m-\frac{n}{2}-\frac{1}{2}-|\alpha|} f_\alpha\|_{L^p(\Omega)}, \tag{1.6}$$

for $p > \frac{n}{n-1}$.

The bounds above are sharp, in the sense that the solution of the polyharmonic equation in an arbitrary domain generally does not exhibit more regularity. Indeed, assume that $n \in [3, 2m+1] \cap \mathbb{N}$ is odd and let $\Omega \subset \mathbb{R}^n$ be the punctured unit ball $B_1 \setminus \{O\}$, where $B_r = \{x \in \mathbb{R}^n : |x| < r\}$. Consider a function $\eta \in C_0^\infty(B_{1/2})$ such that $\eta = 1$ on $B_{1/4}$. Then let

$$u(x) := \eta(x)\, \partial_x^{m-\frac{n}{2}-\frac{1}{2}} \Gamma(x) = C\eta(x)\, \partial_x^{m-\frac{n}{2}-\frac{1}{2}} (|x|^{2m-n}), \qquad x \in B_1 \setminus \{O\}, \tag{1.7}$$

where ∂_x stands for a derivative in the direction of x_i for some $i = 1, ..., n$. It is straightforward to check that $u \in \mathring{W}^{m,2}(\Omega)$ and $(-\Delta)^m u \in C_0^\infty(\Omega)$. While

$\nabla^{m-\frac{n}{2}+\frac{1}{2}}u$ is bounded, the derivatives of the order $m-\frac{n}{2}+\frac{3}{2}$ are not, and moreover, $\nabla^{m-\frac{n}{2}+\frac{1}{2}}u$ is not continuous at the origin. Therefore, the estimates (1.5) are optimal in general domains.

We also derive full analogues of (1.2), (1.3), (1.5), (1.6) and accompanying lower order bounds in even dimensions. In that case, the optimal regularity turns out to be of the order $m-\frac{n}{2}$.

Finally, we would like to mention that the Green function estimates in this paper generalize the earlier developments in [13], where the biharmonic Green function was treated, and [15], where the pointwise estimates on polyharmonic Green function have been established in dimensions $2m+1$ and $2m+2$ for $m>2$ and dimensions $5,6,7$ for $m=2$.

2. Preliminaries

The Green function estimates in the present paper are based, in particular, on the recent results for locally polyharmonic functions that will appear in [14]. We record them below without the proof. Here and throughout the paper $B_r(Q)$ and $S_r(Q)$ denote, respectively, the ball and the sphere with radius r centered at Q and $C_{r,R}(Q)=B_R(Q)\setminus\overline{B_r(Q)}$. When the center is at the origin, we write B_r in place of $B_r(O)$, and similarly $S_r:=S_r(O)$ and $C_{r,R}:=C_{r,R}(O)$. Also, $\nabla^m u$ stands for a vector of all derivatives of u of the order m.

Proposition 2.1. *Let Ω be a bounded domain in $\mathbb{R}^n$, $2\le n\le 2m+1$, $Q\in\mathbb{R}^n\setminus\Omega$, and $R>0$. Suppose*

$$(-\Delta)^m u=f \text{ in } \Omega, \quad f\in C_0^\infty(\Omega\setminus B_{4R}(Q)), \quad u\in \mathring{W}^{m,2}(\Omega). \tag{2.1}$$

Then

$$\frac{1}{\rho^{2\lambda+n-1}}\int_{S_\rho(Q)\cap\Omega}|u(x)|^2\,d\sigma_x\le\frac{C}{R^{2\lambda+n}}\int_{C_{R,4R}(Q)\cap\Omega}|u(x)|^2\,dx \quad \text{for every} \quad \rho<R, \tag{2.2}$$

where C is a constant depending on m and n only, and

$$\lambda=m-n/2+1/2 \text{ when } n \text{ is odd}, \qquad \lambda=m-n/2 \text{ when } n \text{ is even}. \tag{2.3}$$

Moreover, for every $x\in B_{R/4}(Q)\cap\Omega$

$$|\nabla^i u(x)|^2\le C\frac{|x-Q|^{2\lambda-2i}}{R^{n+2\lambda}}\int_{C_{R/4,4R}(Q)\cap\Omega}|u(y)|^2\,dy, \qquad 0\le i\le\lambda \tag{2.4}$$

where λ is given by (2.3).

In addition, using the Kelvin transform, estimates near the origin for solutions of elliptic equations can be translated into estimates at infinity. In particular, Proposition 2.1 leads to the following result (also proved in [14]).

Proposition 2.2. *Let Ω be a bounded domain in $\mathbb{R}^n$, $2 \le n \le 2m+1$, $Q \in \mathbb{R}^n \setminus \Omega$, $r > 0$ and assume that*

$$(-\Delta)^m u = f \ \text{in}\ \Omega, \quad f \in C_0^\infty(B_{r/4}(Q) \cap \Omega), \quad u \in \mathring{W}^{m,2}(\Omega). \tag{2.5}$$

Then

$$\rho^{2\lambda+n+1-4m} \int_{S_\rho(Q)\cap\Omega} |u(x)|^2 \, d\sigma_x \le C\, r^{2\lambda+n-4m} \int_{C_{r/4,r}(Q)\cap\Omega} |u(x)|^2 \, dx, \tag{2.6}$$

for any $\rho > r$ and λ given by (2.3).

