THE CONNECTIONS BETWEEN DIRICHLET, REGULARITY AND NEUMANN PROBLEMS FOR SECOND ORDER ELLIPTIC OPERATORS WITH COMPLEX BOUNDED MEASURABLE COEFFICIENTS

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Abstract. The present paper discusses the relations between regularity, Dirichlet, and Neumann problems. Among other results, we prove that the solvability of the regularity problem does not imply the solvability of the dual Dirichlet problem for general elliptic operators with complex bounded measurable coefficients. This is strikingly different from the case of real operators, for which such an implication was established in 1993 by C. Kenig, J. Pipher [Invent. Math. 113] and since then has served as an integral part of many results.

1. Introduction

The theory of elliptic boundary value problems on Lipschitz domains has long and celebrated history. See, e.g., [17] for en excellent account of major results. Until recently, however, it has been primarily restricted to elliptic operators with real symmetric coefficients. A few exceptions include the perturbation results in [9], [10], with applications to the Kato square root problem, and the study of real non-symmetric operators in [18], [21]. The Kato problem has later been resolved in full generality ([5]), and since then the interest to the elliptic problems for rough complex coefficients was constantly rising. In particular, the recent papers [2] and [4] revealed new solvability results for Dirichlet, Neumann and Regularity problems in L^2.

In 1993 C. Kenig and J. Pipher proved that the solvability of the regularity problem in L^p for an elliptic operator L with real coefficients implies the solvability of the Dirichlet problem in $L^{p'}$ for its adjoint L^*, $1 < p < \infty$, $\frac{1}{p} + \frac{1}{p'} = 1$ ([19]). This fact has been routinely used in the study of elliptic boundary value problems, and by now became a part of the "standard theory". In the present paper we show that *it fails* for some elliptic operators with complex bounded measurable coefficients, even in the time-independent case.

Moreover, the property that the solvability of the regularity problem implies the solvability of the Neumann problem fails as well. To the best of our knowledge, this has been previously established only for $p > 2$ ([20]), and the case $p < 2$ treated in the present paper was stated as an open problem in [20], p. 249.

Let us turn to the details. Let A be an $n \times n$ matrix with entries

$$(1.1)\quad a_{jk} : \mathbb{R}^n \longrightarrow \mathbb{C}, \quad a_{jk} \in L^\infty(\mathbb{R}^n), \quad j = 1, ..., n, \quad k = 1, ..., n,$$

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satisfying the ellipticity condition

$$\lambda|\xi|^2 \leq \Re e A\xi \cdot \bar{\xi} \quad \text{and} \quad |A\xi \cdot \bar{\zeta}| \leq \Lambda|\xi||\zeta|, \quad \forall\, \xi, \zeta \in \mathbb{C}^n, \tag{1.2}$$

for some constants $0 < \lambda \leq \Lambda < \infty$. Then the second order divergence form operator is given by

$$Lf := -\mathrm{div}(A\nabla f), \tag{1.3}$$

interpreted in the weak sense via a sesquilinear form. Throughout the discussion $n \geq 3$ unless otherwise specified.

Next, let us denote by $\mathbb{A} = \{\mathsf{a}_{ij}\}_{i,j=1}^{n+1}$ the block matrix such that $\mathsf{a}_{ij} = a_{ij}$ for $1 \leq i, j \leq n$, $\mathsf{a}_{n+1,n+1} = 1$, $\mathsf{a}_{n+1,j} = 0$, $j = 1, ..., n$ and $\mathsf{a}_{i,n+1} = 0$, $i = 1, ..., n$. If A is elliptic then so is $\mathbb{A}$, and hence $\mathbb{A}$ gives rise to an elliptic operator $\mathbb{L} = -\mathrm{div}_{x,t}\mathbb{A}\nabla_{x,t}$ in $\mathbb{R}^{n+1}$. It is not hard to check that $e^{-t\sqrt{L}}f(x)$, $t > 0$, $x \in \mathbb{R}^n$, is a solution to the equation $\mathbb{L}u = 0$ in $\mathbb{R}^{n+1}_+$ with the boundary data f on $\mathbb{R}^n$. Once again, the equation $\mathbb{L}u = 0$ is understood in the weak sense, that is, $\mathbb{L}u = 0$ for $u \in W^{1,2}_{loc}(\mathbb{R}^{n+1}_+)$ if $\int_{\mathbb{R}^{n+1}_+} A\nabla u \cdot \overline{\nabla\psi}\, dx = 0$ for every $\psi \in C_0^\infty(\mathbb{R}^{n+1}_+)$.

The family of operators $\{e^{-t\sqrt{L}}\}_{t>0}$ is the Poisson semigroup of L. It is well-defined in $L^2(\mathbb{R}^n)$ for every L as above via the theory of maximal accretive operators (see Section 2 for details). In the case when $L = -\Delta$, it provides the usual Poisson extension, or Poisson integral, of f in $\mathbb{R}^{n+1}_+$.

We say that the Dirichlet problem (D_p), $1 < p < \infty$, is solvable for the operator $\mathbb{L}$ in $\mathbb{R}^{n+1}_+$ if for every $f \in L^2(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$ the solution to the equation $\mathbb{L}u = 0$ with the boundary data f given by the Poisson extension $u(x,t) = e^{-t\sqrt{L}}f(x)$, $x \in \mathbb{R}^n$, $t > 0$, satisfies the non-tangential maximal function estimate

$$\|\mathcal{N}_p u\|_{L^p(\mathbb{R}^n)} \leq C\|f\|_{L^p(\mathbb{R}^n)}. \tag{1.4}$$

Similarly, the regularity problem (R_p), $1 < p < \infty$, is solvable for the operator $\mathbb{L}$ in $\mathbb{R}^{n+1}_+$ if for every $f \in L^2(\mathbb{R}^n) \cap \mathring{W}^{1,p}(\mathbb{R}^n)$ the solution to the equation $\mathbb{L}u = 0$ with the boundary data f given by the Poisson extension $u(x,t) = e^{-t\sqrt{L}}f(x)$, $x \in \mathbb{R}^n$, $t > 0$, satisfies the non-tangential maximal function estimate

$$\|\mathcal{N}_2(\nabla_{x,t}u)\|_{L^p(\mathbb{R}^n)} \leq C\|\nabla_x f\|_{L^p(\mathbb{R}^n)}. \tag{1.5}$$

The aforementioned L^p-based non-tangential maximal function is defined as

$$\mathcal{N}_p f(x) := \sup_{(y,t)\in\Gamma_\kappa(x)} \left(\fint\!\!\fint_{D((y,t),\kappa t)} |u(z,s)|^p\, dzds \right)^{\frac{1}{p}}, \qquad x \in \mathbb{R}^n, \quad 1 < p < \infty, \tag{1.6}$$

where $\Gamma_\kappa(x) := \{(y,t) \in \mathbb{R}^{n+1}_+ : |x - y| < \kappa t\}$, $x \in \mathbb{R}^n$, $D((y,t),\kappa t)$ is a ball in $\mathbb{R}^{n+1}$ centered at $(y,t) \in \mathbb{R}^{n+1}_+$ with the radius κt, and $0 < \kappa < 1$ is some small constant. The notation $\nabla_{x,t}$ stands for the full gradient in $\mathbb{R}^{n+1}$, and ∇_x corresponds to the gradient in x. We will simply write ∇ for both whenever the variables of differentiation are clear from the context. Finally, $\mathring{W}^{1,p}(\mathbb{R}^n)$, $1 < p < \infty$, stands for the Sobolev space given by the completion of $C_0^\infty(\mathbb{R}^n)$ in the norm $\|g\|_{\mathring{W}^{1,p}(\mathbb{R}^n)} = \|\nabla g\|_{L^p(\mathbb{R}^n)}$.

In the present paper we show that for every block operator $\mathbb{L}$ as above the range of p for which the regularity problem (R_p) is solvable in $\mathbb{R}^{n+1}_+$ includes $\left(\max\left\{\frac{2n}{n+4}, 1\right\}, 2\right]$. On the other hand, for every $r > \frac{2n}{n-2}$ there exists an elliptic operator L in $\mathbb{R}^n$ whose Poisson semigroup does not obey the estimate (1.4). Hence, *for every* $p \in \left(\max\left\{\frac{2n}{n+4}, 1\right\}, \frac{2n}{n+2}\right)$*, there exists a block operator* $\mathbb{L}$ *such that* (R_p) *for* $\mathbb{L}$ *is solvable but* $(D_{p'})$ *for* $\mathbb{L}^*$ *is not.*

The approach we develop stems from the theory of Hardy spaces associated to general elliptic operators [14], [15], some earlier L^p results for the corresponding heat semigroup and Riesz transform [3], and the examples concerning failure of the de Giorgi-Nash-Moser bounds for weak solutions of elliptic differential equations [11].

Let us discuss informally some intuition behind the main results of this paper. First of all, the reverse Riesz transform estimates in [3] ascertain that

$$\|\sqrt{L}f\|_{L^p(\mathbb{R}^n)} \leq C\|\nabla f\|_{L^p(\mathbb{R}^n)}, \qquad \forall\, f \in \mathring{W}^{1,p}(\mathbb{R}^n), \quad p \in \left(\max\left\{\tfrac{2n}{n+4}, 1\right\}, 2\right]. \tag{1.7}$$

On the other hand,

$$\mathbb{A}(x)\,\nabla_{x,t}u(x,t)\cdot\vec{N}(x) = -\partial_t u(x,t) = -\partial_t e^{-t\sqrt{L}}f(x) = \sqrt{L}e^{-t\sqrt{L}}f(x), \qquad x \in \mathbb{R}^n, \tag{1.8}$$

where $\vec{N}$ is an outward unit normal to the boundary of the domain, that is, in our case, the unit vector in the direction opposite to t. Therefore, passing to the limit as $t \to 0$ in (1.8), we reveal that $\sqrt{L}f$ gives the conormal derivative of u on $\mathbb{R}^n$. Hence, (1.7) implies that on the boundary the conormal derivative of the solution is bounded by the tangential derivative ∇f (cf. Rellich identity). This suggests that the regularity problem must be solvable.

We note, however, that the estimate (1.5) is stronger than (1.7). It is, in fact, closer in spirit to the Riesz transform characterization of Hardy spaces based on L [15], since the L^p norm of the non-tangential maximal function naturally brings up the norm in the Hardy space. Starting from these considerations, the actual estimate (1.5) will be established in Section 4.

As for the Dirichlet problem, the bound (1.4) is closely related to the uniform estimate

$$\|e^{-t\sqrt{L}}f\|_{L^p(\mathbb{R}^n)} \leq C\|f\|_{L^p(\mathbb{R}^n)} \quad \text{for all} \quad t > 0. \tag{1.9}$$

In the recent work [15], resting on [11] and some ideas from [6], the authors showed that for every $p > \frac{2n}{n-2}$ there exists an elliptic operator L such that its heat semigroup $\{e^{-tL}\}_{t>0}$ is not uniformly bounded in L^p. This fact led us to believe that for such p and L neither (1.9) nor (1.4) would be satisfied.

Indeed, in Section 3 we prove that any of the estimates (1.9) or (1.4) with $p > \frac{2n}{n-2}$ implies that

$$L^{-1} : L^q(\mathbb{R}^n) \longrightarrow L^r(\mathbb{R}^n), \qquad \tfrac{n}{q} - \tfrac{n}{r} = 2, \qquad \tfrac{2n}{n+2} \leq q < r < p. \tag{1.10}$$

Departing from (1.10), one can show that in a unit ball B_1 the $\mathring{W}^{1,2}(B_1)$ solution to the equation $Lu = f$, $f \in C_0^\infty(B_1)$, belongs to all L^r spaces, $\frac{2n}{n-2} \leq r < p$, which contradicts the calculations in [11] and ultimately yields the negative results for the Dirichlet problem.

Finally, let us turn to the Neumann case. We say that the Neumann problem (N_p), $1 < p < \infty$, is solvable for the operator $\mathbb{L}$ in $\mathbb{R}^{n+1}_+$ if for every $g \in L^2(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$ the solution to the equation $\mathbb{L}u = 0$ given by the Poisson extension $u(x,t) = e^{-t\sqrt{L}}f(x)$, $x \in \mathbb{R}^n$, $t > 0$, for the function $f = (\sqrt{L})^{-1}g$, satisfies the non-tangential maximal function estimate

$$\|\mathcal{N}_2(\nabla_{x,t}u)\|_{L^p(\mathbb{R}^n)} \leq C\|g\|_{L^p(\mathbb{R}^n)}. \tag{1.11}$$

Note that according to (1.8) the function $g = \sqrt{L}f$ represents the conormal derivative of the solution at the boundary, that is, the Neumann data.

It is known that even for the Laplacian on Lipschitz domains the solvability of (R_p) does not imply solvability of (N_p), at least when $p > 2$ ([20]). As we mentioned earlier, for $p < 2$ this question remained open. On the other hand, in [4] the authors established that for real symmetric, constant or block operators, as well as for their small perturbations (R_2) for the operator L is equivalent to (N_2) for L^*. They use some auxiliary, slightly unconventional, Neumann datum, but in our context of block operators it coincides with the usual one. Furthermore, according to [21], for real, t-independent matrices in dimension two $(D_{p'})$ for L^* implies (R_p) for L and, at the same time, (N_p) for $\widetilde{L}^*$, where $\widetilde{L}^*$ is the operator associated to matrix $A^*/\det(A)$, A being the matrix of L.

Here we show that the regularity and Neumann problems are not necessarily solvable simultaneously, even for $p < 2$. More precisely, the solvability of (R_p) for $\mathbb{L}$ does not necessarily imply solvability of (N_p) for $\mathbb{L}^*$.

To conclude, we would like to point out that the known results for general (complex) elliptic operators or even real non-symmetric ones often address the operators with the coefficients independent on the transversal direction ([4], [2],[21], [18]). Moreover, some regularity in t is necessary for the solvability of boundary problems (see [8]). The counterexample in the present paper is built for a block operator, which is obviously t-independent. Hence, it remains relevant in the setting of aforementioned papers. It is also worth mentioning that both in [4] and in [2] the authors work with the perturbation of the entire "package" of simultaneous solvability of Dirichlet, Neumann and Regularity problems. Part of the motivation of this work came from the effort to understand whether for $p \neq 2$ anything in the Dirichlet-Neumann-Regularity package generally comes "for free", whether solvability of one of the problems always implies solvability of some other.

2. Preliminaries

Since much of the discussion in the paper will be revolving around the solution to an elliptic problem for the block operator given by the Poisson semigroup and the corresponding heat semigroup, we shall briefly list their main properties.

The operator L defined by (1.1)–(1.3) can be viewed as an accretive operator in $L^2(\mathbb{R}^n)$. It is of type ω on $L^2(\mathbb{R}^n)$ for some $\omega \in [0, \pi/2)$. In particular, $-L$ generates a complex semigroup (referred to as a heat semigroup) which extends to an analytic semigroup $\{e^{-zL}\}$ of contractions on $L^2(\mathbb{R}^n)$ in the sector $\Sigma^0_{\pi/2-\omega} = \{z \in \mathbb{C} \setminus \{0\} : |\arg z| < \pi/2 - \omega\}$. Furthermore, L possesses a maximal accretive square root $\sqrt{L}$ which generates an L^2-contracting semigroup $\{e^{-t\sqrt{L}}\}_{t>0}$ (hereafter referred to the Poisson semigroup of L). See [16], [23] for details.

We say that the family of operators $\{S_t\}_{t>0}$ is bounded in $L^p(\mathbb{R}^n)$, $1 \leq p \leq \infty$, if the operator S_t is bounded in $L^p(\mathbb{R}^n)$ for every $t > 0$ and the norm of S_t in $L^p(\mathbb{R}^n)$ does not depend on $t > 0$.

Proposition 2.1. *Let L be an elliptic operator on $\mathbb{R}^n$, $n \geq 2$, satisfying (1.1)–(1.3). For any $n \geq 3$ there exist numbers $p_-(L)$ and $p_+(L)$ with*

$$(2.1) \qquad 1 \leq p_-(L) < \tfrac{2n}{n+2} \qquad \textit{and} \qquad \tfrac{2n}{n-2} < p_+(L) \leq \infty,$$

such that the heat semigroup $\{e^{-tL}\}_{t>0}$ *is bounded in* $L^p(\mathbb{R}^n)$ *whenever* $p_-(L) < p < p_+(L)$*. If* $n = 2$*, the heat semigroup is bounded in all* $L^p(\mathbb{R}^n)$*,* $1 < p < \infty$*.*

Moreover, the bounds (2.1) *are sharp, in the following sense. Given any* $\widetilde{p}_-$ *with* $1 \leq \widetilde{p}_- < \frac{2n}{n+2}$*,* $n \geq 3$*, there exists an operator L such that the heat semigroup* $\{e^{-tL}\}_{t>0}$ *is not bounded in* $L^{\widetilde{p}_-}$*. And similarly, given any* $\widetilde{p}_+$ *with* $\frac{2n}{n-2} < \widetilde{p}_+ \leq \infty$*, there exists an operator L such that the heat semigroup* $\{e^{-tL}\}_{t>0}$ *is not bounded in* $L^{\widetilde{p}_+}$*.*

The estimates (2.1) on the range of boundedness of the heat semigroup were established in [3]. Their sharpness was proved in [15] by an argument relying on some ideas from [6] and the example from [11]. The latter will be discussed in details in §3, and will play a crucial role in our study of the Dirichlet problem. In the remainder of the paper, we will denote by $(p_-(L), p_+(L))$ the *maximal* interval of exponents such that heat semigroup is bounded in $L^p(\mathbb{R}^n)$ for all p in this range. In particular, $p_-(L) := 1$ and $p_+(L) := \infty$ when $n = 2$.

We say that family of operators $\{S_t\}_{t>0}$ satisfies $L^p - L^q$ off-diagonal estimates, $1 < p, q < \infty$, if for arbitrary closed sets $E, F \subset \mathbb{R}^n$

$$\|S_t f\|_{L^q(F)} \leq C t^{\frac{1}{2}\left(\frac{n}{q}-\frac{n}{p}\right)} e^{-\frac{\operatorname{dist}(E,F)^2}{ct}} \|f\|_{L^p(E)}, \tag{2.2}$$

for every $t > 0$ and every $f \in L^p(\mathbb{R}^n)$ supported in E.

Lemma 2.2. *For every p and q such that* $p_-(L) < p \leq q < p_+(L)$ *the families* $\{e^{-tL}\}_{t>0}$*,* $\{tLe^{-tL}\}_{t>0}$ *satisfy* $L^p - L^q$ *off-diagonal estimates. In particular, the operators* e^{-tL}*,* tLe^{-tL}*,* $t > 0$*, map* $L^p(\mathbb{R}^n)$ *to* $L^q(\mathbb{R}^n)$ *with the norm controlled by* $Ct^{\frac{1}{2}\left(\frac{n}{q}-\frac{n}{p}\right)}$*.*

This result was essentially established in [3]. See also [15] for a detailed discussion.

Note that $tLe^{-tL} = -t\partial_t e^{-tL}$, $t > 0$. The properties of the derivatives in x of the heat semigroup are somewhat different. They are closely connected to the properties of the corresponding Riesz transform.

Proposition 2.3. *Let L be an elliptic operator on* $\mathbb{R}^n$*,* $n \geq 2$*, satisfying* (1.1)–(1.3)*, and let* $p_-(L)$ *denote, as before, the lower bound for the interval of boundedness of the heat semigroup. Then there exists a number* $\varepsilon(L) > 0$ *such that the family* $\{\sqrt{t}\,\nabla_x e^{-tL}\}_{t>0}$ *is bounded in* $L^p(\mathbb{R}^n)$ *whenever* $p_-(L) < p < 2 + \varepsilon(L)$*.*

The bound $\varepsilon(L) > 0$ *is sharp for all* $n \geq 2$*, in the sense that for every* $p > 2$ *there exists an operator L such that the family* $\{\sqrt{t}\,\nabla_x e^{-tL}\}_{t>0}$ *is not bounded in* $L^p(\mathbb{R}^n)$*.*

Moreover, the family $\{\sqrt{t}\,\nabla_x e^{-tL}\}_{t>0}$ *satisfies* $L^p - L^q$ *off-diagonal estimates whenever* $p_-(L) < p \leq q < 2 + \varepsilon(L)$*. In particular, the operators* $\sqrt{t}\,\nabla_x e^{-tL}$*,* $t > 0$*, map* $L^p(\mathbb{R}^n)$ *to* $L^q(\mathbb{R}^n)$ *with the norm controlled by* $Ct^{\frac{1}{2}\left(\frac{n}{q}-\frac{n}{p}\right)}$*.*

The Proposition was proved in [3]. The sharpness of the bound $\varepsilon(L) > 0$ relies on the Meyer's counterexample (see [6]). In fact, it was also shown in [3] that the lower bound for the interval of boundedness of the family $\{\sqrt{t}\,\nabla_x e^{-tL}\}_{t>0}$ must coincide with the lower bound of the interval boundedness of the heat semigroup. Hence, without any ambiguity we can denote by $(p_-(L), 2 + \varepsilon(L))$ the maximal interval of exponents such that the family $\{\sqrt{t}\,\nabla_x e^{-tL}\}_{t>0}$ is bounded in $L^p(\mathbb{R}^n)$ for all p in this range.

We shall also need the following version of the off-diagonal bounds in the Sobolev spaces. Let us denote by $B(y, t)$ the ball in $\mathbb{R}^n$ centered at $y \in \mathbb{R}^n$ with radius $t > 0$. Then the following estimate holds.

Lemma 2.4. *Let L be an elliptic operator on $\mathbb{R}^n$, $n \geq 2$, satisfying (1.1)–(1.3). Then for any $x \in \mathbb{R}^n$ and $t > 0$,*

$$\left(\fint_{B(x,t)} |\nabla_y e^{-t^2 L} f(y)|^q \, dy \right)^{\frac{1}{q}} \leq C \sum_{j=1}^{\infty} 2^{-jN} \left(\fint_{B(x,2^{j+1}t)} |\nabla f(y)|^p \, dy \right)^{\frac{1}{p}}, \tag{2.3}$$

where p, q are such that $p_-(L) < p \leq q < 2 + \varepsilon(L)$ and N is any natural number.

The Lemma has been proved for $q = 2$ in [3] as a part of the argument for Lemma 4.8, and the proof for general q is virtually verbatim.

We would like also to list the direct and reverse estimates on the Riesz transform. The main result in this regard is the Kato estimate (proved in [5]) which ascertains that the domain of the square root of an elliptic operator is the Sobolev space $W^{1,2}(\mathbb{R}^n)$ and

$$\| \sqrt{L} f \|_{L^2(\mathbb{R}^n)} \approx \| \nabla f \|_{L^2(\mathbb{R}^n)} \quad \text{for every} \quad f \in \mathring{W}^{1,2}(\mathbb{R}^n). \tag{2.4}$$

However, later on these estimates were extended to other L^p spaces. The precise results are as follows.

Proposition 2.5. *Let L be an elliptic operator on $\mathbb{R}^n$, satisfying (1.1)–(1.3). The interior of the maximum interval of exponents p in $(1, \infty)$ such that*

$$\nabla L^{-1/2} : L^p(\mathbb{R}^n) \longrightarrow L^p(\mathbb{R}^n), \tag{2.5}$$

coincides with $(p_-(L), 2 + \varepsilon(L))$. Conversely,

$$L^{1/2} : \mathring{W}^{1,p}(\mathbb{R}^n) \longrightarrow L^p(\mathbb{R}^n), \quad \textit{whenever} \quad \max\left\{1, \frac{np_-(L)}{n + p_-(L)}\right\} < p < p_+(L). \tag{2.6}$$

Clearly, (2.5) corresponds to the estimate

$$\|\nabla L^{-1/2} f\|_{L^p(\mathbb{R}^n)} \leq C \|f\|_{L^p(\mathbb{R}^n)} \quad \forall \ f \in L^p(\mathbb{R}^n), \tag{2.7}$$

while (2.6) amounts to

$$\|L^{1/2} f\|_{L^p(\mathbb{R}^n)} \leq C \|\nabla f\|_{L^p(\mathbb{R}^n)} \quad \forall \ f \in \mathring{W}^{1,p}(\mathbb{R}^n), \tag{2.8}$$

Following [3], we refer to (2.7) and (2.8) as direct and reverse estimates on the Riesz transform, respectively.

A part of the proposition above regarding (2.5), (2.7) is due to [7], [13], [3]. The reverse estimates (2.6), (2.8) were treated in [3].

To finish with the preliminaries, we recall the well-known Caccioppoli inequality.

Lemma 2.6. *Let L be an elliptic operator defined by (1.1)–(1.3). Suppose $Lu = 0$ in the ball $B_{2r}(x_0)$. Then there exists $C = C(\lambda, \Lambda) > 0$ such that*

$$\int_{B_r(x_0)} |\nabla u(x)|^2 \, dx \leq \frac{C}{r^2} \int_{B_{2r}(x_0)} |u(x)|^2 \, dx. \tag{2.9}$$

Finally, we would like to say a few words about the definitions of (D_p), (N_p) and (R_p). The non-tangential maximal function $\mathcal{N}_p$ we use in the definition of the Dirichlet problem is a substitute of the classical $\mathcal{N}_\infty$, given by supremum over a cone. Since in the present context even the solution to the Dirichlet problem is not necessarily locally bounded, one measures the supremum of averages of u over the interior balls rather than the supremum of u itself. In the L^2 context this already has been done in [4], [2].

Indeed, the gradient of the solution generally fails to be locally bounded even for real symmetric operators. The use of $\mathcal{N}_2$ for the regularity and Neumann problems in this context goes back to [19] and is, by now, traditional.

One could, in principle, use the $\mathcal{N}_2$ maximal function rather than $\mathcal{N}_p$ in the definition of the Dirichlet problem as well. However, this would allow for solutions of the Dirichlet problem (D_p) that are not locally in L^p which seems somewhat unnatural. In the end of Section 3 we will show that there exists a solution u with the L^p data for which $\|\mathcal{N}_2 u\|_{L^p(\mathbb{R}^n)}$ is finite, but $u \notin L^p_{loc}$, to justify the validity of this concern. Note that this problem does not arise for the real symmetric operators. In that setting, even if ∇u is not locally in L^p, the solution u itself is still bounded and belongs to all L^p_{loc}, $1 < p \leq \infty$, by the de Giorgi-Nash-Moser theory.

Next, the equality $\mathbb{L}u = 0$ in the definitions of the Dirichlet and regularity problems is understood in the weak sense. Since for $f \in L^2(\mathbb{R}^n)$ the Poisson semigroup $e^{-t\sqrt{L}}f \in L^2(\mathbb{R}^n)$ uniformly in $t > 0$, we have $u = e^{-\cdot\sqrt{L}}f \in L^2_{loc}(\mathbb{R}^{n+1}_+)$ and therefore, $u \in W^{1,2}_{loc}(\mathbb{R}^{n+1}_+)$ by Caccioppoli inequality. Hence, the usual weak definition makes sense. Furthermore, the limit as $t \to 0$ of $e^{-t\sqrt{L}}f$ is equal to f in $L^2(\mathbb{R}^n)$. This follows from the standard results of holomorphic functional calculus of L (see [1]), and this is the sense in which we initially understand the boundary data of u. If, for example, $L = -\Delta$ and u is, respectively, the Poisson extension of a continuous function f then the limit exists pointwise a.e., and gives the usual restriction to the boundary (see [24], p.62).

As regards the Neumann problem, for every $g \in L^2(\mathbb{R}^n)$ the function $f = (\sqrt{L})^{-1}g$ exists and belongs to $\mathring{W}^{1,2}(\mathbb{R}^n)$ by (2.4). We can use the boundedness of the heat semigroup in $\mathring{W}^{1,2}(\mathbb{R}^n)$ to establish boundedness of the Poisson semigroup in $\mathring{W}^{1,2}(\mathbb{R}^n)$ (see, e.g., (3.3)–(3.4) for an analogous argument), and the former fact can be found, e.g., in [3]. Therefore, $e^{-t\sqrt{L}}f$ exists for every $t > 0$ and belongs to $\mathring{W}^{1,2}(\mathbb{R}^n)$ with the norm independent on t. Then $\nabla_{x,t}e^{-t\sqrt{L}}f$ is back in $L^2(\mathbb{R}^n)$ by (2.4). This also formally justifies the calculation in (1.8). Then the limit as $t \to 0$ is taken in L^2 sense, as above for the Dirichlet problem.

Since we concentrate on counterexamples in the present paper and do not strive for general theory, we do not discuss further possible consequences of (N_p), (D_p), (R_p), such as convergence of solutions in the non-tangential sense, existence and uniqueness of solutions for any given data in L^p etc.

3. Counterexample to the solvability of the Dirichlet problem

As we explained in the introduction, the counterexample for the Dirichlet problem is built on the observation that for a block matrix in $\mathbb{R}^{n+1}$ the solution is given by the Poisson semigroup, and the Poisson semigroup is not necessarily bounded in L^p for p sufficiently far from 2, nor it satisfies the non-tangential maximal function estimate (1.4). To a large extent the argument here follows the lines of the corresponding one in [15], which in turn rested on [11] and [6]. In [15] we proved that for every $r > \frac{2n}{n-2}$ there is an elliptic operator whose heat semigroup is not bounded in L^r. Here we aim to show that the Poisson semigroup of such an operator does not satisfy (1.4).