Furthermore, for any $x \in \Omega \setminus B_{4r}(Q)$

$$|\nabla^i u(x)|^2 \le C \, \frac{r^{2\lambda+n-4m}}{|x-Q|^{2\lambda+2n-4m+2i}} \int_{C_{r/4,4r}(Q)\cap\Omega} |u(y)|^2 \, dy, \qquad 0 \le i \le \lambda. \tag{2.7}$$

3. Estimates for the Green function

Following [1] we point out that the fundamental solution for the m-Laplacian is a linear combination of the characteristic singular solution (defined below) and any m-harmonic function in $\mathbb{R}^n$. The characteristic singular solution is

$$C_{m,n}|x|^{2m-n}, \qquad \text{if } n \text{ is odd, or if } n \text{ is even with } n \ge 2m+2, \tag{3.1}$$

$$C_{m,n}|x|^{2m-n}\log|x|, \qquad \text{if } n \text{ is even with } n \le 2m. \tag{3.2}$$

The exact expressions for constants $C_{m,n}$ can be found in [1], p.8. For the purposes of this paper we will use the fundamental solution given by

$$\Gamma(x) = C_{m,n} \begin{cases} |x|^{2m-n}, & \text{if } n \text{ is odd}, \\ |x|^{2m-n}\log\frac{\operatorname{diam}\Omega}{|x|}, & \text{if } n \text{ is even and } n \le 2m, \\ |x|^{2m-n}, & \text{if } n \text{ is even and } n \ge 2m+2. \end{cases} \tag{3.3}$$

Theorem 3.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Then there exist constants C, C' depending on m and n only such that for every $x, y \in \Omega$ the following estimates hold. If $n \in [3, 2m+1] \cap \mathbb{N}$ is odd then*

$$|\nabla_x^i \nabla_y^j G(x,y)| \le C \min\left\{1, \left(\frac{d(x)}{|x-y|}\right)^{m-\frac{n}{2}+\frac{1}{2}-i}, \left(\frac{d(y)}{|x-y|}\right)^{m-\frac{n}{2}+\frac{1}{2}-j}\right\} \times \frac{1}{|x-y|^{n-2m+i+j}}, \tag{3.4}$$

whenever $0 \le i,j \le m-\frac{n}{2}+\frac{1}{2}$ *are such that* $i+j \ge 2m-n$*, and*

$$|\nabla_x^i \nabla_y^j G(x,y)| \le C \min\left\{1, \left(\frac{d(x)}{|x-y|}\right)^{m-\frac{n}{2}+\frac{1}{2}-i}, \left(\frac{d(y)}{|x-y|}\right)^{m-\frac{n}{2}+\frac{1}{2}-j}\right\} \times \\ \times \frac{1}{|x-y|^{n-2m+i+j}} \min\left\{\frac{|x-y|}{d(x)}, \frac{|x-y|}{d(y)}, 1\right\}^{n-2m+i+j} \quad (3.5)$$

if $0 \le i,j \le m-\frac{n}{2}+\frac{1}{2}$ *are such that* $i+j \le 2m-n$*.*

If $n \in [2,2m] \cap \mathbb{N}$ *is even, then*

$$|\nabla_x^i \nabla_y^j G(x,y)| \le C \min\left\{1, \left(\frac{d(x)}{|x-y|}\right)^{m-\frac{n}{2}-i}, \left(\frac{d(y)}{|x-y|}\right)^{m-\frac{n}{2}-j}\right\} \times \\ \times \frac{1}{|x-y|^{n-2m+i+j}} \min\left\{\frac{|x-y|}{d(x)}, \frac{|x-y|}{d(y)}, 1\right\}^{n-2m+i+j} \times \\ \times \log\left(1+\frac{\min\{d(x),d(y)\}}{|x-y|}\right), \quad (3.6)$$

for all $0 \le i,j \le m-\frac{n}{2}$*.*

Furthermore, the estimates on the regular part of the Green function $S(x,y) = G(x,y) - \Gamma(x-y)$*,* $x,y \in \Omega$*, are as follows. If* $n \in [3,2m+1] \cap \mathbb{N}$ *is odd then*

$$|\nabla_x^i \nabla_y^j S(x,y)| \le \frac{C}{\max\{d(x),d(y),|x-y|\}^{n-2m+i+j}}, \quad (3.7)$$

whenever $0 \le i,j \le m-\frac{n}{2}+\frac{1}{2}$ *are such that* $i+j \ge 2m-n$*, and*

$$|\nabla_x^i \nabla_y^j S(x,y)| \le \frac{C}{|x-y|^{n-2m+i+j}} \min\left\{\frac{|x-y|}{d(x)}, \frac{|x-y|}{d(y)}, 1\right\}^{n-2m+i+j}, \quad (3.8)$$

if $0 \le i,j \le m-\frac{n}{2}+\frac{1}{2}$ *are such that* $i+j \le 2m-n$*.*

If $n \in [2,2m] \cap \mathbb{N}$ *is even, then*

$$|\nabla_x^i \nabla_y^j S(x,y)| \le \frac{C}{|x-y|^{n-2m+i+j}} \min\left\{\frac{|x-y|}{d(x)}, \frac{|x-y|}{d(y)}, 1\right\}^{n-2m+i+j} \times \\ \times \log\left(1+\frac{\operatorname{diam}\Omega}{\max\{d(x),d(y),|x-y|\}}\right), \quad (3.9)$$

for all $0 \le i,j \le m-\frac{n}{2}$*.*

Proof. Let us start with some auxiliary calculations. Let α be a multi-index of length less than or equal to λ, where λ is given by (2.3). Then $\partial_y^\alpha \Gamma(x-y)$ can be written as

$$\partial_y^\alpha \Gamma(x-y) = P^\alpha(x-y) \log\frac{\operatorname{diam}\Omega}{|x-y|} + Q^\alpha(x-y). \quad (3.10)$$

When the dimension is odd, $P^\alpha \equiv 0$. If the dimension is even (and less than or equal to $2m$ by the assumptions of the theorem) then P^α is a homogeneous polynomial of order $2m-n-|\alpha|$ as long as $|\alpha| \le 2m-n$. In any case, Q^α is a homogeneous function of order $2m-n-|\alpha|$.