Proposition 3.1. *Let L be an elliptic operator on $\mathbb{R}^n$ defined by (1.1)–(1.3), and assume that for some $q \in [2,\infty)$ the Poisson semigroup $\{e^{-t\sqrt{L}}\}_{t>0}$ extends to an operator on $L^q(\mathbb{R}^n)$ with the estimate*

$$\sup_{t>0} \fint_{|t-s|<\kappa t} \|e^{-s\sqrt{L}} f\|_{L^q(\mathbb{R}^n)}\, ds \le C\|f\|_{L^q(\mathbb{R}^n)}, \quad f \in L^q(\mathbb{R}^n), \tag{3.1}$$

for some $\kappa < 1$. Then

$$L^{-\alpha} : L^p(\mathbb{R}^n) \longrightarrow L^r(\mathbb{R}^n), \qquad \alpha = \frac{1}{2}\left(\frac{n}{p} - \frac{n}{r}\right), \tag{3.2}$$

for $p_-(L) < p < r < q$.

The Proposition has an analogue for the heat semigroup and we follow its proof in [3], tracking the necessary modifications.

Proof. First of all, for any p, r such that $p_-(L) < p < r < p_+(L)$ the heat semigroup is $L^p - L^r$ bounded with the norm controlled by $Ct^{\frac{1}{2}\left(\frac{n}{r}-\frac{n}{p}\right)}$. This fact together with the subordination formula

$$e^{-t\sqrt{L}} f = C\int_0^\infty \frac{e^{-u}}{\sqrt{u}}\, e^{-\frac{t^2 L}{4u}} f\, du, \tag{3.3}$$

can be used to prove that within the same range for p and r the Poisson semigroup $\{e^{-t\sqrt{L}}\}_{t>0}$ is $L^p - L^r$ bounded with the norm controlled by $Ct^{\frac{n}{r}-\frac{n}{p}}$. Indeed, by Minkowski inequality

$$\begin{aligned} \left\|e^{-t\sqrt{L}} f\right\|_{L^r(\mathbb{R}^n)} &\le C\int_0^\infty \frac{e^{-u}}{\sqrt{u}} \left\|e^{-\frac{t^2L}{4u}} f\right\|_{L^r(\mathbb{R}^n)} du \\ &\le C\int_0^\infty \frac{e^{-u}}{\sqrt{u}} \left(\frac{t^2}{4u}\right)^{\frac{1}{2}\left(\frac{n}{r}-\frac{n}{p}\right)} \|f\|_{L^p(\mathbb{R}^n)}\, du \le Ct^{\frac{n}{r}-\frac{n}{p}}\|f\|_{L^p(\mathbb{R}^n)}, \quad t > 0. \end{aligned} \tag{3.4}$$

Hence, for all $p_-(L) < p < r < p_+(L)$

$$\fint_{|t-s|<\kappa t} \|e^{-s\sqrt{L}} f\|_{L^r(\mathbb{R}^n)}\, ds \le Ct^{\frac{n}{r}-\frac{n}{p}}\|f\|_{L^p(\mathbb{R}^n)}, \qquad t > 0. \tag{3.5}$$

Furthermore, using interpolation of this property with (3.1), composition (the multiplication property of the semigroup) and (3.4) one can establish that (3.5) holds for all p, r such that $p_L < p \le r < q$.

Then by the H^∞ functional calculus (see [23], [1]) for any $\alpha > 0$ we can write

$$f = \frac{1}{\Gamma(2\alpha)}\int_0^\infty t^{2\alpha-1} L^\alpha e^{-t\sqrt{L}} f\, dt, \tag{3.6}$$

where the integral converges strongly in $L^2(\mathbb{R}^n)$. Now let

$$T_{\varepsilon,R}f := \frac{1}{\Gamma(2\alpha)}\int_\varepsilon^R t^{2\alpha-1}e^{-t\sqrt{L}}f\,dt. \tag{3.7}$$

We shall prove that the operators $T_{\varepsilon,R}$ are $L^p - L^r$ bounded with the norm independent on ε, R for all p, r as in (3.2), and then pass to the limit to establish (3.2).

Let $p_-(L) < p < r^- < r < r^+ < q$ and assume that $f \in L^p(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ with $\|f\|_{L^p(\mathbb{R}^n)} = 1$. Then for any $a > \varepsilon$

$$\begin{aligned}
\left\|\int_\varepsilon^a t^{2\alpha-1}e^{-t\sqrt{L}}f\,dt\right\|_{L^{r^-}(\mathbb{R}^n)} &\le \int_\varepsilon^a t^{2\alpha-1}\left\|e^{-t\sqrt{L}}f\right\|_{L^{r^-}(\mathbb{R}^n)}dt \\
&\le \int_\varepsilon^a \fint_{s\in\left(\frac{t}{1+\kappa},\frac{t}{1-\kappa}\right)} t^{2\alpha-1}\left\|e^{-t\sqrt{L}}f\right\|_{L^{r^-}(\mathbb{R}^n)}ds dt \le C\int_{s\in\left(\frac{\varepsilon}{1+\kappa},\frac{a}{1-\kappa}\right)} s^{2\alpha-1}\fint_{|t-s|<\kappa s}\left\|e^{-t\sqrt{L}}f\right\|_{L^{r^-}(\mathbb{R}^n)}dtds \\
&\le C\int_{s\in\left(\frac{\varepsilon}{1+\kappa},\frac{a}{1-\kappa}\right)} s^{\frac{n}{p}-\frac{n}{r}-1}s^{\frac{n}{r^-}-\frac{n}{p}}\|f\|_{L^p(\mathbb{R}^n)}\,ds \le Ca^{\frac{n}{r^-}-\frac{n}{r}}.
\end{aligned} \tag{3.8}$$

Similarly, for any $a < R$

$$\left\|\int_a^R t^{2\alpha-1}e^{-t\sqrt{L}}f\,dt\right\|_{L^{r^+}(\mathbb{R}^n)} \le Ca^{\frac{n}{r^+}-\frac{n}{r}}. \tag{3.9}$$

Next, for all $\lambda > 0$

$$|\{x\in\mathbb{R}^n : T_{\varepsilon,R}f(x) > \lambda\}| \le |\{x\in\mathbb{R}^n : T_{\varepsilon,a}f(x) > \lambda/2\}| + |\{x\in\mathbb{R}^n : T_{a,R}f(x) > \lambda/2\}|, \tag{3.10}$$

where we let $T_{\varepsilon,a} = 0$ if $a < \varepsilon$ and $T_{a,R} = 0$ when $a > R$. Then by (3.8)–(3.9) and Chebyshev inequality the expression above is bounded by

$$C\lambda^{-r^-}a^{r^-\left(\frac{n}{r^-}-\frac{n}{r}\right)} + C\lambda^{-r^+}a^{r^+\left(\frac{n}{r^+}-\frac{n}{r}\right)}. \tag{3.11}$$

Now we choose a such that $a^n = \lambda^{-r}$. Then

$$|\{x\in\mathbb{R}^n : T_{\varepsilon,R}f(x) > \lambda\}| \le C\lambda^{-r}, \tag{3.12}$$

and hence, $T_{\varepsilon,R} \in L^{r,\infty}(\mathbb{R}^n)$. Combining this result with the Marcinkiewicz interpolation theorem, we conclude that for all p, r as in (3.2)

$$\|T_{\varepsilon,R}f\|_{L^r(\mathbb{R}^n)} \le C\|f\|_{L^p(\mathbb{R}^n)}, \quad \text{for all} \quad f\in L^p(\mathbb{R}^n)\cap L^2(\mathbb{R}^n), \tag{3.13}$$

with the constant C independent of ε, R.

It remains to pass to the limit as $\varepsilon \to 0$ and $R \to \infty$. Once again, we follow the procedure used in [3]. First of all, if $0 \le \alpha \le 1/2$ then L^α is an isomorphism from $\mathring{W}^{2\alpha,2}(\mathbb{R}^n)$ onto $L^2(\mathbb{R}^n)$, where $\mathring{W}^{2\alpha,2}(\mathbb{R}^n)$ is the Sobolev space given by the completion of $C_0^\infty(\mathbb{R}^n)$ in the norm $\|g\|_{\mathring{W}^{2\alpha,2}(\mathbb{R}^n)} = \|(-\Delta)^\alpha g\|_{L^2(\mathbb{R}^n)}$. For α complex with $\Re\alpha = 0$ it follows from the H^∞ functional calculus in L^2, for α complex with $\Re\alpha = 1/2$ it follows combining the functional calculus in L^2 with the Kato estimate

(2.4), and for the full range of α we then employ Stein's interpolation theorem. This observation, combined with the convergence of the integral in (3.7) in L^2 for all $f \in L^2$, implies that

$$\|L^{-\alpha} f - T_{\varepsilon,R} f\|_{\mathring{W}^{2\alpha,2}(\mathbb{R}^n)} \approx \left\| f - \frac{1}{\Gamma(2\alpha)} \int_\varepsilon^R t^{2\alpha-1} e^{-t\sqrt{L}} L^\alpha f\, dt \right\|_{L^2(\mathbb{R}^n)} \to 0 \text{ as } \varepsilon \to 0,\ R \to \infty. \tag{3.14}$$

Hence, by Sobolev embedding $\|L^{-\alpha} f - T_{\varepsilon,R} f\|_{L^s(\mathbb{R}^n)} \to 0$ for $s = \frac{2n}{n-4\alpha}$, $f \in L^2(\mathbb{R}^n)$. Together with the estimate (3.13) this completes the limiting procedure. Finally, when $\alpha > 1/2$, we simply write $L^{-\alpha}$ as a composition of smaller powers of L. $\square$

Proposition 3.2. [11] *Let $n \geq 3$. For any $q < n/2$ and $\lambda > 0$ there is an $n \times n$ matrix $A = A(q, \lambda)$ satisfying (1.1)–(1.2) and such that*

$$u(x) = \frac{x_1}{|x|^q} e^{i\lambda \ln |x|} \tag{3.15}$$

solves the equation $Lu = -\mathrm{div}(A\nabla u) = 0$ in $\mathbb{R}^n \setminus \{0\}$.

The example above was obtained in [11]. To be more precise, the matrix A defining the operator L has a form

$$A = \left\{ (\alpha + i)\delta_{jk} + \beta \frac{x_j x_k}{|x|^2} \right\}_{j,k=1}^n, \tag{3.16}$$

where $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{C}$ are some constants. It was established that for any fixed $\alpha \in \mathbb{R}$, $\lambda \neq 0$, $q \neq 0$ there exists $\beta = \beta(\alpha, q, \lambda)$ such that u in (3.15) solves the equation $-\mathrm{div}(A\nabla u) = 0$, and moreover, if $\alpha > 0$ is sufficiently small and $\beta = \beta(\alpha, q, \lambda)$, then the corresponding matrix A satisfies the ellipticity conditions.

Proposition 3.2 shows, in particular, that there is a weak solution to an elliptic equation which is not locally Hölder continuous, and hence, does not satisfy de Giorgi-Nash-Moser estimates. Parenthetically, we point out that for $n \geq 5$ this fact has been established earlier in [22], by a different method. However, the approach in [22] is not as explicit, and not suitable for the purposes of this paper.

Having at hand Proposition 3.2, it is not hard to pass to the following result.

Lemma 3.3. *For every $r > \frac{2n}{n-2}$ there exists an elliptic operator L and $f \in C_0^\infty(\mathbb{R}^n)$ such that $u = L^{-1} f \notin L^r_{loc}(\mathbb{R}^n)$.*

Proof. If $r > \frac{2n}{n-2}$ then $1 + \frac{n}{r} < \frac{n}{2}$. Pick any q such that $1 + \frac{n}{r} < q < \frac{n}{2}$. Then, according to Proposition 3.2, there exists an elliptic operator L such that u given by (3.15) is a solution of $Lu = 0$ in $\mathbb{R}^n \setminus \{0\}$.

Next, take some $\phi \in C_0^\infty(\mathbb{R}^n)$, supported in the unit ball B_1, such that $\phi = 1$ in the ball of radius $1/2$ centered at the origin. Then $\nabla\phi \in C_0^\infty(B_1)$ and $\nabla\phi = 0$ in a neighborhood of 0. Since the only singularity of u (and of A) is at 0, we have, in the usual weak sense,

$$L(u\phi) = -\mathrm{div}(A\nabla(u\phi)) = -\mathrm{div}(Au\nabla\phi) + A\nabla u \cdot \nabla\phi =: f \in C_0^\infty(B_1), \tag{3.17}$$

where the second equality follows from the fact that $Lu = 0$.

Note that $f \in C_0^\infty(\mathbb{R}^n)$, but however, $L^{-1}f = u\phi$ does not belong to L^r in any neighborhood of the origin, since $r(1-q)+n<0$ by our choice of q. $\square$

Proposition 3.4. *For every $r > \frac{2n}{n-2}$ there exists a block elliptic operator $\mathbb{L}$ in $\mathbb{R}^{n+1}_+$ such that the solution to the Dirichlet problem given by the corresponding Poisson semigroup does not obey the estimate*

$$\sup_{t>0} \fint_{|t-s|<\kappa t} \|u(\cdot,s)\|_{L^r(\mathbb{R}^n)}\, ds \le C\|f\|_{L^r(\mathbb{R}^n)}, \quad \text{for all} \quad f \in L^r(\mathbb{R}^n)\cap L^2(\mathbb{R}^n). \tag{3.18}$$

In particular, the corresponding (D_r) is not solvable in $\mathbb{R}^{n+1}$.

Proof. Fix $r > \frac{2n}{n-2}$. If the estimate (3.18) is satisfied for $u(x,t) = e^{-t\sqrt{L}}f(x)$, $x \in \mathbb{R}^n$, $t>0$, for some elliptic operator L, then by Proposition 3.1

$$L^{-1} : L^{p_1}(\mathbb{R}^n) \longrightarrow L^{p_2}(\mathbb{R}^n), \quad \text{whenever} \quad p_-(L) < p_1 < p_2 < r \quad \text{with} \quad 1 = \frac{1}{2}\left(\frac{n}{p_1} - \frac{n}{p_2}\right). \tag{3.19}$$

Thus, given any $f \in C_0^\infty(\mathbb{R}^n)$, the function $L^{-1}f$ must belong to all $L^{p_2}(\mathbb{R}^n)$, $\frac{2n}{n-2} < p_2 < r$. However, for every such p_2 there is a counterexample provided by Lemma 3.3. Any of those counterexamples will give rise to a block operator $\mathbb{L}$ for which (3.18) fails. This proves the first statement of the Proposition.

Now assume that (D_r) for the operator $\mathbb{L}$ is solvable. By definition, this means that $u(\cdot,t) = e^{-t\sqrt{L}}f$, $t>0$, satisfies the non-tangential maximal function estimate (1.4) in L^r.

However, for every $t>0$

$$\begin{aligned}
\|u(\cdot,t)\|_{L^r(\mathbb{R}^n)} &= \left(\int_{x\in\mathbb{R}^n} |u(x,t)|^r\, dx\right)^{\frac{1}{r}} = \left(\int_{x\in\mathbb{R}^n} \frac{C}{t^n}\int_{y\in\mathbb{R}^n:\, |x-y|<c\kappa t} |u(x,t)|^r\, dydx\right)^{\frac{1}{r}} \\
&= \left(\int_{y\in\mathbb{R}^n} \frac{C}{t^n}\int_{x\in\mathbb{R}^n:\, |x-y|<c\kappa t} |u(x,t)|^r\, dxdy\right)^{\frac{1}{r}} = C\left\|\left(\fint_{x\in\mathbb{R}^n:\, |x-\cdot|<c\kappa t} |u(x,t)|^r\, dx\right)^{\frac{1}{r}}\right\|_{L^r(\mathbb{R}^n)}.
\end{aligned} \tag{3.20}$$

Then

$$\begin{aligned}
\sup_{t>0} \fint_{|t-s|<c\kappa t} \|u(\cdot,s)\|_{L^r(\mathbb{R}^n)}\, ds &\le C \sup_{t>0}\left(\fint_{|t-s|<c\kappa t} \|u(\cdot,s)\|^r_{L^r(\mathbb{R}^n)}\, ds\right)^{\frac{1}{r}} \\
&\le C\sup_{t>0}\left\|\left(\fint_{|t-s|<c\kappa t}\fint_{|x-\cdot|<c\kappa t} |u(x,s)|^r\, dxds\right)^{\frac{1}{r}}\right\|_{L^r(\mathbb{R}^n)} \le C\|\mathcal{N}_r f\|_{L^r(\mathbb{R}^n)},
\end{aligned} \tag{3.21}$$

provided that c is small enough, depending on the dimension only. Hence, the estimate ((1.4)) in L^r implies (3.18) with $c\kappa$ in place of κ. This leads us to contradiction, and finishes the proof of the Proposition. $\square$

Finally, we return to the issue raised in the end of Section 2 in connection with the choice of the non-tangential maximal function. Below we demonstrate that finiteness of $\|\mathcal{N}_2 u\|_{L^r(\partial\Omega)}$ does not necessarily imply finiteness of $\|\mathcal{N}_r u\|_{L^r(\partial\Omega)}$, even when u is a solution to the Dirichlet problem for an elliptic equation with the boundary data in $L^r(\partial\Omega)$. In fact, in our example Ω is a unit ball in $\mathbb{R}^n$

centered at the origin, f is continuous on $\partial\Omega$, and u is continuous in a neighborhood of $\partial\Omega$. Thus, the Dirichlet data can be interpreted in the usual sense of restriction to the boundary: $f = u\big|_{\partial\Omega}$. Also, the proper analogue of the non-tangential maximal function for a bounded domain $\Omega \subset \mathbb{R}^n$ is given by

$$(3.22)\quad \mathcal{N}_p f(x) := \sup_{y\in\Gamma(x)} \left(\fint_{|z-y|\leq\kappa\delta(y)} |u(z)|^p \, dz \right)^{\frac{1}{p}}, \qquad x \in \partial\Omega, \quad 1 < p < \infty,$$

where $\delta(x)$, $x \in \Omega$, denotes the distance from x to the boundary $\partial\Omega$, $\Gamma_\kappa(x) := \{y \in \Omega : |x-y| < \sqrt{\kappa^2+1}\,\delta(y)\}$, $x \in \partial\Omega$, is a family of the non-tangential approach regions, and $\kappa = \kappa(\partial\Omega)$ is a small constant between 0 and 1.

Lemma 3.5. *For any $r > \frac{2n}{n-2}$ there exists a bounded smooth domain $\Omega \subset \mathbb{R}^n$, an elliptic operator L and $f \in L^r(\partial\Omega)$ such that a solution u to the Dirichlet problem $Lu = 0$, $u\big|_{\partial\Omega} = f$, satisfies $\|\mathcal{N}_2 u\|_{L^r(\partial\Omega)} < \infty$, but $u \notin L^r_{loc}(\Omega)$, in particular, $\|\mathcal{N}_r u\|_{L^r(\partial\Omega)}$ is not finite.*

Proof. The example goes back to Proposition 3.2. Let Ω be a unit ball in $\mathbb{R}^n$ (denoted by B_1). Let q be such that $1 + \frac{n}{r} < q < \frac{n}{2}$, and u and L satisfy Proposition 3.2. Then $u \notin L^r_{loc}(\Omega)$. Let us prove that $\|\mathcal{N}_2 u\|_{L^r(\partial\Omega)}$ is, nonetheless, finite.

First of all, there exists a constant $C > 0$ such that

$$(3.23)\quad \|\mathcal{N}_2 u\|_{L^r(\partial\Omega)} \leq C\|u\|_{L^2(\Omega)} + \|\mathcal{N}_2' u\|_{L^r(\partial\Omega)},$$

where $\mathcal{N}_2'$ is defined by (3.22) with the supremum taken over "truncated cones" $\Gamma_\kappa'(x) := \{y \in \Omega \setminus B_{3/4} : |x-y| < \sqrt{\kappa^2+1}\,\delta(y)\}$, $x \in \partial\Omega$.

Since $q < \frac{n}{2} + 1$, the quantity $\|u\|_{L^2(\Omega)}$ is finite. On the other hand, for every $y \in \Omega \setminus B_{3/4}$ the ball $\{z \in \Omega : |z-y| \leq \kappa\delta(y)\}$ is at least at distance $1/4$ from the origin, while its radius is smaller than $1/4$. Hence, in every such ball an L^2-average of $|u(x)| = |x_1|/|x|^q$, is bounded by the value of $|x|^{1-q}$ in the center of the ball. In other words,

$$(3.24)\quad \sup_{y\in\Gamma'(x)} \left(\fint_{|z-y|\leq\kappa\delta(y)} |u(z)|^2 \, dz \right)^{\frac{1}{2}} \leq C \sup_{y\in\Gamma'(x)} |y|^{1-q} \leq C,$$

since $\Gamma'(x)$ always stays away from the origin. Therefore, the second term in (3.23) is also bounded, and the left hand side of (3.23) is finite, as desired.

Note that $f = u\big|_{\partial B_1}$ is simply x_1, in particular, $f \in C(\partial B_1)$. □

The Lemma above suggests that the choice of $\mathcal{N}_p$ is somewhat more natural than the choice of $\mathcal{N}_2$ in the definition of the Dirichlet problem. It has to be mentioned, though, that the uniform bound on the L^p norm of $\mathcal{N}_2$ maximal function of the Poisson semigroup, in the spirit of (1.4), might still imply the uniform bound on $\mathcal{N}_p$ in (1.4). We do not know if this is true or not for $p > 2$.

4. Regularity problem for a block operator

Theorem 4.1. *Let L be an elliptic operator on $\mathbb{R}^n$ defined by (1.1)–(1.3). Then*

$$\|\mathcal{N}_2(\nabla_{x,t}e^{-t\sqrt{L}}f)\|_{L^p(\mathbb{R}^n)} \leq C\|\nabla f\|_{L^p(\mathbb{R}^n)}, \tag{4.1}$$

for every $f \in \mathring{W}^{1,p}(\mathbb{R}^n)$, $\max\left\{\frac{p_-(L)n}{n+p_-(L)}, 1\right\} < p < 2 + \varepsilon(L)$. In particular, (4.1) holds for any elliptic operator L when $\max\left\{\frac{2n}{n+4}, 1\right\} < p \leq 2$.

Here and throughout the paper we will slightly abuse the notation and write

$$\mathcal{N}_p(\nabla_{x,t}e^{-t\sqrt{L}}f)(x) := \sup_{(y,t)\in\Gamma_\kappa(x)}\left(\fint\!\!\fint_{D((y,t),\kappa t)} |\nabla_{z,s}e^{-s\sqrt{L}}f(z)|^p\,dzds\right)^{\frac{1}{p}}, \qquad x\in\mathbb{R}^n, \quad 1<p<\infty. \tag{4.2}$$

In particular, the expression above does not depend on t.

Remark. We will prove the theorem assuming that the implicit constant κ in the definition of $\mathcal{N}_2$ is less than 1/8. It is sufficient for all practical purposes, and it is not a crucial restriction, since the L^p norms of the non-tangential maximal functions corresponding to different values of κ are equivalent. This can be established by a standard argument.

Let us start with a few auxiliary results. One of the main technical difficulties in dealing with $\mathcal{N}_2(\nabla_{x,t}e^{-t\sqrt{L}}f)$ comes from the fact that the Poisson semigroup associated to a general operator L not only does not exhibit pointwise bounds or regularity, but does not even have sufficient L^2 decay. We will often develop parts of the argument for the heat semigroup first, and then use the following estimates on the difference.

Lemma 4.2. *Let L be an elliptic operator on $\mathbb{R}^n$ defined by (1.1)–(1.3). Then*

$$|(\nabla_x e^{-t\sqrt{L}} - \nabla_x e^{-t^2L})f(x)| \leq C\left(\int_0^\infty |v^2\nabla_x L e^{-v^2L}f(x)|^2\,\frac{dv}{v}\right)^{1/2}, \tag{4.3}$$

$$t|(\nabla_x e^{-t\sqrt{L}} - \nabla_x e^{-t^2L})f(x)| \leq C\left(\int_0^\infty |v^3\nabla_x L e^{-v^2L}f(x)|^2\,\frac{dv}{v}\right)^{1/2}, \tag{4.4}$$

$$|(e^{-t\sqrt{L}} - e^{-t^2L})f(x)| \leq C\left(\int_0^\infty |v^2 L e^{-v^2L}f(x)|^2\,\frac{dv}{v}\right)^{1/2}, \tag{4.5}$$

for any $t>0$, $x\in\mathbb{R}^n$.

We learned this trick from Pascal Auscher when working on the characterization of Hardy spaces via the non-tangential maximal function associated to the Poisson semigroup in [14]. The estimate (4.5) has been presented in [14], and here we will concentrate on (4.3)–(4.4).

Proof of Lemma 4.2. Let us start with (4.3). For any $t>0$ and $x\in\mathbb{R}^n$

$$|(\nabla_x e^{-t\sqrt{L}} - \nabla_x e^{-t^2L})f(x)|^2 = \frac{1}{t}\left|\int_0^t \partial_s\left(s^{1/2}(\nabla_x e^{-s\sqrt{L}} - \nabla_x e^{-s^2L})f(x)\right)\,ds\right|^2$$

$$\leq \frac{1}{t}\left|\int_0^t s^{1/2}\partial_s(\nabla_x e^{-s\sqrt{L}} - \nabla_x e^{-s^2L})f(x)\,ds + \frac{1}{2}\int_0^t s^{-1/2}(\nabla_x e^{-s\sqrt{L}} - \nabla_x e^{-s^2L})f(x)\,ds\right|^2$$

$$\leq C\int_0^t |(\nabla_x e^{-s\sqrt{L}} - \nabla_x e^{-s^2L})f(x)|^2\,\frac{ds}{s}$$

$$+C\int_0^t |s\,\nabla_x\partial_s e^{-s\sqrt{L}}f(x)|^2\,\frac{ds}{s} + C\int_0^t |s\,\nabla_x\partial_s e^{-s^2L}f(x)|^2\,\frac{ds}{s}. \tag{4.6}$$

We will show that each of the integrals above is bounded by the square of the right-hand side of (4.3). It is obvious for the last integral in (4.6). As for the second one, by subordination formula (3.3) and Minkowski inequality

$$\left(\int_0^t |s\,\nabla_x\partial_s e^{-s\sqrt{L}}f(x)|^2\,\frac{ds}{s}\right)^{\frac{1}{2}} \leq C\int_0^\infty \frac{e^{-u}}{\sqrt{u}}\left(\int_0^\infty |s\,\nabla_x\partial_s e^{-\frac{s^2L}{4u}}f(x)|^2\,\frac{ds}{s}\right)^{\frac{1}{2}}du$$

$$\leq C\int_0^\infty \frac{e^{-u}}{u^{3/2}}\left(\int_0^\infty |s^2\,\nabla_x L e^{-\frac{s^2L}{4u}}f(x)|^2\,\frac{ds}{s}\right)^{\frac{1}{2}}du \leq C\int_0^\infty \frac{e^{-u}}{u^{1/2}}\left(\int_0^\infty |v^2\,\nabla_x L e^{-v^2L}f(x)|^2\,\frac{dv}{v}\right)^{\frac{1}{2}}du$$

$$\leq C\left(\int_0^\infty |v^2\,\nabla_x L e^{-v^2L}f(x)|^2\,\frac{dv}{v}\right)^{\frac{1}{2}}, \tag{4.7}$$

where we used the change of variables $v := s/\sqrt{4u}$. Similarly,

$$\left(\int_0^t |(\nabla_x e^{-s\sqrt{L}} - \nabla_x e^{-s^2L})f(x)|^2\,\frac{ds}{s}\right)^{\frac{1}{2}} = C\left(\int_0^\infty\left|\int_0^\infty \frac{e^{-u}}{\sqrt{u}}(\nabla_x e^{-\frac{s^2L}{4u}} - \nabla_x e^{-s^2L})f(x)\,du\right|^2\frac{ds}{s}\right)^{1/2}$$

$$= C\left(\int_0^\infty\left|\int_0^\infty \frac{e^{-u}}{\sqrt{u}}\int_s^{s/\sqrt{4u}} \partial_v\nabla_x e^{-v^2L}f(x)\,dvdu\right|^2\frac{ds}{s}\right)^{1/2}$$

$$= C\left(\int_0^\infty\left|\int_0^\infty \frac{e^{-u}}{\sqrt{u}}\int_s^{s/\sqrt{4u}} v\nabla_x L e^{-v^2L}f(x)\,dvdu\right|^2\frac{ds}{s}\right)^{1/2}. \tag{4.8}$$

Observe that

$$\left|\int_0^{1/4}\frac{e^{-u}}{\sqrt{u}}\int_s^{s/\sqrt{4u}} v\nabla_x L e^{-v^2L}f(x)\,dvdu\right| \leq \int_s^\infty\left|\int_0^{s^2/(4v^2)}\frac{e^{-u}}{\sqrt{u}}\,du\right|\,|v\nabla_x L e^{-v^2L}f(x)|\,dv$$

$$\leq Cs\int_s^\infty |\nabla_x L e^{-v^2L}f(x)|\,dv \leq Cs^{\frac{1}{2}}\left(\int_s^\infty |v\nabla_x L e^{-v^2L}f(x)|^2\,dv\right)^{\frac{1}{2}}, \tag{4.9}$$

while

$$\left|\int_{1/4}^\infty\frac{e^{-u}}{\sqrt{u}}\int_{s/\sqrt{4u}}^{s} v\nabla_x L e^{-v^2L}f(x)\,dvdu\right| = \int_0^s\left|\int_{s^2/(4v^2)}^\infty\frac{e^{-u}}{\sqrt{u}}\,du\right|\,|v\nabla_x L e^{-v^2L}f(x)|\,dv$$

$$\leq \frac{C}{s}\int_0^s |v^2\nabla_x Le^{-v^2L}f(x)|\,dv \leq \frac{C}{s^{1/2}}\left(\int_0^s |v^2\nabla_x Le^{-v^2L}f(x)|^2\,dv\right)^{\frac12}. \tag{4.10}$$

Hence,

$$\begin{aligned}&|(\nabla_x e^{-t\sqrt{L}} - \nabla_x e^{-t^2L})f(x)|\\ &\leq C\left(\int_0^\infty\int_s^\infty |v\nabla_x Le^{-v^2L}f(x)|^2\,dvds\right)^{1/2} + \left(\int_0^\infty\int_0^s |v^2\nabla_x Le^{-v^2L}f(x)|^2\,dv\,\frac{ds}{s^2}\right)^{1/2}\\ &+C\left(\int_0^\infty |v^2\nabla_x Le^{-v^2L}f(x)|^2\,\frac{dv}{v}\right)^{1/2} \leq C\left(\int_0^\infty |v^2\nabla_x Le^{-v^2L}f(x)|^2\,\frac{dv}{v}\right)^{1/2},\end{aligned} \tag{4.11}$$

as desired. This competes the proof of (4.3).