Consider a function η such that

$$\eta \in C_0^\infty(B_{1/2}) \quad \text{and} \quad \eta = 1 \quad \text{in} \quad B_{1/4}, \tag{3.11}$$

and define

$$\mathcal{R}_\alpha(x,y) := \partial_y^\alpha\, G(x,y) - \eta\left(\frac{x-y}{d(y)}\right)\left(P^\alpha(x-y)\log\frac{d(y)}{|x-y|} + Q^\alpha(x-y)\right), \tag{3.12}$$

for $x, y \in \Omega$. Also, let us denote

$$\begin{aligned} f_\alpha(x,y) &:= (-\Delta_x)^m \mathcal{R}_\alpha(x,y) \\ &= -\left[(-\Delta_x)^m, \eta\left(\frac{x-y}{d(y)}\right)\right]\left(P^\alpha(x-y)\log\frac{d(y)}{|x-y|} + Q^\alpha(x-y)\right). \end{aligned} \tag{3.13}$$

It is not hard to see that for every α as above

$$f_\alpha(\cdot,y) \in C_0^\infty(C_{d(y)/4,d(y)/2}(y)) \qquad \text{and} \qquad |f_\alpha(x,y)| \le C d(y)^{-n-|\alpha|}, \quad x,y \in \Omega. \tag{3.14}$$

Then for every fixed $y \in \Omega$ the function $x \mapsto \mathcal{R}_\alpha(x,y)$ is a solution of the boundary value problem

$$(-\Delta_x)^m \mathcal{R}_\alpha(x,y) = f_\alpha(x,y) \text{ in } \Omega, \quad f_\alpha(\cdot,y) \in C_0^\infty(\Omega), \quad \mathcal{R}_\alpha(\cdot,y) \in \mathring{W}^{m,2}(\Omega), \tag{3.15}$$

so that

$$\|\nabla_x^m \mathcal{R}_\alpha(\cdot,y)\|_{L^2(\Omega)} = \|\mathcal{R}_\alpha(\cdot,y)\|_{W^{m,2}(\Omega)} \le C\|f_\alpha(\cdot,y)\|_{W^{-m,2}(\Omega)}, \qquad 0 \le |\alpha| \le \lambda. \tag{3.16}$$

Here $W^{-m,2}(\Omega)$ stands for the Banach space dual of $\mathring{W}^{m,2}(\Omega)$, i.e.

$$\|f_\alpha(\cdot,y)\|_{W^{-m,2}(\Omega)} = \sup_{v \in \mathring{W}^{m,2}(\Omega):\, \|v\|_{\mathring{W}^{m,2}(\Omega)}=1} \int_\Omega f_\alpha(x,y) v(x)\, dx. \tag{3.17}$$

Recall that by Hardy's inequality

$$\left\|\frac{v}{|\cdot - Q|^m}\right\|_{L^2(\Omega)} \le C\,\|\nabla^m v\|_{L^2(\Omega)} \quad \text{for every} \quad v \in \mathring{W}^{m,2}(\Omega), \quad Q \in \partial\Omega. \tag{3.18}$$

Then for some $y_0 \in \partial\Omega$ such that $|y-y_0| = d(y)$ and any v in (3.17)

$$\begin{aligned} \int_\Omega f_\alpha(x,y)v(x)\,dx &\le C\left\|\frac{v}{|\cdot - y_0|^m}\right\|_{L^2(\Omega)} \|f_\alpha(\cdot,y)|\cdot - y_0|^m\|_{L^2(\Omega)} \\ &\le C d(y)^m\, \|\nabla^m v\|_{L^2(\Omega)}\, \|f_\alpha(\cdot,y)\|_{L^2(C_{d(y)/4,d(y)/2}(y))}, \end{aligned} \tag{3.19}$$

and therefore, by (3.14)

$$\|\nabla_x^m \mathcal{R}_\alpha(\cdot,y)\|_{L^2(\Omega)} \le C d(y)^{m-|\alpha|-n/2}. \tag{3.20}$$

Now we split the discussion into a few cases.

Case I: $|x-y| \geq Nd(y)$ **or** $|x-y| \geq Nd(x)$ for some large N to be specified later.

Let us first assume that $|x-y| \geq Nd(y)$. As before, we denote by y_0 some point on the boundary such that $|y-y_0| = d(y)$. Then by (3.14)–(3.15) the function $x \mapsto \mathcal{R}_\alpha(x,y)$ is m-harmonic in $\Omega \setminus B_{3d(y)/2}(y_0)$. Hence, by Proposition 2.2 with $r = 6d(y)$

$$|\nabla_x^i \mathcal{R}_\alpha(x,y)|^2 \leq C\, \frac{d(y)^{2\lambda+n-4m}}{|x-y_0|^{2\lambda+2n-4m+2i}} \int_{C_{3d(y)/2,24d(y)}(y_0)} |\mathcal{R}_\alpha(z,y)|^2\, dz, \quad (3.21)$$

provided that $0 \leq i \leq \lambda$ and $|x-y| \geq 4r + d(y)$, i.e $N \geq 25$. The right-hand side of (3.21) is bounded by

$$\begin{aligned} C\, &\frac{d(y)^{2\lambda+n-2m}}{|x-y_0|^{2\lambda+2n-4m+2i}} \int_{C_{3d(y)/2,24d(y)}(y_0)} \frac{|\mathcal{R}_\alpha(z,y)|^2}{|z-y_0|^{2m}}\, dz \\ &\leq C\, \frac{d(y)^{2\lambda+n-2m}}{|x-y_0|^{2\lambda+2n-4m+2i}} \int_\Omega |\nabla_z^m \mathcal{R}_\alpha(z,y)|^2\, dz \\ &\leq C\, \frac{d(y)^{2\lambda-2|\alpha|}}{|x-y|^{2\lambda+2n-4m+2i}}, \end{aligned} \quad (3.22)$$

by Hardy's inequality and (3.20). Therefore,

$$|\nabla_x^i \mathcal{R}_\alpha(x,y)|^2 \leq C\, \frac{d(y)^{2\lambda-2|\alpha|}}{|x-y|^{2\lambda+2n-4m+2i}}, \quad \text{when} \quad |x-y| \geq Nd(y). \quad 0 \leq i, |\alpha| \leq \lambda. \quad (3.23)$$

Since for $N \geq 25$ the condition $|x-y| \geq Nd(y)$ guarantees that $\eta\left(\frac{x-y}{d(y)}\right) = 0$ and hence, $\mathcal{R}_\alpha(x,y) = \partial_y^\alpha G(x,y)$ when $|x-y| \geq Nd(y)$, the estimate (3.23) with $j := |\alpha|$ implies

$$|\nabla_x^i \nabla_y^j G(x,y)|^2 \leq C\, \frac{d(y)^{2\lambda-2j}}{|x-y|^{2\lambda+2n-4m+2i}}, \quad \text{when} \quad |x-y| \geq Nd(y), \quad 0 \leq i,j \leq \lambda. \quad (3.24)$$

Also, by the symmetry of the Green function we automatically deduce that

$$|\nabla_x^i \nabla_y^j G(x,y)|^2 \leq C\, \frac{d(x)^{2\lambda-2i}}{|x-y|^{2\lambda+2n-4m+2j}}, \quad \text{when} \quad |x-y| \geq Nd(x), \quad 0 \leq i,j \leq \lambda. \quad (3.25)$$

In particular, (3.24) and (3.25) combined give the estimate

$$|\nabla_x^i \nabla_y^j G(x,y)| \leq \frac{C}{|x-y|^{n-2m+i+j}}, \quad \text{when} \quad |x-y| \geq N \min\{d(x), d(y)\}, \quad (3.26)$$

for $0 \leq i,j \leq \lambda$.

Now further consider several cases. If n is odd, then

$$|\nabla_x^i \nabla_y^j \Gamma(x-y)| \leq \frac{C}{|x-y|^{n-2m+i+j}} \qquad \text{for all} \qquad x,y \in \Omega, \quad i,j \geq 0, \quad (3.27)$$

while if n is even, then

$$|\nabla_x^i \nabla_y^j \Gamma(x-y)| \le C_1\,|x-y|^{-n+2m-i-j} \log \frac{\operatorname{diam}(\Omega)}{|x-y|} + C_2\,|x-y|^{-n+2m-i-j}, \quad (3.28)$$

for all $x, y \in \Omega$ and $0 \le i + j \le 2m - n$.

Combining this with (3.26) we deduce that for $n \le 2m+1$ odd

$$|\nabla_x^i \nabla_y^j S(x,y)| \le \frac{C}{|x-y|^{n-2m+i+j}} \quad \text{when} \quad |x-y| \ge N \min\{d(x), d(y)\}, \quad (3.29)$$

while if n is even, then

$$|\nabla_x^i \nabla_y^j S(x,y)| \le C\,|x-y|^{-n+2m-i-j} \left(C' + \log \frac{\operatorname{diam}(\Omega)}{|x-y|} \right) \quad (3.30)$$

provided that $|x-y| \ge N \min\{d(x), d(y)\}$ and $0 \le i, j \le \lambda$.

Case II: $|x-y| \le N^{-1} d(y)$ **or** $|x-y| \le N^{-1} d(x)$.

Assume that $|x-y| \le N^{-1} d(y)$. For such x we have $\eta\left(\frac{x-y}{d(y)}\right) = 1$ and therefore

$$\mathcal{R}_\alpha(x,y) = \partial_y^\alpha\, G(x,y) - P^\alpha(x-y) \log \frac{d(y)}{|x-y|} - Q^\alpha(x-y). \quad (3.31)$$

Hence, if n is odd,

$$\mathcal{R}_\alpha(x,y) = \partial_y^\alpha \left(G(x,y) - \Gamma(x-y) \right), \quad \text{when} \quad |x-y| \le N^{-1} d(y), \quad (3.32)$$

and if n is even,

$$\mathcal{R}_\alpha(x,y) = \partial_y^\alpha \left(G(x,y) - \Gamma(x-y) \right) + P^\alpha(x-y) \log \frac{\operatorname{diam}\Omega}{d(y)}, \quad (3.33)$$

when $|x-y| \le N^{-1} d(y)$. By the interior estimates for solutions of elliptic equations

$$|\nabla_x^i \mathcal{R}_\alpha(x,y)|^2 \le \frac{C}{d(y)^{n+2i}} \int_{B_{d(y)/8}(x)} |\mathcal{R}_\alpha(z,y)|^2\, dz, \quad \text{for any } i \le m, \quad (3.34)$$

since the function $\mathcal{R}_\alpha$ is m-harmonic in $B_{d(y)/4}(y) \supset B_{d(y)/8}(x)$. Now we bound the expression above by

$$\frac{C}{d(y)^{n+2i-2m}} \int_{B_{d(y)/4}(y)} \frac{|\mathcal{R}_\alpha(z,y)|^2}{|z-y_0|^{2m}}\, dz \le \frac{C}{d(y)^{n+2i-2m}} \, \|\nabla_x^m \mathcal{R}(\cdot,y)\|^2_{L^2(\Omega)}$$
$$\le \frac{C}{d(y)^{2n-4m+2i+2|\alpha|}}, \quad (3.35)$$

with $0 \le |\alpha| \le \lambda$.

Let us now focus on the case of n odd. It follows from (3.32) and (3.34) – (3.35) that

$$|\nabla_x^i \nabla_y^j S(x,y)| \le \frac{C}{d(y)^{n-2m+i+j}}, \quad 0 \le i \le m, \quad 0 \le j \le \lambda, \quad |x-y| \le N^{-1} d(y), \quad (3.36)$$

and hence, by symmetry,

$$|\nabla_x^i \nabla_y^j S(x,y)| \le \frac{C}{d(x)^{n-2m+i+j}}, \quad 0 \le i \le \lambda, \quad 0 \le j \le m, \quad |x-y| \le N^{-1} d(x). \tag{3.37}$$

However, we have

$$|x-y| \le N^{-1} d(y) \quad \Longrightarrow \quad (N-1)\, d(y) \le N d(x) \le (N+1)\, d(y), \tag{3.38}$$

i.e. $d(y) \approx d(x)$ whenever $|x-y|$ is less than or equal to either $N^{-1}d(y)$ or $N^{-1}d(x)$. Therefore, when the dimension is odd,

$$|\nabla_x^i \nabla_y^j S(x,y)| \le \frac{C}{\max\{d(x), d(y)\}^{n-2m+i+j}}, \tag{3.39}$$

provided that $|x-y| \le N^{-1} \max\{d(x), d(y)\}$, $0 \le i, j \le \lambda$, $i + j \ge 2m - n$, and

$$|\nabla_x^i \nabla_y^j S(x,y)| \le C \min\{d(x), d(y)\}^{2m-n-i-j}, \tag{3.40}$$

for $|x-y| \le N^{-1} \max\{d(x), d(y)\}$, $0 \le i, j \le \lambda$, $i + j \le 2m - n$.