Turning to (4.4), we estimate analogously to (4.6)

$$\begin{aligned}&|t(\nabla_x e^{-t\sqrt{L}} - \nabla_x e^{-t^2L})f(x)|^2 = \frac1t\left|\int_0^t \partial_s\left(s^{3/2}(\nabla_x e^{-s\sqrt{L}} - \nabla_x e^{-s^2L})f(x)\right)ds\right|^2\\ &\leq C\int_0^t |s(\nabla_x e^{-s\sqrt{L}} - \nabla_x e^{-s^2L})f(x)|^2\,\frac{ds}{s}\\ &\qquad +C\int_0^t |s^2\,\nabla_x\partial_s e^{-s\sqrt{L}}f(x)|^2\,\frac{ds}{s} + C\int_0^t |s^2\,\nabla_x\partial_s e^{-s^2L}f(x)|^2\,\frac{ds}{s}.\end{aligned} \tag{4.12}$$

The last integral above is trivially bounded by the square of the right-hand side of (4.4). Furthermore, closely following (4.7), we have

$$\begin{aligned}&\left(\int_0^t |s^2\,\nabla_x\partial_s e^{-s\sqrt{L}}f(x)|^2\,\frac{ds}{s}\right)^{\frac12} \leq C\int_0^\infty \frac{e^{-u}}{u^{3/2}}\left(\int_0^\infty |s^3\,\nabla_x Le^{-\frac{s^2L}{4u}}f(x)|^2\,\frac{ds}{s}\right)^{\frac12}du\\ &\leq C\int_0^\infty e^{-u}\left(\int_0^\infty |v^3\,\nabla_x Le^{-v^2L}f(x)|^2\,\frac{dv}{v}\right)^{\frac12}du \leq C\left(\int_0^\infty |v^3\,\nabla_x Le^{-v^2L}f(x)|^2\,\frac{dv}{v}\right)^{\frac12},\end{aligned} \tag{4.13}$$

with $v = s/\sqrt{4u}$. Also,

$$\begin{aligned}&\left(\int_0^t |s(\nabla_x e^{-s\sqrt{L}} - \nabla_x e^{-s^2L})f(x)|^2\,\frac{ds}{s}\right)^{\frac12}\\ &= C\left(\int_0^\infty\left|\int_0^\infty e^{-u}\,\frac{s}{\sqrt{u}}\int_s^{s/\sqrt{4u}} v\nabla_x Le^{-v^2L}f(x)\,dvdu\right|^2\frac{ds}{s}\right)^{1/2}.\end{aligned} \tag{4.14}$$

According to (4.9), a part of the inside integral above corresponding to $u \in (0,1/4)$ is controlled by $s^{\frac32}\left(\int_s^\infty |v\nabla_x Le^{-v^2L}f(x)|^2\,dv\right)^{\frac12}$. Furthermore,

$$\left|\int_{1/4}^\infty e^{-u}\,\frac{s}{\sqrt{u}}\int_{s/\sqrt{4u}}^s v\nabla_x Le^{-v^2L}f(x)\,dvdu\right| \leq C\left|\int_{1/4}^\infty\int_{s/\sqrt{4u}}^s v^2\nabla_x Le^{-v^2L}f(x)\,dvdu\right|$$

$$\leq C\int_0^s \left|\int_{s^2/(4v^2)}^\infty e^{-u}\,du\right| |v^2\nabla_x Le^{-v^2L}f(x)|\,dv \leq C\int_0^s e^{-s^2/(4v^2)}\,|v^2\nabla_x Le^{-v^2L}f(x)|\,dv$$

$$\leq C\,s^{1/2}\left(\int_0^s e^{-s^2/(2v^2)}|v^2\nabla_x Le^{-v^2L}f(x)|^2\,dv\right)^{\frac{1}{2}}. \tag{4.15}$$

Therefore, the left-hand side of (4.14) is bounded by

$$C\left(\int_0^\infty\int_s^\infty |v\nabla_x Le^{-v^2L}f(x)|^2\,dv\,s^2ds\right)^{1/2}+\left(\int_0^\infty\int_0^s e^{-s^2/(2v^2)}|v^2\nabla_x Le^{-v^2L}f(x)|^2\,dv\,ds\right)^{1/2}$$

$$\leq C\left(\int_0^\infty |v^3\nabla_x Le^{-v^2L}f(x)|^2\,\frac{dv}{v}\right)^{1/2}, \tag{4.16}$$

which together with (4.12) and (4.13) finishes the argument for (4.4). □

In the course of the proof we will also need boundedness in L^p of certain square functions, in particular, those coming from Lemma 4.2.

Lemma 4.3. *Let L be an elliptic operator on $\mathbb{R}^n$ defined by* (1.1)–(1.3). *Then*

$$\left\|\left(\int_0^\infty |v^2Le^{-v^2L}f|^2\,\frac{dv}{v}\right)^{1/2}\right\|_{L^p(\mathbb{R}^n)} \leq C\|f\|_{L^p(\mathbb{R}^n)},\ \ \forall\, f\in L^p(\mathbb{R}^n),\ \ p_-(L)<p<p_+(L), \tag{4.17}$$

$$\left\|\left(\int_0^\infty |v^2\nabla_x\sqrt{L}e^{-v^2L}f|^2\,\frac{dv}{v}\right)^{1/2}\right\|_{L^p(\mathbb{R}^n)} \leq C\|f\|_{L^p(\mathbb{R}^n)},\ \ \forall\, f\in L^p(\mathbb{R}^n),\ \ p_-(L)<p<2+\varepsilon(L). \tag{4.18}$$

$$\left\|\left(\int_0^\infty |v^5\nabla_x L^2e^{-v^2L}f|^2\,\frac{dv}{v}\right)^{1/2}\right\|_{L^p(\mathbb{R}^n)} \leq C\|f\|_{L^p(\mathbb{R}^n)},\ \ \forall\, f\in L^p(\mathbb{R}^n),\ \ p_-(L)<p<2+\varepsilon(L). \tag{4.19}$$

The argument is analogous to the one for the square function bounds in [3]. We omit the proof.

Now we are ready for the

Proof of Theorem 4.1. **Step I.** The estimate on the derivative of $e^{-t\sqrt{L}}f$ in t in (4.1) reduces to the characterization of Hardy spaces associated to L via the non-tangential maximal function associated to the Poisson semigroup in [14] and their Riesz transform characterization established in [15]. In [14] we introduced a concept of the Hardy space H^1_L associated to a given elliptic operator L and proved that the L^1-norm of the non-tangential maximal function associated to the Poisson semigroup is an equivalent norm in H^1_L, that is,

$$\left\|\sup_{t>0}\left(\fint_{B(\cdot,t)}\left|e^{-t\sqrt{L}}g(x)\right|^2\,dx\right)^{\frac{1}{2}}\right\|_{L^1(\mathbb{R}^n)} \approx \|g\|_{H^1_L(\mathbb{R}^n)} \quad \text{for every} \quad g\in H^1_L(\mathbb{R}^n). \tag{4.20}$$

In particular,

$$\mathcal{N}_{Pois}g(x) := \sup_{t>0}\left(\fint_{B(x,t)}\left|e^{-t\sqrt{L}}g(x)\right|^2\,dx\right)^{\frac{1}{2}},\quad x\in\mathbb{R}^n, \tag{4.21}$$

is a bounded operator from $H^1_L(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$. We claim that, in addition,

$$\mathcal{N}_{Pois} : L^p(\mathbb{R}^n) \longrightarrow L^p(\mathbb{R}^n), \quad 2 < p < p_+(L). \tag{4.22}$$

Let us postpone momentarily the proof of this fact, and first show how this information leads to the bound on the derivative in t in (4.1).

Indeed, (4.22) entails that

$$\mathcal{N}_{Pois} : H^p_L(\mathbb{R}^n) \longrightarrow L^p(\mathbb{R}^n), \quad 2 < p < p_+(L), \tag{4.23}$$

since the spaces H^p_L coincide with L^p for $p \in (p_-(L), p_+(L))$. Then, using the interpolation property of H^p_L spaces proved in [15] and (4.20), we conclude that (4.23) holds for all $p \in [1, p_+(L))$. For further reference note that, in particular,

$$\left\| \sup_{t>0} \Big(\fint_{B(\cdot,t)} \left| e^{-t\sqrt{L}} g(x) \right|^2 dx \Big)^{\frac{1}{2}} \right\|_{L^p(\mathbb{R}^n)} \leq C\|g\|_{L^p(\mathbb{R}^n)}, \quad \forall\, g \in L^p(\mathbb{R}^n), \quad p_-(L) < p < p_+(L). \tag{4.24}$$

However, one can see from the definition or a representation (1.6) that

$$\begin{aligned}
\mathcal{N}_2(\partial_t e^{-t\sqrt{L}} f)(x) &\leq C \sup_{t>0} \left(\fint\!\!\fint_{D((x,t),2\kappa t)} |e^{-s\sqrt{L}} \sqrt{L} f(z)|^2 \, dz ds \right)^{\frac{1}{2}} \\
&\leq C \sup_{t>0} \left(\fint_{|s-t|<2\kappa t} \fint_{|z-x|<2\kappa t} |e^{-s\sqrt{L}} \sqrt{L} f(z)|^2 \, dz ds \right)^{\frac{1}{2}} \\
&\leq C \sup_{t>0} \sup_{s:\, |s-t|<2\kappa t} \left(\fint_{|z-x|<2\kappa s/(1-2\kappa)} |e^{-s\sqrt{L}} \sqrt{L} f(z)|^2 \, dz \right)^{\frac{1}{2}} \\
&\leq C \sup_{s>0} \Big(\fint_{B(x,s)} \left| e^{-s\sqrt{L}} \sqrt{L} f(y) \right|^2 dy \Big)^{\frac{1}{2}} \leq C |\mathcal{N}_{Pois}(\sqrt{L} f)(x)|, \qquad x \in \mathbb{R}^n,
\end{aligned} \tag{4.25}$$

provided that the constant κ in the definition of $\mathcal{N}_2$ is less that 1/4. As we pointed out, (4.23) is valid for all $p \in [1, p_+(L))$, and hence, the inequality (4.25) implies that

$$\|\mathcal{N}_2(\partial_t e^{-t\sqrt{L}} f)\|_{L^p(\mathbb{R}^n)} \leq C \left\| \sqrt{L} f \right\|_{H^p_L(\mathbb{R}^n)}, \quad 1 \leq p < p_+(L), \tag{4.26}$$

whenever $\sqrt{L} f \in H^p_L(\mathbb{R}^n)$. Finally, the Riesz transform characterization of Hardy spaces established in [15] ascertains that for every $f \in \mathring{W}^{1,p}(\mathbb{R}^n)$, $\max\left\{\frac{p_-(L)n}{n+p_-(L)}, 1\right\} < p < 2 + \varepsilon(L)$, the function $\sqrt{L} f$ belongs to $H^p_L(\mathbb{R}^n)$ and

$$\left\| \sqrt{L} f \right\|_{H^p_L(\mathbb{R}^n)} \leq C\|\nabla f\|_{L^p(\mathbb{R}^n)}, \quad \forall\, f \in \mathring{W}^{1,p}(\mathbb{R}^n), \quad \max\left\{ \frac{p_-(L)n}{n + p_-(L)}, 1 \right\} < p < 2 + \varepsilon(L). \tag{4.27}$$

Now the combination of (4.26) and (4.27) gives the desired estimate. It remains to justify (4.22). Observe that by Lemma 2.2,

$$\left\| \sup_{t>0} \Big(\fint_{B(\cdot,t)} \left| e^{-t^2 L} f(x) \right|^2 dx \Big)^{\frac{1}{2}} \right\|_{L^p(\mathbb{R}^n)} \leq \left\| \sup_{t>0} \Big(\fint_{B(\cdot,t)} \left| e^{-t^2 L} (\chi_{B(\cdot,4t)} f)(x) \right|^2 dx \Big)^{\frac{1}{2}} \right\|_{L^p(\mathbb{R}^n)}$$

$$+\left\|\sup_{t>0}\sum_{j=2}^{\infty}\Big(\fint_{B(\cdot,t)}\left|e^{-t^2L}(\chi_{B(\cdot,2^{j+1}t)\setminus B(\cdot,2^jt)}f)(x)\right|^2\,dx\Big)^{\frac12}\right\|_{L^p(\mathbb{R}^n)}$$

$$\leq C\left\|\sup_{t>0}\Big(\fint_{B(\cdot,4t)}|f(x)|^2\,dx\Big)^{\frac12}\right\|_{L^p(\mathbb{R}^n)}+\left\|\sup_{t>0}\sum_{j=2}^{\infty}\frac{1}{t^{n/2}}e^{-\frac{(2^jt)^2}{ct^2}}\Big(\int_{B(\cdot,2^{j+1}t)\setminus B(\cdot,2^jt)}|f(x)|^2\,dx\Big)^{\frac12}\right\|_{L^p(\mathbb{R}^n)}. \tag{4.28}$$

The right-hand side of (4.28) is, in turn, bounded by

$$C\,\|\mathcal{M}_2(f)\|_{L^p(\mathbb{R}^n)}\leq C\,\|f\|_{L^p(\mathbb{R}^n)}\quad\text{whenever}\quad 2<p<p_+(L). \tag{4.29}$$

Here $\mathcal{M}_2$ denotes an L^2-based version of the Hardy-Littlewood maximal function, that is,

$$\mathcal{M}_qf(x)=\sup_{t>0}\left(\fint_{B(x,t)}|f(y)|^q\,dy\right)^{\frac1q},\qquad x\in\mathbb{R}^n,\qquad 1\leq q<\infty. \tag{4.30}$$

Writing it via the usual Hardy-Littlewood maximal function, one can easily show that $\mathcal{M}_q$ is bounded in all $L^p(\mathbb{R}^n)$, $q<p<\infty$, which justifies the inequality in (4.29).