As for the Green function itself, we then have for $|x-y| \le N^{-1} \max\{d(x), d(y)\}$

$$|\nabla_x^i \nabla_y^j G(x,y)| \le \frac{C}{|x-y|^{n-2m+i+j}}, \quad \text{if } i + j \ge 2m - n, \tag{3.41}$$

and

$$|\nabla_x^i \nabla_y^j G(x,y)| \le C \min\{d(x), d(y)\}^{2m-n-i-j}, \quad \text{if } i + j \le 2m - n, \tag{3.42}$$

with i, j such that $0 \le i, j \le \lambda$.

Similar considerations apply to the case when the dimension is even, leading to the following results:

$$|\nabla_x^i \nabla_y^j S(x,y)| \le \frac{C}{d(y)^{n-2m+i+j}} \left(C' + \log \frac{\operatorname{diam} \Omega}{d(y)} \right), \tag{3.43}$$

for $0 \le i \le m$, $0 \le j \le \lambda$, $|x-y| \le N^{-1} d(y)$, and

$$|\nabla_x^i \nabla_y^j S(x,y)| \le \frac{C}{d(x)^{n-2m+i+j}} \left(C' + \log \frac{\operatorname{diam} \Omega}{d(x)} \right), \tag{3.44}$$

for $0 \le i \le \lambda$, $0 \le j \le m$, $|x-y| \le N^{-1} d(x)$. In particular, in view of (3.38), and the fact that $2m - n - i - j \ge 0$ whenever $0 \le i, j \le \lambda$ and n is even, we have

$$|\nabla_x^i \nabla_y^j S(x,y)| \le C \min\{d(x), d(y)\}^{2m-n-i-j} \left(C' + \log \frac{\operatorname{diam} \Omega}{\max\{d(x), d(y)\}} \right), \tag{3.45}$$

for $|x-y| \le N^{-1} \max\{d(x), d(y)\}$, $0 \le i, j \le \lambda$.

Passing to the Green function estimates, (3.31) and (3.34)–(3.35) lead to the bound

$$|\nabla_x^i \nabla_y^j G(x,y)| \le C\, d(y)^{2m-n-i-j} \left(C' + \log \frac{d(y)}{|x-y|} \right), \tag{3.46}$$

for $0 \leq i, j \leq \lambda$, $|x-y| \leq N^{-1} d(y)$. Hence, by symmetry,

$$|\nabla_x^i \nabla_y^j G(x,y)| \leq C\, d(x)^{2m-n-i-j} \left(C' + \log \frac{d(x)}{|x-y|} \right), \tag{3.47}$$

for $0 \leq i, j \leq \lambda$, $|x-y| \leq N^{-1} d(x)$, and therefore,

$$|\nabla_x^i \nabla_y^j G(x,y)| \leq C \min\{d(x), d(y)\}^{2m-n-i-j} \left(C' + \log \frac{\min\{d(x), d(y)\}}{|x-y|} \right), \tag{3.48}$$

for all $0 \leq i, j \leq \lambda$, and $|x-y| \leq N^{-1} \max\{d(x), d(y)\}$.

Finally, it remains to consider

Case III: $|x-y| \approx d(y) \approx d(x)$, or more precisely, the situation when

$$N^{-1}\, d(x) \leq |x-y| \leq N d(x) \quad \text{and} \quad N^{-1}\, d(y) \leq |x-y| \leq N d(y). \tag{3.49}$$

In this case we use the m-harmonicity of $x \mapsto G(x,y)$ in $B_{d(x)/(2N)}(x)$. Let $x_0 \in \partial\Omega$ be such that $|x - x_0| = d(x)$. By the interior estimates,

$$\begin{aligned}
|\nabla_x^i \nabla_y^{|\alpha|} G(x,y)|^2 &\leq \frac{C}{d(x)^{n+2i}} \int_{B_{d(x)/(2N)}(x)} |\nabla_y^{|\alpha|} G(z,y)|^2\, dz \\
&\leq \frac{C}{d(x)^{n+2i}} \int_{B_{d(x)/(2N)}(x)} |\nabla_y^{|\alpha|} \Gamma(z-y)|^2\, dz \\
&\qquad + \frac{C}{d(x)^{n+2i-2m}} \int_{B_{2d(x)}(x_0)} \frac{|\mathcal{R}_\alpha(z,y)|^2}{|z-x_0|^{2m}}\, dz \\
&\leq \frac{C}{d(x)^{n+2i}} \int_{B_{d(x)/(2N)}(x)} |\nabla_y^{|\alpha|} \Gamma(z-y)|^2\, dz \\
&\qquad + \frac{C}{d(x)^{n+2i-2m}} \int_{\Omega} |\nabla_z^m \mathcal{R}_\alpha(z,y)|^2\, dz \\
&\leq \frac{C}{d(x)^{2n-4m+2i+2|\alpha|}} + \frac{C}{d(x)^{n-2m+2i} d(y)^{n-2m+2|\alpha|}},
\end{aligned} \tag{3.50}$$

provided that $0 \leq i \leq m$, $0 \leq |\alpha| \leq \lambda$ and n is odd. The right-hand side of (3.50) also provided the estimate on derivatives of the Green function holds when n is even, upon observing that

$$\begin{aligned}
\frac{C}{d(x)^{n+2i}} \int_{B_{d(x)/(2N)}(x)} |\nabla_y^{|\alpha|} G(z,y)|^2\, dz &\leq \frac{C}{d(x)^{n+2i}} \int_{B_{d(x)/(2N)}(x)} |P^\alpha(z-y)|^2\, dz \\
&\quad + \frac{C}{d(x)^{n+2i}} \int_{B_{d(x)/(2N)}(x)} |Q^\alpha(z-y)|^2\, dz \\
&\quad + \frac{C}{d(x)^{n+2i-2m}} \int_{B_{2d(x)}(x_0)} \frac{|\mathcal{R}_\alpha(z,y)|^2}{|z-x_0|^{2m}}\, dz,
\end{aligned} \tag{3.51}$$

since the absolute value of $\log \frac{|z-y|}{d(y)}$ is bounded by a constant for z, x, y as in (3.51), (3.49).

Hence, for x, y satisfying (3.49) we have

$$\begin{aligned} |\nabla_x^i \nabla_y^j G(x,y)| &\leq \frac{C}{\min\{d(x), d(y), |x-y|\}^{n-2m+i+j}} \\ &\approx \frac{C}{\max\{d(x), d(y), |x-y|\}^{n-2m+i+j}}, \end{aligned} \tag{3.52}$$

for $0 \leq i, j \leq \lambda$.

When n is odd, the same argument implies the following estimate on a regular part of Green function

$$\begin{aligned} |\nabla_x^i \nabla_y^j S(x,y)| &\leq \frac{C}{\min\{d(x), d(y), |x-y|\}^{n-2m+i+j}} \\ &\approx \frac{C}{\max\{d(x), d(y), |x-y|\}^{n-2m+i+j}}, \end{aligned} \tag{3.53}$$

for $0 \leq i, j \leq \lambda$, and x, y satisfying (3.49). If n is even, however, we are led to a bound

$$\begin{aligned} |\nabla_x^i \nabla_y^j S(x,y)| &\leq \frac{C}{\min\{d(x), d(y), |x-y|\}^{n-2m+i+j}} \times \\ &\quad \times \left(C' + \log \frac{\operatorname{diam} \Omega}{\max\{d(x), d(y), |x-y|\}^{n-2m+i+j}} \right) \\ &\approx \frac{C}{\max\{d(x), d(y), |x-y|\}^{n-2m+i+j}} \times \\ &\quad \times \left(C' + \log \frac{\operatorname{diam} \Omega}{\max\{d(x), d(y), |x-y|\}^{n-2m+i+j}} \right) \end{aligned} \tag{3.54}$$

for $0 \leq i, j \leq \lambda$.

The final bounds for the Green function are a combination of estimates (3.24), (3.25), (3.41), (3.42), (3.48), (3.52). It helps to observe that the regions of $(x, y) \in \Omega \times \Omega$ in (3.24), (3.25) are disjoint from those in (3.41), (3.42), (3.48). The condition $|x - y| \leq N^{-1} \max\{d(x), d(y)\}$ excludes the possibility of $|x - y| \leq N^{-1} \max\{d(x), d(y)\}$. This is, in particular, due to (3.38). Also, the bound (3.52) is the same as (3.24), (3.25), (3.41), (3.42) for the case when $d(x)$, $d(y)$ and $|x - y|$ are all comparable. Hence, it can be suitably absorbed. Finally, it is straightforward to check that

$$C' + \log \frac{\min\{d(x), d(y)\}}{|x-y|} \approx \log \left(1 + \frac{\min\{d(x), d(y)\}}{|x-y|} \right) \tag{3.55}$$

for $|x - y| \leq N^{-1} \max\{d(x), d(y)\}$.

Analogously, the desired estimates on the regular part of the Green function can be drawn from (3.29), (3.30), (3.39), (3.40). (3.45), (3.53), (3.54). $\square$

4. Applications: estimates on solutions of the Dirichlet problem

Green function estimates proved in Section 3 allow us to investigate the solutions of the Dirichlet problem for the polyharmonic equation for a wide class of data.

Proposition 4.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and assume that $n \in [3, 2m+1] \cap \mathbb{N}$ is odd. Consider the boundary value problem*

$$(-\Delta)^m u = \sum_{|\alpha| \le m - \frac{n}{2} + \frac{1}{2}} c_\alpha \partial^\alpha f_\alpha \in W^{-m,2}(\Omega), \quad u \in \mathring{W}^{m,2}(\Omega). \tag{4.1}$$

Then the solution satisfies the estimate

$$|\nabla^{m-\frac{n}{2}+\frac{1}{2}} u(x)| \le C \sum_{|\alpha| \le m - \frac{n}{2} + \frac{1}{2}} \int_\Omega d(y)^{m-\frac{n}{2}+\frac{1}{2}-|\alpha|} \frac{|f_\alpha(y)|}{|x-y|}\, dy, \ x \in \Omega, \tag{4.2}$$

whenever the integrals on the right-hand side of (4.2) *are finite. The constant C in* (4.2) *depends on m and n only.*

In particular, there exists a constant $C_\Omega > 0$ depending on m, n and the domain Ω such that

$$\|\nabla^{m-\frac{n}{2}+\frac{1}{2}} u\|_{L^\infty(\Omega)} \le C_\Omega \sum_{|\alpha| \le m - \frac{n}{2} + \frac{1}{2}} \|d(\cdot)^{m-\frac{n}{2}-\frac{1}{2}-|\alpha|} f_\alpha\|_{L^p(\Omega)}, \tag{4.3}$$

for $p > \frac{n}{n-1}$, provided that the norms on the right-hand side of (4.3) *are finite.*

Proof. Indeed, the integral representation formula

$$u(x) = \int_\Omega G(x,y) \sum_{|\alpha| \le m - \frac{n}{2} + \frac{1}{2}} c_\alpha \partial^\alpha f_\alpha(y)\, dy, \qquad x \in \Omega, \tag{4.4}$$

follows directly from the definition of Green function. It implies that

$$\nabla^{m-\frac{n}{2}+\frac{1}{2}} u(x) = \sum_{|\alpha| \le m - \frac{n}{2} + \frac{1}{2}} c_\alpha (-1)^{|\alpha|} \int_\Omega \nabla_x^{m-\frac{n}{2}+\frac{1}{2}} \partial_y^\alpha G(x,y) f_\alpha(y)\, dy. \tag{4.5}$$

Furthermore, due to the estimate (3.4) with $i = j = m - \frac{n}{2} + \frac{1}{2}$ we have

$$\int_\Omega \Big| \nabla_x^{m-\frac{n}{2}+\frac{1}{2}} \nabla_y^{m-\frac{n}{2}+\frac{1}{2}} G(x,y) \Big|\, |f(y)|\, dy \le C \int_\Omega \frac{|f(y)|}{|x-y|}\, dy, \tag{4.6}$$

while the bounds in (3.5) can be used to show that for every $j \le m - \frac{n}{2} - \frac{1}{2}$

$$\int_\Omega \Big| \nabla_x^{m-\frac{n}{2}+\frac{1}{2}} \nabla_y^{j} G(x,y) \Big|\, |f(y)|\, dy \le C \int_\Omega \min\left\{1, \left(\frac{d(y)}{|x-y|}\right)^{m-\frac{n}{2}+\frac{1}{2}-j}\right\} \times$$

$$\times \frac{1}{|x-y|^{\frac{n}{2}-m+\frac{1}{2}+j}} \min\left\{\frac{|x-y|}{d(x)}, \frac{|x-y|}{d(y)}, 1\right\}^{\frac{n}{2}-m+\frac{1}{2}+j} |f(y)|\, dy. \tag{4.7}$$