Let us now estimate the difference between the heat and the Poisson semigroups. Thanks to Lemma 4.2, we know that

$$\left\|\sup_{t>0}\Big(\fint_{B(\cdot,t)}\left|(e^{-t\sqrt{L}}-e^{-t^2L})f(x)\right|^2\,dx\Big)^{\frac12}\right\|_{L^p(\mathbb{R}^n)}\leq C\left\|\sup_{t>0}\Big(\fint_{B(\cdot,t)}\int_0^\infty|v^2Le^{-v^2L}f(x)|^2\,\frac{dv}{v}\,dx\Big)^{\frac12}\right\|_{L^p(\mathbb{R}^n)}$$

$$\leq C\left\|\mathcal{M}_2\left(\Big(\int_0^\infty|v^2Le^{-v^2L}f|^2\,\frac{dv}{v}\Big)^{\frac12}\right)\right\|_{L^p(\mathbb{R}^n)}\leq C\left\|\Big(\int_0^\infty|v^2Le^{-v^2L}f|^2\,\frac{dv}{v}\Big)^{1/2}\right\|_{L^p(\mathbb{R}^n)}, \tag{4.31}$$

whenever $p>2$. The square function above is bounded in L^p for all $p\in(p_-(L),p_+(L))$ by Lemma 4.3, and that finishes the argument for (4.22).

Unfortunately, the approach described above does not apply to the estimate on $\mathcal{N}_2(\nabla_xe^{-t\sqrt{L}}f)$ in (4.1), since the gradient in x does not commute with the operator $e^{-t\sqrt{L}}$, $t>0$. Below we will build a different argument, strategically resembling the one for the Riesz transform characterization in [14], but aimed directly at (4.1).

We have to show that

$$\|\mathcal{N}_2(\nabla_xe^{-t\sqrt{L}}f)\|_{L^p(\mathbb{R}^n)}\leq C\|\nabla f\|_{L^p(\mathbb{R}^n)}, \tag{4.32}$$

for every $f\in\mathring{W}^{1,p}(\mathbb{R}^n)$, $\max\left\{\frac{p_-(L)n}{n+p_-(L)},1\right\}<p<2+\varepsilon(L)$. For future reference, we note that analogously to (4.25), we have

$$\mathcal{N}_2(\nabla_xe^{-t\sqrt{L}}f)(x)\leq C\sup_{s>0}\Big(\fint_{B(x,s)}\left|\nabla_ye^{-s\sqrt{L}}f(y)\right|^2\,dy\Big)^{\frac12},\qquad x\in\mathbb{R}^n, \tag{4.33}$$

for $\kappa<1/4$. Whenever convenient, we will estimate the right-hand side of (4.33) in place of $\mathcal{N}_2(\nabla_xe^{-t\sqrt{L}}f)(x)$, $x\in\mathbb{R}^n$, without further comments.

Step II. Let us now use the ideas of (4.28)–(4.31) to show that the estimate (4.32) holds for $p \in (2, 2+\varepsilon(L))$. Indeed, according to Lemma 2.4,

$$\Big\| \sup_{t>0} \Big(\fint_{B(\cdot,t)} \big|\nabla_x e^{-t^2 L} f(x)\big|^2 \, dx \Big)^{\frac12} \Big\|_{L^p(\mathbb{R}^n)} \leq C \Big\| \sup_{t>0} \sum_{j=1}^{\infty} 2^{-jN} \Big(\fint_{B(\cdot,2^{j+1}t)} |\nabla_x f(x)|^2 \, dx \Big)^{\frac12} \Big\|_{L^p(\mathbb{R}^n)}$$

$$\leq C \sum_{j=1}^{\infty} 2^{-jN} \, \|\mathcal{M}_2(\nabla f)\|_{L^p(\mathbb{R}^n)} \leq C \, \|\nabla f\|_{L^p(\mathbb{R}^n)}\,, \quad \text{for} \quad 2 < p < p_+(L). \tag{4.34}$$

As for the difference between the heat and the Poisson semigroups, we follow (4.31) and invoke Lemmas 4.2 and 4.3 to obtain

$$\Big\| \sup_{t>0} \Big(\fint_{B(\cdot,t)} \big|(\nabla_x e^{-t\sqrt{L}} - \nabla_x e^{-t^2 L}) f(x)\big|^2 \, dx \Big)^{\frac12} \Big\|_{L^p(\mathbb{R}^n)}$$

$$\leq C \Big\| \mathcal{M}_2 \Big(\Big(\Big(\int_0^\infty |v^2 \nabla_x L e^{-v^2 L} f|^2 \, \frac{dv}{v} \Big)^{1/2} \Big) \Big\|_{L^p(\mathbb{R}^n)}$$

$$\leq C \Big\| \Big(\int_0^\infty |v^2 \nabla_x \sqrt{L} e^{-v^2 L} (\sqrt{L} f)|^2 \, \frac{dv}{v} \Big)^{1/2} \Big\|_{L^p(\mathbb{R}^n)} \leq C \| \sqrt{L} f \|_{L^p(\mathbb{R}^n)} \leq C \|\nabla f\|_{L^p(\mathbb{R}^n)}, \tag{4.35}$$

with $2 < p < 2+\varepsilon(L)$. Clearly, a combination of (4.34) and (4.35) gives (4.32) for $p \in (2, 2+\varepsilon(L))$, as desired.

Step III. Let r be a real number on the interval $(p_-(L), 2]$. We claim that it is now enough to show that for every such r and $q := \max\big\{1, \frac{rn}{n+r}\big\}$

$$\Big\| \mathcal{N}_2(\nabla_x e^{-t\sqrt{L}} f) \Big\|_{L^{q,\infty}(\mathbb{R}^n)} \leq C \|\nabla f\|_{L^q(\mathbb{R}^n)}, \qquad f \in C_0^\infty(\mathbb{R}^n), \tag{4.36}$$

or, equivalently, to show that for f and q as above

$$\Big| \Big\{ x \in \mathbb{R}^n : \, \mathcal{N}_2(\nabla_x e^{-t\sqrt{L}} f)(x) > \alpha \Big\} \Big| \leq \frac{C}{\alpha^q} \int_{\mathbb{R}^n} |\nabla f(y)|^q \, dy, \quad \forall \, \alpha > 0. \tag{4.37}$$

Indeed, (4.36) implies that the operator

$$\mathcal{N}^{\nabla}_{Pois} f(x) := \mathcal{N}_2(\nabla_x e^{-t\sqrt{L}} f)(x), \quad x \in \mathbb{R}^n, \tag{4.38}$$

initially defined on $C_0^\infty(\mathbb{R}^n)$, extends to a bounded operator

$$\mathcal{N}^{\nabla}_{Pois} : \mathring{W}^{1,q}(\mathbb{R}^n) \longrightarrow L^{q,\infty}(\mathbb{R}^n). \tag{4.39}$$

On the other hand, in Step II we showed that $\mathcal{N}^{\nabla}_{Pois}$ extends to a bounded operator from $\mathring{W}^{1,q}(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$, $2 < q < 2+\varepsilon(L)$, and hence, from $\mathring{W}^{1,q}(\mathbb{R}^n)$ to $L^{q,\infty}(\mathbb{R}^n)$ for all such q. Then, by interpolation, the mapping properties in (4.39) hold for all q such that $\max\big\{1, \frac{rn}{n+r}\big\} \leq q < 2+\varepsilon(L)$,

$r \in (p_-(L), 2]$, in particular, for all q satisfying $\max\left\{1, \frac{p_-(L)n}{n+p_-(L)}\right\} < q < 2 + \varepsilon(L)$. Then, once again invoking interpolation, we arrive at

$$\text{(4.40)}\quad \mathcal{N}^{\nabla}_{Pois} : \mathring{W}^{1,p}(\mathbb{R}^n) \longrightarrow L^p(\mathbb{R}^n) \quad \text{whenever} \quad \max\left\{1, \frac{p_-(L)n}{n+p_-(L)}\right\} < q < 2 + \varepsilon(L),$$

which yields (4.32).

Step IV. Now we turn to (4.37). According to [3], Lemma 4.12, every function $f \in \mathcal{S}'(\mathbb{R}^n)$ with $\|\nabla f\|_{L^q(\mathbb{R}^n)} < \infty$, $1 \leq q \leq \infty$, can be decomposed as follows. Given any $\alpha > 0$ there exists a collection of cubes $\{Q_i\}_{i\in\mathbb{Z}}$ with finite overlap, a function g and a family of functions $\{b_i\}_{i\in\mathbb{Z}}$ such that

$$\text{(4.41)}\qquad \operatorname{supp} b_i \subset Q_i, \qquad \|\nabla b_i\|_{L^q(\mathbb{R}^n)} \leq C\alpha |Q_i|^{1/q}, \qquad \forall\, i \in \mathbb{Z},$$

$$\text{(4.42)}\qquad \|\nabla g\|_{L^q(\mathbb{R}^n)} \leq C\|\nabla f\|_{L^q(\mathbb{R}^n)}, \quad \|\nabla g\|_{L^\infty(\mathbb{R}^n)} \leq C\alpha,$$

and

$$\text{(4.43)}\quad f = g + \sum_{i\in\mathbb{Z}} b_i, \quad \text{with} \quad \sum_{i\in\mathbb{Z}} |Q_i| \leq C\alpha^{-q}\|\nabla f\|^q_{L^q(\mathbb{R}^n)}.$$

Let us denote

$$\text{(4.44)}\quad A_g := \left\{x \in \mathbb{R}^n : I_g(x) > \alpha/2\right\} \quad \text{and} \quad A_b := \{x \in \mathbb{R}^n : I_b(x) > \alpha/2\},$$

where

$$I_g(x) := \mathcal{N}^{\nabla}_{Pois} g(x) \quad \text{and} \quad I_b(x) = \mathcal{N}^{\nabla}_{Pois}\Big(\sum_{i\in\mathbb{Z}} b_i\Big)(x), \qquad \text{for all } x \in \mathbb{R}^n.$$

Then the expression on the left-hand side of (4.37) is bounded by $|A_g| + |A_b|$.

The size of the set A_g can be estimated in a fairly straightforward way. We have

$$\text{(4.45)}\quad |A_g| \leq \frac{C}{\alpha^p}\int_{\mathbb{R}^n} \left|I_g(x)\right|^p\, dx \leq \frac{C}{\alpha^p}\int_{\mathbb{R}^n} |\nabla g(x)|^p\, dx \leq \frac{C}{\alpha^q}\|\nabla f\|^q_{L^q(\mathbb{R}^n)},$$

where p is an arbitrary number between 2 and $2 + \varepsilon(L)$. Indeed, the first inequality above is Chebyshev inequality, the second one follows from the results of Step II – the boundedness of the corresponding non-tangential maximal function from $\mathring{W}^{1,p}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$, $p \in (2, 2+\varepsilon(L))$, and the third inequality comes out by interpolation of two statements in (4.42).

Step V. In order to bound A_b we split I_b into several parts. First of all, let

$$\text{(4.46)}\qquad I_b^1(x) := \sup_{t>0}\left(\iint_{D((x,t),2\kappa t)} \Big|\nabla_z e^{-s\sqrt{L}}\Big(\sum_{i\in\mathbb{Z}:\, l(Q_i)\leq 2t} b_i\Big)(z)\Big|^2\, dzds\right)^{\frac{1}{2}},$$

$$\text{(4.47)}\qquad I_b^2(x) := \sup_{t>0}\left(\fint_{B(x,t)} \Big|\nabla_y e^{-t\sqrt{L}}\Big(\sum_{i\in\mathbb{Z}:\, l(Q_i)>t} b_i\Big)(y)\Big|^2\, dy\right)^{\frac{1}{2}},$$

where $x \in \mathbb{R}^n$, so that $I_b(x) \leq I_b^1(x) + I_b^2(x)$ for any $x \in \mathbb{R}^n$. Indeed, we first substitute the supremum over a cone in the maximal function by supremum over t as in the first line of (4.25), then separate the parts of the sum of b_i's corresponding to $i \in \mathbb{Z} : l(Q_i) \leq 2t$ and $i \in \mathbb{Z} : l(Q_i) > 2t$. The first of the emerging expressions is denoted by I_b^1, and the second is further estimated following computations in (4.25). One just has to note that for $s \in ((1-2\kappa)t, (1+2\kappa)t))$ the set of $i \in \mathbb{Z}$: $l(Q_i) > 2t$ is a subset of $\{i \in \mathbb{Z} : l(Q_i) > 2s/(1+2\kappa)\} \subset \{i \in \mathbb{Z} : l(Q_i) > s\}$ for $\kappa < 1/4$.

Consider I_b^1 first. Using the fact that $t \approx s$ in (4.46) and Caccioppoli inequality, we write

$$\begin{aligned}
I_b^1(x) &\leq C \sup_{t>0} \Big(\fint\!\!\fint_{D((x,t),2\kappa t)} \Big| s\nabla_{z,s} e^{-s\sqrt{L}} \Big(\sum_{i\in\mathbb{Z}:\, l(Q_i)\leq 2t} \frac{b_i}{l(Q_i)} \Big)(z) \Big|^2 dz ds \Big)^{\frac{1}{2}} \\
&\leq C \sup_{t>0} \Big(\fint\!\!\fint_{D((x,t),4\kappa t)} \Big| e^{-s\sqrt{L}} \Big(\sum_{i\in\mathbb{Z}:\, l(Q_i)\leq 2t} \frac{b_i}{l(Q_i)} \Big)(z) \Big|^2 dz ds \Big)^{\frac{1}{2}} \\
&\leq C N_{Pois} \Big(\sum_{i\in\mathbb{Z}} \frac{b_i}{l(Q_i)} \Big)(x), \qquad x \in \mathbb{R}^n, \qquad (4.48)
\end{aligned}$$

once again using the argument of (4.25) with 2κ in place of κ (recall that $\kappa < 1/8$).