We split the latter integral to the cases $|x-y| \leq N^{-1}d(y)$ and $|x-y| \geq N^{-1}d(y)$ with $N \geq 25$ (as in Theorem 3.1). Recall that according to (3.38) in the first case $d(x) \approx d(y)$ and therefore

$$\min\left\{\frac{|x-y|}{d(x)}, \frac{|x-y|}{d(y)}, 1\right\} \approx \frac{|x-y|}{d(y)} \quad \text{when} \quad |x-y| \leq N^{-1}d(y), \tag{4.8}$$

while in the second case $d(x) \leq |x-y| + d(y) \leq (1+N)|x-y|$, so that

$$\min\left\{\frac{|x-y|}{d(x)}, \frac{|x-y|}{d(y)}, 1\right\} \approx C \quad \text{when} \quad |x-y| \geq N^{-1}d(y). \tag{4.9}$$

Hence, the expression on the right-hand side of (4.7) can be further estimated by

$$\begin{aligned}
& C\int_{y\in\Omega:\,|x-y|\leq N^{-1}d(y)} \frac{1}{|x-y|^{\frac{n}{2}-m+\frac{1}{2}+j}} \left(\frac{|x-y|}{d(y)}\right)^{\frac{n}{2}-m+\frac{1}{2}+j} |f(y)|\,dy \\
&\quad + C\int_{y\in\Omega:\,|x-y|\geq N^{-1}d(y)} \left(\frac{d(y)}{|x-y|}\right)^{m-\frac{n}{2}+\frac{1}{2}-j} \frac{1}{|x-y|^{\frac{n}{2}-m+\frac{1}{2}+j}} |f(y)|\,dy \\
& \leq C\int_{y\in\Omega:\,|x-y|\leq N^{-1}d(y)} d(y)^{m-\frac{n}{2}-\frac{1}{2}-j}|f(y)|\,dy \\
& + C\int_{y\in\Omega:\,|x-y|\geq N^{-1}d(y)} d(y)^{m-\frac{n}{2}+\frac{1}{2}-j} \frac{|f(y)|}{|x-y|}\,dy \\
& \leq C\int_{\Omega} d(y)^{m-\frac{n}{2}+\frac{1}{2}-j} \frac{|f(y)|}{|x-y|}\,dy,
\end{aligned} \tag{4.10}$$

as desired.

This finishes the proof of (4.2) and (4.3) follows from it via the mapping properties of the Riesz potential. □

Proposition 4.1 has a natural analogue for the case when the dimension is even. The details are as follows.

Proposition 4.2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and assume that $n \in [2, 2m] \cap \mathbb{N}$ is even. Consider the boundary value problem*

$$(-\Delta)^m u = \sum_{|\alpha|\leq m-\frac{n}{2}} c_\alpha \partial^\alpha f_\alpha \in W^{-m,2}(\Omega), \quad u \in \mathring{W}^{m,2}(\Omega). \tag{4.11}$$

Then the solution satisfies the estimate

$$|\nabla^{m-\frac{n}{2}} u(x)| \leq C \sum_{|\alpha|\leq m-\frac{n}{2}} \int_\Omega d(y)^{m-\frac{n}{2}-|\alpha|} \log\left(1 + \frac{d(y)}{|x-y|}\right) |f_\alpha(y)|\,dy, \tag{4.12}$$

for all $x \in \Omega$, whenever the integrals on the right-hand side of (4.12) are finite. The constant C in (4.12) depends on m and n only.

In particular, for every $\varepsilon \in (0,1)$ there exists a constant $C_{\Omega,\varepsilon} > 0$ depending on m, n, ε and the domain Ω such that

$$\|\nabla^{m-\frac{n}{2}} u\|_{L^\infty(\Omega)} \quad \leq \quad C_{\Omega,\varepsilon} \sum_{|\alpha|\leq m-\frac{n}{2}} \left\| d(y)^{m-\frac{n}{2}-|\alpha|+\varepsilon} f_\alpha \right\|_{L^p(\Omega)}, \qquad (4.13)$$

for all $p > \frac{n}{n-\varepsilon}$, provided that the norms on the right-hand side of (4.13) are finite.

Proof. The argument is fairly close to the proof of Proposition 4.1. We write

$$\left|\nabla^{m-\frac{n}{2}} u(x)\right| \quad \leq \quad C \sum_{|\alpha|\leq m-\frac{n}{2}} \int_\Omega \left|\nabla_x^{m-\frac{n}{2}} \nabla_y^{|\alpha|} G(x,y)\right| |f_\alpha(y)|\, dy, \qquad (4.14)$$

for every $x \in \Omega$, and split the integrals on the right-hand side according to whether $|x-y| \leq N^{-1} d(y)$ or $|x-y| \geq N^{-1} d(y)$, $N \geq 25$. Then using (4.8) and (4.9) we bound each term on the right-hand side of (4.14) by

$$\begin{aligned}
& C \int_{y\in\Omega:\, |x-y|\leq N^{-1}d(y)} \frac{1}{|x-y|^{\frac{n}{2}-m+|\alpha|}} \left(\frac{|x-y|}{d(y)}\right)^{\frac{n}{2}-m+|\alpha|} \times \\
& \qquad \times \log\left(1+\frac{d(y)}{|x-y|}\right) |f_\alpha(y)|\, dy \\
& + C \int_{y\in\Omega:\, |x-y|\geq N^{-1}d(y)} \left(\frac{d(y)}{|x-y|}\right)^{m-\frac{n}{2}-|\alpha|} \frac{1}{|x-y|^{\frac{n}{2}-m+|\alpha|}} \times \\
& \qquad \times \log\left(1+\frac{\min\{d(y),d(x)\}}{|x-y|}\right) |f_\alpha(y)|\, dy, \qquad (4.15)
\end{aligned}$$

However, if $|x-y| \geq N^{-1} d(y)$ and hence, $d(x) \leq (N+1)|x-y|$, we have

$$\log\left(1+\frac{\min\{d(y),d(x)\}}{|x-y|}\right) \approx C \approx \log\left(1+\frac{d(y)}{|x-y|}\right). \qquad (4.16)$$

Therefore, both terms in (4.15)are bounded by

$$C \int_\Omega d(y)^{m-\frac{n}{2}-j} \log\left(1+\frac{d(y)}{|x-y|}\right) |f_\alpha(y)|\, dy, \qquad (4.17)$$

which leads to (4.12).