Recall that $r \in (p_-(L), 2]$ is a real number such that $q = \max\left\{1, \frac{rn}{n+r}\right\}$. Hence, by (4.48) and (4.24) we have

$$(4.49)\quad |\{x \in \mathbb{R}^n : I_b^1(x) > \alpha/4\}| \leq \frac{C}{\alpha^r} \int_{\mathbb{R}^n} |I_b^1(x)|^r \, dx \leq \frac{C}{\alpha^r} \Big\| \sum_{i\in\mathbb{Z}} \frac{b_i}{l(Q_i)} \Big\|_{L^r(\mathbb{R}^n)}^r .$$

However, by Hölder inequality for sequences

$$(4.50)\quad \frac{C}{\alpha^r} \Big\| \sum_{i\in\mathbb{Z}} \frac{|b_i|}{l(Q_i)} \Big\|_{L^r(\mathbb{R}^n)}^r \leq \frac{C}{\alpha^r} \Big\| \Big(\sum_{i\in\mathbb{Z}} \frac{|b_i|^r}{l(Q_i)^r} \Big)^{1/r} \Big(\sum_{i\in\mathbb{Z}} \chi_{Q_i} \Big)^{1-1/r} \Big\|_{L^r(\mathbb{R}^n)}^r \leq \frac{C}{\alpha^r} \int_{\mathbb{R}^n} \sum_{i\in\mathbb{Z}} \frac{|b_i|^r}{l(Q_i)^r} \, dx,$$

using the fact that the cubes $\{Q_i\}_{i\in\mathbb{Z}}$ have finite overlap, i.e. there exists some fixed constant C such that $\sum_{i\in\mathbb{Z}} \chi_{Q_i}(x) \leq C$ for all $x \in \mathbb{R}^n$. Furthermore, due to (4.41) and Poincaré inequality,

$$(4.51)\quad \|b_i\|_{L^r(\mathbb{R}^n)} \leq C\|\nabla b_i\|_{L^q(\mathbb{R}^n)} \leq C\alpha |Q_i|^{1/q} = C\alpha |l(Q_i)|^{1+n/r},$$

if $q = \frac{rn}{n+r}$. If, on the other hand, $q = 1 > \frac{rn}{n+r}$, then

$$(4.52)\quad \|b_i\|_{L^r(Q_i)} \leq C|Q_i|^{\frac{1}{r}-\frac{n-1}{n}} \|b_i\|_{L^{\frac{n}{n-1}}(Q_i)} \leq C|Q_i|^{\frac{1}{r}-\frac{n-1}{n}} \|\nabla b_i\|_{L^1(Q_i)} \leq C\alpha |l(Q_i)|^{1+n/r},$$

using Hölder inequality for the first bound above. Hence, in both cases,

$$(4.53)\quad |\{x \in \mathbb{R}^n : I_b^1(x) > \alpha/4\}| \leq C \sum_{i\in\mathbb{Z}} |Q_i| \leq C\alpha^{-q} \|\nabla f\|_{L^q(\mathbb{R}^n)}^q .$$

Step VI. Now we pass to the estimate on I_b^2. It will break further into a few cases. Consider first the case of the heat semigroup. To this end, let us denote

$$(4.54)\quad T_i f(x) := \sup_{0<t\leq l(Q_i)} \Big(\fint_{B(x,t)} \Big| \nabla_y e^{-t^2 L} f(y) \Big|^2 dy \Big)^{\frac{1}{2}}, \quad i \in \mathbb{Z}, \quad x \in \mathbb{R}^n, \quad f \in L^2(\mathbb{R}^n),$$

so that

$$\sup_{t>0}\Big(\fint_{B(x,t)}\Big|\nabla_y e^{-t^2L}\Big(\sum_{i\in\mathbb{Z}:\, l(Q_i)>t} b_i\Big)(y)\Big|^2\,dy\Big)^{\frac12}\le\sum_{i\in\mathbb{Z}}T_ib_i(x),\qquad x\in\mathbb{R}^n. \tag{4.55}$$

We start with the observation that

$$\begin{aligned}
&\left|\left\{x\in\mathbb{R}^n:\ \sup_{t>0}\Big(\fint_{B(x,t)}\Big|\nabla_y e^{-t^2L}\Big(\sum_{i\in\mathbb{Z}:\, l(Q_i)>t} b_i\Big)(y)\Big|^2\,dy\Big)^{\frac12}>\alpha/8\right\}\right|\\
&\quad\le\sum_{i\in\mathbb{Z}}|4Q_i|+\left\{x\in\mathbb{R}^n\setminus\cup_{i\in\mathbb{Z}}4Q_i:\ \left|\sum_{i\in\mathbb{Z}}T_ib_i(x)\right|>\alpha/8\right\}\\
&\quad\le\frac{C}{\alpha^q}\,\|\nabla f\|^q_{L^q(\mathbb{R}^n)}+\frac{C}{\alpha^p}\int_{\mathbb{R}^n}\Big|\sum_{i\in\mathbb{Z}}T_ib_i(x)\chi_{\mathbb{R}^n\setminus 4Q_i}(x)\Big|^p\,dx\\
&\quad\le\frac{C}{\alpha^q}\,\|\nabla f\|^q_{L^q(\mathbb{R}^n)}+\frac{C}{\alpha^p}\left(\sum_{i\in\mathbb{Z}}\int_{\mathbb{R}^n\setminus 4Q_i}T_ib_i(x)u(x)\,dx\right)^p\\
&\quad\le\frac{C}{\alpha^q}\,\|\nabla f\|^q_{L^q(\mathbb{R}^n)}+\frac{C}{\alpha^p}\left(\sum_{i\in\mathbb{Z}}\sum_{j=3}^{\infty}\|T_ib_i\|_{L^p(2^jQ_i\setminus 2^{j-1}Q_i)}\|u\|_{L^{p'}(2^jQ_i\setminus 2^{j-1}Q_i)}\right)^p, \qquad (4.56)
\end{aligned}$$

for some $u\in L^{p'}(\mathbb{R}^n)$ such that $\|u\|_{L^{p'}(\mathbb{R}^n)}=1$, $\frac{1}{p}+\frac{1}{p'}=1$. Throughout this argument (Step VI) we assume that p is an arbitrary number in $(1,\infty)$. In fact, for the purposes of the result in Step VI we only need $p=2$, but later, in Step VIII we will be referring to some of the calculations developed at this stage and there it will be crucial to be able to choose $p>2$.

For every $i\in\mathbb{Z}$, $j\ge3$,

$$\begin{aligned}
\|T_ib_i\|_{L^p(2^jQ_i\setminus 2^{j-1}Q_i)} &= \left(\int_{2^jQ_i\setminus 2^{j-1}Q_i}\sup_{0<t\le l(Q_i)}\left(\fint_{B(x,t)}\Big|\nabla_y e^{-t^2L}b_i(y)\Big|^2\,dy\right)^{p/2}\,dx\right)^{1/p}\\
&\le C\left(\int_{2^jQ_i\setminus 2^{j-1}Q_i}\sup_{0<t\le l(Q_i)}\left(\frac{1}{t^{n+2}}\int_{2^{j+1}Q_i\setminus 2^{j-2}Q_i}\Big|t\nabla_y e^{-t^2L}b_i(y)\Big|^2\,dy\right)^{p/2}\,dx\right)^{1/p}\\
&\le C\sup_{0<t\le l(Q_i)}\frac{(2^jl(Q_i))^{\frac{n}{p}}}{t^{\frac{n}{2}+1}}\left(\int_{2^{j+1}Q_i\setminus 2^{j-2}Q_i}\Big|t\nabla_y e^{-t^2L}b_i(y)\Big|^2\,dy\right)^{1/2}\\
&\le C\sup_{0<t\le l(Q_i)}\left(2^jl(Q_i)\right)^{\frac{n}{p}}\ t^{-\frac{n}{r}-1}\,e^{-\frac{(2^jl(Q_i))^2}{ct^2}}\,\|b_i\|_{L^r(\mathbb{R}^n)} \qquad (4.57)
\end{aligned}$$

where we used Proposition 2.3, and (4.41). From the estimates (4.51)–(4.52) we deduce that the last expression in (4.57) is bounded by $C\alpha 2^{j\left(\frac{n}{p}-M\right)} l(Q_i)^{\frac{n}{p}}$ for any $M > \frac{n}{r} + 1$. Therefore,

$$\frac{C}{\alpha^p}\left(\sum_{i\in\mathbb{Z}}\sum_{j=3}^{\infty}\|T_i b_i\|_{L^p(2^jQ_i\setminus 2^{j-1}Q_i)}\|u\|_{L^{p'}(2^jQ_i\setminus 2^{j-1}Q_i)}\right)^p$$
$$\leq C\left(\sum_{i\in\mathbb{Z}}\sum_{j=3}^{\infty}2^{j\left(\frac{n}{p}-M\right)} l(Q_i)^{\frac{n}{p}}\|u\|_{L^{p'}(2^jQ_i\setminus 2^{j-1}Q_i)}\right)^p$$
$$\leq C\left(\sum_{i\in\mathbb{Z}}\sum_{j=3}^{\infty}2^{j(n-M)} l(Q_i)^{n}\left(\fint_{2^jQ_i}|u(x)|^{p'}\,dx\right)^{\frac{1}{p'}}\right)^p \leq C\left(\sum_{i\in\mathbb{Z}}\int_{Q_i}\mathcal{M}_{p'}(u)(y)\,dy\right)^p, \tag{4.58}$$

provided that M is sufficiently large. Since the cubes $\{Q_i\}_{i\in\mathbb{Z}}$ have a finite overlap, the last expression in (4.58) does not exceed $C\left(\int_{\bigcup_{i\in\mathbb{Z}} Q_i}\mathcal{M}_{p'}(u)(y)\,dy\right)^p$.

Recall now that the Hardy-Littlewood maximal function is weak type $(1,1)$ and hence, the operator $\mathcal{M}_{p'}$ is weak type (p',p'). Hence, by Kolmogorov condition (see, e.g., [12], p. 51),

$$\left(\int_{\bigcup_{i\in\mathbb{Z}} Q_i}\mathcal{M}_{p'}(u)(y)\,dy\right)^p \leq C\left(\Big|\bigcup_{i\in\mathbb{Z}} Q_i\Big|^{1-\frac{1}{p'}}\|u\|_{L^{p'}(\mathbb{R}^n)}\right)^p \leq C\sum_{i\in\mathbb{Z}}|Q_i| \leq \frac{C}{\alpha^q}\|\nabla f\|^q_{L^q(\mathbb{R}^n)}. \tag{4.59}$$

Combining this with (4.56), we deduce that

$$\left|\left\{x\in\mathbb{R}^n:\ \sup_{t>0}\Big(\fint_{B(x,t)}\Big|\nabla_y e^{-t^2L}\Big(\sum_{i\in\mathbb{Z}:\,l(Q_i)\leq t} b_i\Big)(y)\Big|^2\,dy\Big)^{\frac12} > \alpha/8\right\}\right| \leq \frac{C}{\alpha^q}\|\nabla f\|^q_{L^q(\mathbb{R}^n)}. \tag{4.60}$$

Step VII. It remains to estimate

$$\left|\left\{x\in\mathbb{R}^n:\ \sup_{t>0}\Big(\fint_{B(x,t)}\Big|(\nabla_y e^{-t^2L}-\nabla_y e^{-t\sqrt{L}})\Big(\sum_{i\in\mathbb{Z}:\,l(Q_i)>t} b_i\Big)(y)\Big|^2\,dy\Big)^{\frac12} > \alpha/8\right\}\right|$$
$$\leq \left|\left\{x\in\mathbb{R}^n:\ \sup_{t>0}\Big(\fint_{B(x,t)}\int_0^\infty\Big|v^2\nabla_x L e^{-v^2L}\Big(\sum_{i\in\mathbb{Z}:\,l(Q_i)>t} b_i\Big)(y)\Big|^2\,\frac{dv}{v}\,dy\Big)^{\frac12} > \alpha/8\right\}\right|. \tag{4.61}$$

This also splits further, according to whether $2l(Q_i) > v$ or $2l(Q_i)\leq v$. We start with the second case. First of all,

$$\sup_{t>0}\Big(\fint_{B(x,t)}\int_0^\infty\Big|v^2\nabla_x L e^{-v^2L}\Big(\sum_{i\in\mathbb{Z}:\,t<l(Q_i)\leq v/2} b_i\Big)(y)\Big|^2\,\frac{dv}{v}\,dy\Big)^{\frac12}$$
$$\leq C\sup_{t>0}\Big(\fint_{B(x,t)}\int_0^\infty\Big|v^3\nabla_x L e^{-v^2L}\Big(\sum_{i\in\mathbb{Z}:\,t<l(Q_i)\leq v/2}\frac{b_i}{l(Q_i)}\Big)(y)\Big|^2\,\frac{dv}{v}\,dy\Big)^{\frac12}$$

$$\leq C \sup_{t>0} \Big(\int_{2t}^{\infty} \fint_{B(x,t)} \Big| v^3 \nabla_x L e^{-v^2 L} \Big(\sum_{i\in\mathbb{Z}} \frac{b_i}{l(Q_i)} \Big)(y) \Big|^2 \, dy \, \frac{dv}{v} \Big)^{\frac{1}{2}}, \tag{4.62}$$

for every $x \in \mathbb{R}^n$. Since $v > 2t$, we can extract $e^{-t^2 L}$ from $e^{-v^2 L}$. More precisely, the last expression in (4.62) can be written as

$$C \sup_{t>0} \left(\int_{2t}^{\infty} \fint_{B(x,t)} \Big| \nabla_x e^{-t^2 L} \Big(v^3 L e^{-(v^2-t^2)L} \Big(\sum_{i\in\mathbb{Z}} \frac{b_i}{l(Q_i)} \Big) \Big)(y) \Big|^2 \, dy \, \frac{dv}{v} \right)^{\frac{1}{2}}, \tag{4.63}$$

which according to Lemma 2.4 is bounded by

$$C \sup_{t>0} \sum_{j=1}^{\infty} 2^{-jN} \left(\int_{2t}^{\infty} \left(\fint_{B(x,2^{j+1}t)} \Big| \nabla_x \Big(v^3 L e^{-(v^2-t^2)L} \Big(\sum_{i\in\mathbb{Z}} \frac{b_i}{l(Q_i)} \Big) \Big)(y) \Big|^p \, dy \right)^{\frac{2}{p}} \frac{dv}{v} \right)^{\frac{1}{2}}, \tag{4.64}$$

for any $p \in (p_-(L), 2]$, $N > \frac{n}{p} + 1$. However,

$$\begin{aligned}
\Big| \nabla_x \Big(v^3 L e^{-(v^2-t^2)L} \Big(\sum_{i\in\mathbb{Z}} \frac{b_i}{l(Q_i)} \Big) \Big) \Big| &= \Big| v^3 \nabla_x L \int_{v^2-t^2}^{\infty} \partial_s e^{-sL} \Big(\sum_{i\in\mathbb{Z}} \frac{b_i}{l(Q_i)} \Big) \, ds \Big| \\
&\leq \Big| v^3 \int_{v^2-t^2}^{\infty} \nabla_x L^2 e^{-sL} \Big(\sum_{i\in\mathbb{Z}} \frac{b_i}{l(Q_i)} \Big) \, ds \Big| \leq v^3 \int_{v^2-t^2}^{\infty} \Big| \nabla_x L^2 e^{-sL} \Big(\sum_{i\in\mathbb{Z}} \frac{b_i}{l(Q_i)} \Big) \Big| \, ds \\
&\leq v^3 \int_{v^2/2}^{\infty} \Big| \nabla_x L^2 e^{-sL} \Big(\sum_{i\in\mathbb{Z}} \frac{b_i}{l(Q_i)} \Big) \Big| \, ds,
\end{aligned} \tag{4.65}$$

since $v > 2t$. Using these considerations along with the Minkowski inequality, we can control the expresiion in (4.64) by

$$\begin{aligned}
&C \sup_{t>0} \sum_{j=1}^{\infty} 2^{-jN} \left(\fint_{B(x,2^{j+1}t)} \left(\int_{2t}^{\infty} \Big(v^3 \int_{v^2/2}^{\infty} \Big| \nabla_x L^2 e^{-sL} \Big(\sum_{i\in\mathbb{Z}} \frac{b_i}{l(Q_i)} \Big)(y) \Big| \, ds \Big)^2 \frac{dv}{v} \right)^{\frac{p}{2}} dy \right)^{\frac{1}{p}} \\
&\leq C \mathcal{M}_p \left(\left(\int_{2t}^{\infty} \Big(v^3 \int_{v^2/2}^{\infty} \Big| \nabla_x L^2 e^{-sL} \Big(\sum_{i\in\mathbb{Z}} \frac{b_i}{l(Q_i)} \Big) \Big| \, ds \Big)^2 \frac{dv}{v} \right)^{\frac{1}{2}} \right)(y).
\end{aligned} \tag{4.66}$$

Going further,

$$\begin{aligned}
&\left(\int_{2t}^{\infty} \Big(v^3 \int_{v^2/2}^{\infty} \Big| \nabla_x L^2 e^{-sL} \Big(\sum_{i\in\mathbb{Z}} \frac{b_i}{l(Q_i)} \Big) \Big| \, ds \Big)^2 \frac{dv}{v} \right)^{\frac{1}{2}} \\
&\quad \leq \left(\int_{2t}^{\infty} v^5 \Big(\int_{v^2/2}^{\infty} \Big| s \nabla_x L^2 e^{-sL} \Big(\sum_{i\in\mathbb{Z}} \frac{b_i}{l(Q_i)} \Big) \Big|^2 \, ds \Big) \Big(\int_{v^2/2}^{\infty} \frac{1}{s^2} \, ds \Big) \, dv \right)^{\frac{1}{2}}
\end{aligned}$$

$$\leq \Big(\int_{2t}^{\infty} v^3 \int_{v^2/2}^{\infty} \Big| s\nabla_x L^2 e^{-sL} \Big(\sum_{i\in\mathbb{Z}} \frac{b_i}{l(Q_i)} \Big) \Big|^2 \, ds dv \Big)^{\frac{1}{2}}$$

$$\leq \Big(\int_0^{\infty} \int_0^{\sqrt{2s}} v^3 \, dv \, \Big| s\nabla_x L^2 e^{-sL} \Big(\sum_{i\in\mathbb{Z}} \frac{b_i}{l(Q_i)} \Big) \Big|^2 ds \Big)^{\frac{1}{2}} \leq \Big(\int_0^{\infty} \Big| s^{5/2}\nabla_x L^2 e^{-sL} \Big(\sum_{i\in\mathbb{Z}} \frac{b_i}{l(Q_i)} \Big) \Big|^2 \frac{ds}{s} \Big)^{\frac{1}{2}}. \tag{4.67}$$

Therefore,

$$\Big| \Big\{ x \in \mathbb{R}^n : \sup_{t>0} \Big(\fint_{B(x,t)} \int_0^{\infty} \Big| v^2 \nabla_x L e^{-v^2 L} \Big(\sum_{i\in\mathbb{Z}:\, t<l(Q_i)\leq v/2} b_i \Big)(y) \Big|^2 \frac{dv}{v} dy \Big)^{\frac{1}{2}} > \alpha/16 \Big\} \Big|$$

$$\leq \frac{C}{\alpha^r} \Big\| \sup_{t>0} \Big(\fint_{B(x,t)} \int_0^{\infty} \Big| v^2 \nabla_x L e^{-v^2 L} \Big(\sum_{i\in\mathbb{Z}:\, t<l(Q_i)\leq v/2} b_i \Big)(y) \Big|^2 \frac{dv}{v} dy \Big)^{\frac{1}{2}} \Big\|_{L^r(\mathbb{R}^n)}^r$$

$$\leq \frac{C}{\alpha^r} \Big\| \mathcal{M}_p \Big(\Big(\int_0^{\infty} \Big| s^{5/2} \nabla_x L^2 e^{-sL} \Big(\sum_{i\in\mathbb{Z}} \frac{b_i}{l(Q_i)} \Big) \Big|^2 \frac{ds}{s} \Big)^{\frac{1}{2}} \Big) \Big\|_{L^r(\mathbb{R}^n)}^r, \tag{4.68}$$

for any $p \in (p_-(L), 2]$. We take $p \in (p_-(L), r)$. Using the boundedness of $\mathcal{M}_p$ in L^r and (4.19), we conclude that the left -hand side of (4.68) is bounded by

$$\frac{C}{\alpha^r} \Big\| \sum_{i\in\mathbb{Z}} \frac{b_i}{l(Q_i)} \Big\|_{L^r(\mathbb{R}^n)} \leq C \sum_{i\in\mathbb{Z}} |Q_i| \tag{4.69}$$

analogously to (4.50)–(4.52). Hence,

$$\Big| \Big\{ x \in \mathbb{R}^n : \sup_{t>0} \Big(\fint_{B(x,3\kappa t)} \int_0^{\infty} \Big| v^2 \nabla_x L e^{-v^2 L} \Big(\sum_{i\in\mathbb{Z}:\, t<l(Q_i)\leq v/2} b_i \Big)(y) \Big|^2 \frac{dv}{v} \, dy \Big)^{\frac{1}{2}} > \alpha/16 \Big\} \Big|$$

$$\leq C \sum_{i\in\mathbb{Z}} |Q_i| \leq C\alpha^{-q} \|\nabla f\|_{L^q(\mathbb{R}^n)}^q, \tag{4.70}$$

as desired.

Step VIII. The case $l(Q_i) > v/2$ can be treated analogously to the argument in Step VI. We write

$$\sup_{t>0} \Big(\fint_{B(x,t)} \int_0^{\infty} \Big| v^2 \nabla_x L e^{-v^2 L} \Big(\sum_{i\in\mathbb{Z}:\, t<l(Q_i),\, v<2l(Q_i)} b_i \Big)(y) \Big|^2 \frac{dv}{v} \, dy \Big)^{\frac{1}{2}} \leq \sum_{i\in\mathbb{Z}} \widetilde{T}_i b_i(x), \quad x \in \mathbb{R}^n, \tag{4.71}$$

with

$$\widetilde{T}_i f(x) := \sup_{0<t\leq l(Q_i)} \Big(\fint_{B(x,t)} \int_0^{2l(Q_i)} \Big| v^2 \nabla_x L e^{-v^2 L} f(y) \Big|^2 \frac{dv}{v} \, dy \Big)^{\frac{1}{2}}, \quad x \in \mathbb{R}^n, \quad f \in L^2(\mathbb{R}^n). \tag{4.72}$$

Due to the argument built in Step VI, it is sufficient to establish that

$$\|\widetilde{T}_i b_i\|_{L^p(2^j Q_i \setminus 2^{j-1} Q_i)} \leq C\alpha 2^{-Nj} l(Q_i)^{\frac{n}{p}}, \quad i \in \mathbb{Z}, \quad j \geq 3, \tag{4.73}$$

for a sufficiently large N and some $p \in (1,\infty)$. We will show (4.73) for $p \in (2, 2+\varepsilon(L))$. Indeed,

$$\|\widetilde{T}_i b_i\|_{L^p(2^jQ_i\setminus 2^{j-1}Q_i)} = \left(\int_{2^jQ_i\setminus 2^{j-1}Q_i} \sup_{0<t\le l(Q_i)} \left(\fint_{B(x,t)} \int_0^{2l(Q_i)} \left|v^2\nabla_x Le^{-v^2L} b_i(y)\right|^2 \frac{dv}{v}\,dy\right)^{p/2} dx\right)^{1/p}$$

$$\le \left(\int_{2^jQ_i\setminus 2^{j-1}Q_i} \sup_{0<t\le l(Q_i)} \left(\fint_{B(x,t)} \int_0^{2l(Q_i)} \left|\chi_{2^{j+1}Q_i\setminus 2^{j-2}Q_i}(y) v^2\nabla_x Le^{-v^2L} b_i(y)\right|^2 \frac{dv}{v}\,dy\right)^{p/2} dx\right)^{1/p}$$

$$= \left\|\mathcal{M}_2\left(\left(\int_0^{2l(Q_i)} \left|\chi_{2^{j+1}Q_i\setminus 2^{j-2}Q_i} v^2\nabla_x Le^{-v^2L} b_i\right|^2 \frac{dv}{v}\right)^{1/2}\right)\right\|_{L^p(\mathbb{R}^n)}$$

$$\le C\left\|\left(\int_0^{2l(Q_i)} \left|\chi_{2^{j+1}Q_i\setminus 2^{j-2}Q_i} v^2\nabla_x Le^{-v^2L} b_i\right|^2 \frac{dv}{v}\right)^{1/2}\right\|_{L^p(\mathbb{R}^n)}$$

$$= C\left\|\left(\int_0^{2l(Q_i)} \left|v^2\nabla_x Le^{-v^2L} b_i\right|^2 \frac{dv}{v}\right)^{1/2}\right\|_{L^p(2^{j+1}Q_i\setminus 2^{j-2}Q_i)} \tag{4.74}$$

Using a combination of the off-diagonal estimates for $\frac{v}{\sqrt{2}}\nabla_x e^{-(v/\sqrt{2})^2L}$ and $\left(\frac{v}{\sqrt{2}}\right)^2 Le^{-(v/\sqrt{2})^2L}$ and Minkowski inequality, we further bound the expression above by

$$C\left(\int_0^{2l(Q_i)} \left\|v^3\nabla_x Le^{-v^2L} b_i\right\|^2_{L^p(2^{j+1}Q_i\setminus 2^{j-2}Q_i)} \frac{dv}{v^3}\right)^{1/2}$$

$$\le C\left(\int_0^{2l(Q_i)} \left(v^{\frac{n}{p}-\frac{n}{r}} e^{-\frac{(2^j l(Q_i))^2}{cv^2}} \|b_i\|_{L^r(\mathbb{R}^n)}\right)^2 \frac{dv}{v^3}\right)^{1/2}$$

$$\le C2^{-jM} l(Q_i)^{\frac{n}{p}-\frac{n}{r}-1} \|b_i\|_{L^r(\mathbb{R}^n)}, \tag{4.75}$$

for any $M > \frac{n}{p} - \frac{n}{r} - 1$. This bound, combined with (4.51)–(4.52), leads to (4.73), and finishes the argument. □

As a combination of Theorem 4.1 and Proposition 3.4, we obtain the following Corollary, which is essentially the main result of this paper.

Corollary 4.4. *For every $p \in \left(\max\left\{\frac{2n}{n+4}, 1\right\}, \frac{2n}{n+2}\right)$, there exists an elliptic operator $\mathbb{L}$ in block form with complex bounded measurable coefficients such that (R_p) is solvable for $\mathbb{L}$ but $(D_{p'})$ is not solvable for $\mathbb{L}^*$.*

Proof. The result follows by combining Theorem 4.1 with Proposition 3.4 as soon as we notice that the operator $\mathbb{L}^*$ possesses the same properties as $\mathbb{L}$ itself: it is also an elliptic operator in block form with complex bounded measurable coefficients. □

5. Negative results for the Neumann problem

Proposition 5.1. *For every $p < \frac{2n}{n+2}$, there exists an elliptic operator $\mathbb{L}$ in block form with complex bounded measurable coefficients such that (N_p) is not solvable for $\mathbb{L}$. In particular, for every*

$p \in \left(\max\left\{\frac{2n}{n+4}, 1\right\}, \frac{2n}{n+2}\right)$ *there exists an elliptic operator* $\mathbb{L}$ *such that the regularity problem* (R_p) *is solvable for* $\mathbb{L}^*$ *but* (N_p) *is not solvable for* $\mathbb{L}$.

Proof. Let us fix some $p_0 < \frac{2n}{n+2}$ and assume that the Neumann problem is solvable for every block operator $\mathbb{L}$, that is,

$$\|\mathcal{N}_2(\nabla_{x,t} e^{-t\sqrt{L}} f)\|_{L^{p_0}(\mathbb{R}^n)} \leq C\|g\|_{L^{p_0}(\mathbb{R}^n)}. \tag{5.1}$$

Considering the derivative in t only, and taking into account that $f = (\sqrt{L})^{-1} g$, we are further led to the estimate

$$\left\|\mathcal{N}_2\left(\sqrt{L} e^{-t\sqrt{L}}(\sqrt{L})^{-1} g\right)\right\|_{L^{p_0}(\mathbb{R}^n)} = \left\|\mathcal{N}_2\left(e^{-t\sqrt{L}} g\right)\right\|_{L^{p_0}(\mathbb{R}^n)} \leq C\|g\|_{L^{p_0}(\mathbb{R}^n)}. \tag{5.2}$$

Since $p_0 < 2$, this in particular implies that

$$\left\|\mathcal{N}_{p_0}\left(e^{-t\sqrt{L}} g\right)\right\|_{L^{p_0}(\mathbb{R}^n)} \leq C\|g\|_{L^{p_0}(\mathbb{R}^n)}, \tag{5.3}$$

for all $g \in L^2(\mathbb{R}^n) \cap L^{p_0}(\mathbb{R}^n)$, and thus for all $g \in L^{p_0}(\mathbb{R}^n)$ by density. Hence, using the calculation in (3.20)–(3.21), we deduce that (3.18) holds with $c\kappa$ in place of κ and p_0 in place of r.

By the argument closely following that of Proposition 3.1 we deduce that

$$L^{-\alpha} : L^p(\mathbb{R}^n) \longrightarrow L^r(\mathbb{R}^n), \qquad \alpha = \frac{1}{2}\left(\frac{n}{p} - \frac{n}{r}\right), \tag{5.4}$$

for $p_0 < p < r < p_+(L)$. We just have to be a little more careful with the argument. More precisely, the interpolation of our assumptions with (3.5) for $p_-(L) < p < r < p_+(L)$ will only lead to (3.5) for $p_0 < p < r < p_+(L)$ with the extra restriction that $\left|\frac{1}{p} - \frac{1}{r}\right| \leq \left|\frac{1}{p_-(L)} - \frac{1}{p_+(L)}\right|$, and the semigroup multiplication property is not helping in this case. Therefore, we will have the restriction on the size of α throughout the proof. That does not effect the final result though, since the powers of L can be composed afterwards.

By duality, (5.4) further implies

$$(L^*)^{-\alpha} : L^{r'}(\mathbb{R}^n) \longrightarrow L^{p'}(\mathbb{R}^n), \qquad \alpha = \frac{1}{2}\left(\frac{n}{p'} - \frac{n}{r'}\right), \tag{5.5}$$

where $(p_+(L))' < r' < p' < (p_0)'$, and $(p_+(L))'$ and $(p_0)'$ are dual exponents of $p_+(L)$ and p_0, respectively. Recall that $p_0 < \frac{2n}{n+2}$, so that $(p_0)' > \frac{2n}{n-2}$, and also that we are assuming that (5.2) holds for all elliptic operators, so that (5.5) must hold for all elliptic operators. This will lead to a contradiction with Lemma 3.3 exactly the same way as in the proof of Proposition 3.4. Thus, the Neumann problem (N_{p_0}) is not solvable for some operator $\mathbb{L}$.

However, according to Theorem 4.1 the solvability of regularity problem still holds. This finishes the proof. □

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