Finally, for every $0 < \varepsilon < 1$ there is $C_\varepsilon > 0$ such that $\log(1+x) \leq C_\varepsilon x^\varepsilon$, $x > 0$, which implies that

$$|\nabla^{m-\frac{n}{2}} u(x)| \quad \leq \quad C_\varepsilon \sum_{|\alpha|\leq m-\frac{n}{2}} \int_\Omega d(y)^{m-\frac{n}{2}-|\alpha|} \left(\frac{d(y)}{|x-y|}\right)^\varepsilon |f_\alpha(y)|\, dy, \qquad (4.18)$$

for all $x \in \Omega$, $0 < \varepsilon < 1$.

Then, by the mapping properties of the Riesz potential we recover an estimate

$$\|\nabla^{m-\frac{n}{2}}u\|_{L^\infty(\Omega)} \leq C_{\Omega,\varepsilon} \sum_{|\alpha|\leq m-\frac{n}{2}} \left\| d(y)^{m-\frac{n}{2}-|\alpha|+\varepsilon} f_\alpha \right\|_{L^p(\Omega)}, \; p > \frac{n}{n-\varepsilon}, \quad (4.19)$$

which leads to (4.13). □

References

[1] N. Aronszajn, T. Creese, L. Lipkin, *Polyharmonic functions*, Notes taken by Eberhard Gerlach. Oxford Mathematical Monographs. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1983.

[2] Ch. Coffman, *On the structure of solutions $\Delta^2 u = \lambda u$ which satisfy the clamped plate conditions on a right angle*, SIAM J. Math. Anal. 13 (1982), no. 5, 746–757.

[3] Ch. Coffman, C. Grover, *Obtuse cones in Hilbert spaces and applications to partial differential equations*, J. Funct. Anal. 35 (1980), no. 3, 369–396.

[4] A. Dall'Acqua, G. Sweers, *Estimates for Green function and Poisson kernels of higher-order Dirichlet boundary value problems*, J. Differential Equations 205 (2004), no. 2, 466–487.

[5] R.J. Duffin, *On a question of Hadamard concerning super-biharmonic functions*, J. Math. Physics 27, (1949), 253–258.

[6] R.J. Duffin, *Some problems of mathematics and science*, Bull. Amer. Math. Soc. 80 (1974), 1053–1070.

[7] P.R. Garabedian, *A partial differential equation arising in conformal mapping*, Pacific J. Math. 1, (1951), 485–524.

[8] J. Hadamard, *Sur le problème d'analyse relatif á l'èquilibre des plaques élastiques encastrées*, Mémoire couronné en 1907 par l'Académie: Prix Vaillant, Mémoires présentés par divers savants à l'Académie des Sciences 33, no. 4 (Œuvres 2, 515–629).

[9] V. Kozlov, V. Kondrat'ev, V. Maz'ya, *On sign variability and the absence of "strong" zeros of solutions of elliptic equations*, (Russian) Izv. Akad. Nauk SSSR Ser. Mat. 53 (1989), no. 2, 328–344; translation in Math. USSR-Izv. 34 (1990), no. 2, 337–353.

[10] V. Kozlov, V. Maz'ya, J. Rossmann, *Elliptic boundary value problems in domains with point singularities*, Mathematical Surveys and Monographs, 52. American Mathematical Society, Providence, RI, 1997.

[11] Ju.P. Krasovskiĭ, *Isolation of the singularity in Green's function*, Izv. Akad. Nauk SSSR Ser. Mat. 31 (1967), 977–1010.

[12] Ch. Loewner, *On generation of solutions of the biharmonic equation in the plane by conformal mappings*, Pacific J. Math. 3, (1953), 417–436.

[13] S. Mayboroda, V. Maz'ya, *Boundedness of the gradient of the solution and Wiener test of order one for the biharmonic equation*, Invent. Math., to appear.

[14] S. Mayboroda, V. Maz'ya, *Regularity of solutions to the polyharmonic equation in an arbitrary domain*, in preparation.

[15] V. Maz'ya, *On Wiener's type regularity of a boundary point for higher order elliptic equations*, Nonlinear analysis, function spaces and applications, Vol. 6 (Prague, 1998), 119–155, Acad. Sci. Czech Repub., Prague, 1999.

[16] V. Maz'ya, B.A. Plamenevskiĭ, *Asymptotic behavior of the fundamental solutions of elliptic boundary value problems in domains with conical points*, (Russian) Boundary value problems. Spectral theory (Russian), pp. 100–145, 243, Probl. Mat. Anal., 7, Leningrad. Univ., Leningrad, 1979.

[17] V. Maz'ya, J. Rossmann, *On the Agmon-Miranda maximum principle for solutions of elliptic equations in polyhedral and polygonal domains*, Ann. Global Anal. Geom. 9 (1991), no. 3, 253–303.

[18] V.A. Solonnikov, *The Green's matrices for elliptic boundary value problems. I*, (Russian) Trudy Mat. Inst. Steklov. 110 (1970), 107–145.

[19] V.A. Solonnikov, *The Green's matrices for elliptic boundary value problems. II*, (Russian) Boundary value problems of mathematical physics, 7. Trudy Mat. Inst. Steklov. 116 (1971), 181–216, 237.

[20] G. Szegö, *Remark on the preceding paper of Charles Loewner*, Pacific J. Math. 3, (1953), 437–446.

Svitlana Mayboroda
Department of Mathematics
Purdue University
150 N. University Street
West Lafayette, IN 47907-2067
e-mail: `svitlana@math.purdue.edu`

Vladimir Maz'ya
Department of Mathematical Sciences, M&O
Building, University of Liverpool, Liverpool L69 3BX, UK
Department of Mathematics, Linköping University,
SE-581 83 Linköping, Sweden
e-mail: `vlmaz@mai.liu.se`